Chapter 2

calculus and geometry of curves

2.1 calculus for curves

In this section we describe the calculus for functions with a domain of real numbers and a range of vectors. It is possible to define the derivative in terms of a limiting process, but, little is gained by doing so in this section so I make a more pragmatic definition[1]. We'll begin with $\mathbb{R}^3$,

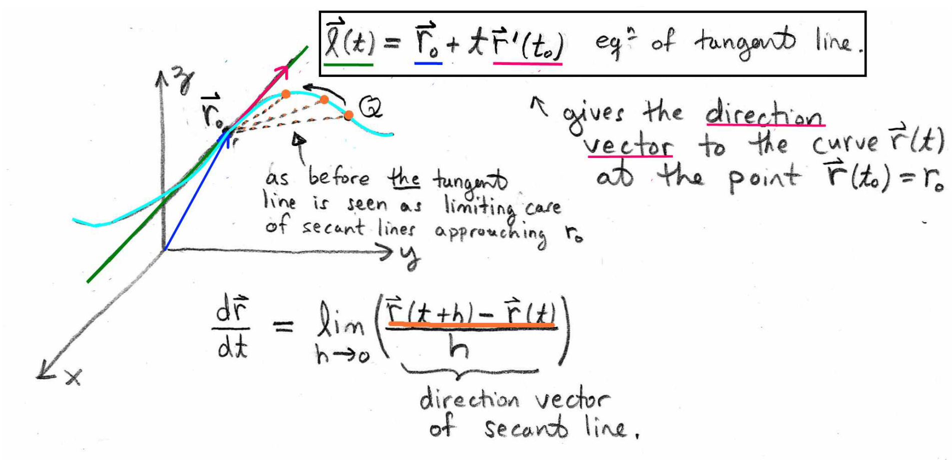
Definition 2.1.1. *calculus of 3-vector-valued functions.*

Suppose $\vec{F}(t) = \langle F_1(t), F_2(t), F_3(t)\rangle$ then

1. If F_1, F_2 and F_3 are differentiable functions near t we define
$$\frac{d\vec{F}}{dt} = \frac{d}{dt}\langle F_1, F_2, F_3\rangle = \left\langle \frac{dF_1}{dt}, \frac{dF_2}{dt}, \frac{dF_3}{dt} \right\rangle.$$
2. If F_1, F_2 and F_3 are integrable functions on $[a, b]$ then we define
$$\int_a^b \vec{F}(t)dt = \int_a^b \langle F_1, F_2, F_3\rangle dt = \left\langle \int_a^b F_1(t)dt, \int_a^b F_2(t)dt, \int_a^b F_3(t)dt \right\rangle.$$
3. We write $\int \vec{f}(t)dt = \vec{F}(t) + \vec{c}$ iff $\frac{d\vec{F}}{dt} = \vec{f}(t)$ and $\vec{c} = \langle c_1, c_2, c_3\rangle$ is a constant vector. Equivalently,
$$\int \vec{f}(t)dt = \left\langle \int f_1(t)dt, \int f_2(t)dt, \int f_3(t)dt \right\rangle.$$

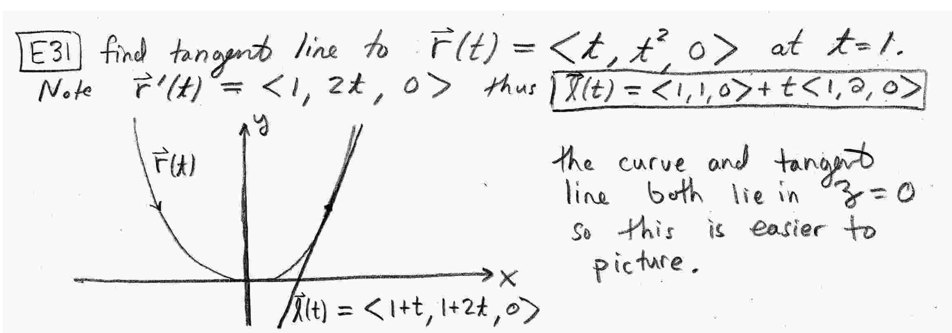
[1] for the purist you can skip ahead to the chapter on differentiation where I describe how to differentiate a general function from $\mathbb{R}^n$ to $\mathbb{R}^m$, this is definition can be derived from that definition with a few basic theorems of advanced calculus.

We also use the prime notation for differentiation of vector valued functions if it is convenient; this means $\vec{A}\,'(t) = d\vec{A}/dt = \frac{d\vec{A}}{dt}$. Higher derivatives are also denoted in the same manner as previously; for example, $\frac{d^2\vec{A}}{dt^2} = \frac{d}{dt}\left[\frac{d\vec{A}}{dt}\right]$. The geometric meaning of the definition is encapsulated in the picture below:



If $t \mapsto \vec{r}(t)$ is some parametrized curve and $t_o \in dom(\vec{r})$ such that $\vec{r}\,'(t_o) \neq 0$ defines the tangent vector to the curve at $\vec{r}(t_o)$. Moreover, a natural parametrization of the tangent line is given by $\vec{l}(s) = \vec{r}_o + s\vec{r}\,'(t_o)$. Recall that the parametric view is natural one in this context. Hopefully we learned this already in the first chapter of these notes.

Example 2.1.2. .



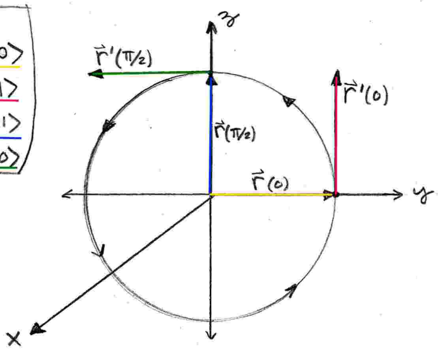
Example 2.1.3. .

E26 Consider $\vec{r}(t) = \langle 0, \cos t, \sin t \rangle$. Find $\vec{r}\,'(t)$ and plot $\vec{r}\,'(0)$ and $\vec{r}\,'(\pi/2)$. Then calculate $\int \vec{r}(t)\,dt$.

$$\vec{r}\,'(t) = \frac{d}{dt}\big(\langle 0, \cos t, \sin t\rangle\big)$$
$$= \left\langle \frac{d}{dt}(0), \frac{d}{dt}(\cos t), \frac{d}{dt}(\sin t)\right\rangle$$
$$= \boxed{\langle 0, -\sin t, \cos t\rangle = \frac{d\vec{r}}{dt}}$$

$$\int \vec{r}(t)\,dt = \left\langle \int 0\,dt, \int \cos t\,dt, \int \sin t\,dt \right\rangle$$
$$= \langle c_1, \sin t + c_2, -\cos t + c_3 \rangle$$
$$= \boxed{\langle 0, \sin t, -\cos t\rangle + C = \int \vec{r}(t)\,dt}$$

Evaluate:
$\vec{r}(0) = \langle 0, 1, 0\rangle$
$\vec{r}\,'(0) = \langle 0, 0, 1\rangle$
$\vec{r}(\pi/2) = \langle 0, 0, 1\rangle$
$\vec{r}\,'(\pi/2) = \langle 0, -1, 0\rangle$



We see that $\vec{r}(t)$ is a circle of radius one in the yz-plane.

Algebraically we can prove this,
$x = 0$
$y = \cos t$
$z = \sin t$
$\Rightarrow y^2 + z^2 = \sin^2 t + \cos^2 t = 1$

Example 2.1.4. .

Consider $r_1(t) = \langle t, t^2, t^3\rangle$ and $r_2(t) = \langle \sin t, \sin 2t, t\rangle$ these intersect at $t=0$ where $r_1(0) = r_2(0) = \langle 0, 0, 0\rangle$. The angle of intersection is the angle between their tangents.

$$r_1'(t) = \langle 1, 2t, 3t^2\rangle \implies r_1'(0) = \langle 1, 0, 0\rangle$$
$$r_2'(t) = \langle \cos t, 2\cos 2t, 1\rangle \implies r_2'(0) = \langle 1, 2, 1\rangle$$

Find angle via dot-product. Note that $r_1'(0)\cdot r_2'(0) = 1$ while $|r_1'(0)| = 1$ and $|r_2'(0)| = \sqrt{1+4+1} = \sqrt{6}$ thus

$$1 = \sqrt{6}\cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{6}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = \boxed{66^\circ}$$

Example 2.1.5. .

E30 Consider the curve $\vec{r}(t) = \langle \cos t, \sin t, \frac{1}{100}\sin(100t) \rangle$, $0 \leq t \leq 2\pi$, plot the curve and the tangent line at $\vec{r}(\pi/4)$. Also find the eq$^{\underline{n}}$ of the tangent line.

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, \tfrac{100}{100}\cos(100t) \rangle$$

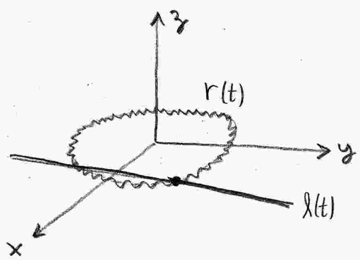
$$\vec{r}(\pi/4) = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle$$

$$\vec{r}\,'(\pi/4) = \langle -1/\sqrt{2}, 1/\sqrt{2}, \cos(25\pi) \rangle = \langle -1/\sqrt{2}, 1/\sqrt{2}, -1 \rangle$$

Thus the eq$^{\underline{n}}$ of the tangent line to $r(\pi/4)$ is

$$\boxed{\vec{l}(t) = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle + t\langle -1/\sqrt{2}, 1/\sqrt{2}, -1 \rangle}$$

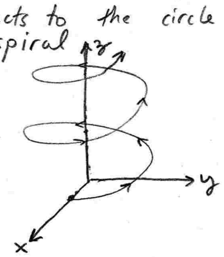
Notice $\frac{-1}{100} \leq z \leq \frac{1}{100}$ so $\approx z = 0$ and the curve is simply $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. Roughly,



Remark: conceptually the "t" in $\vec{r}(t)$ and $\vec{l}(t)$ is distinct. Probably it'd be better to use a different parameter for $\vec{l}$.

Example 2.1.6. .

E27 Consider $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$. Plot and find $\vec{r}\,'(t)$. This projects to the circle $\cos^2 t + \sin^2 t = x^2 + y^2 = 1$ in the xy-plane, its a spiral



$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

notice $\frac{dz}{dt} = 1$ so this spiral rises with slope one for all time.

Example 2.1.7. .

$\boxed{E28}$ Study $\vec{r}(t) = \vec{r}_o + t\vec{v}$ where $\vec{r}_o, \vec{v}$ are fixed vectors independent of time. Let $\vec{v} = \langle a, b, c\rangle$,

$$\vec{r}\,'(t) = \frac{d}{dt}\left(\vec{r}_o + t\vec{v}\right) = \underbrace{\frac{d\vec{r}_o}{dt}}_{0} + \frac{d}{dt}\langle ta, tb, tc\rangle = \langle a, b, c\rangle = \vec{v}.$$

the tangent vector to this curve is $\vec{r}\,'(t) = \vec{v}$ for all time. We find that a line in $\mathbb{R}^3$ has constant direction vectors.

Example 2.1.8. *Let* $\vec{F}(t) = \langle 1,\ t,\ \cos(t)\rangle$.

$$\frac{d\vec{F}}{dt} = \left\langle \frac{d}{dt}(1), \frac{d}{dt}(t), \frac{d}{dt}(\cos(t)) \right\rangle = \ \langle 0,\ 1,\ -\sin(t)\rangle.$$

$$\int \vec{F}(t)dt = \left\langle \int dt, \int tdt, \int \cos(t)dt \right\rangle = \left\langle t + c_1,\ \frac{1}{2}t^2 + c_2,\ \sin(t) + c_3 \right\rangle.$$

$$\int_0^1 \vec{F}(t)dt = \left\langle \int_0^1 dt, \int_0^1 tdt, \int_0^1 \cos(t)dt \right\rangle = \left\langle 1,\ \frac{1}{2},\ \sin(1) \right\rangle.$$

The derivative of an n-vector valued functions of a real variable is likewise calculated component by the component.

Definition 2.1.9. *calculus of n-vector-valued functions.*

Suppose $\vec{F}(t) = \langle F_1(t), F_2(t), \dots, F_n(t)\rangle$ then

1. If $F_1, F_2, \dots, F_n$ are differentiable functions near t we define
$$\frac{d\vec{F}}{dt} = \frac{d}{dt}\langle F_1, F_2, \dots, F_n\rangle = \left\langle \frac{dF_1}{dt}, \frac{dF_2}{dt}, \dots, \frac{dF_n}{dt} \right\rangle.$$

2. If $F_1, F_2, \dots, F_n$ are integrable functions on $[a, b]$ then we define
$$\int_a^b \vec{F}(t)dt = \int_a^b \langle F_1, F_2, \dots, F_n\rangle dt = \left\langle \int_a^b F_1(t)dt, \int_a^b F_2(t)dt, \dots, \int_a^b F_n(t)dt \right\rangle.$$

3. We write $\int \vec{f}(t)dt = \vec{F}(t) + \vec{c}$ iff $\frac{d\vec{F}}{dt} = \vec{f}(t)$ and $\vec{c} = \langle c_1, c_2, \dots, c_n\rangle$ is a constant vector. Equivalently,
$$\int \vec{f}(t)dt = \left\langle \int f_1(t)dt, \int f_2(t)dt, \dots, \int f_n(t)dt \right\rangle.$$

In summation notation the definitions translate to:

$$\frac{d}{dt}\left[\sum_{j=1}^{n} F_j\,\widehat{x}_j\right] = \sum_{j=1}^{n}\frac{dF_j}{dt}\,\widehat{x}_j \qquad \text{and} \qquad \int_a^b \left[\sum_{j=1}^{n} F_j\,\widehat{x}_j\right]dt = \sum_{j=1}^{n}\left[\int_a^b F_j\; dt\right]\widehat{x}_j$$

We differentiate and integrate componentwise.

Example 2.1.10. *Let $\vec{F}(t) = \langle t,\ t^2,\ \dots,\ t^n\rangle$. It follows that,*

$$\frac{d\vec{F}}{dt} = \langle\ 1, 2t, \dots, nt^{n-1}\ \rangle$$

and, for constants $c_1, c_2, \dots, c_n$,

$$\int \vec{F}(t)dt = \langle\ \frac{1}{2}t^2 + c_1, \frac{1}{3}t^3 + c_2, \dots, \frac{1}{n+1}t^{n+1} + c_n\ \rangle.$$

Or we could calculate via summation notation, note that $\vec{F}(t) = \sum_{j=1}^{n} t^j\,\widehat{x}_j$ hence,

$$\frac{d\vec{F}}{dt} = \frac{d}{dt}\sum_{j=1}^{n} t^j\,\widehat{x}_j = \sum_{j=1}^{n}\frac{d}{dt}(t^j)\,\widehat{x}_j = \sum_{j=1}^{n} jt^{j-1}\,\widehat{x}_j.$$

Likewise,

$$\int \vec{F}(t)dt = \int\sum_{j=1}^{n} t^j\,\widehat{x}_j dt = \sum_{j=1}^{n}\int t^j dt\,\widehat{x}_j = \sum_{j=1}^{n}\left(\frac{1}{j+1}t^{j+1} + c_j\right)\widehat{x}_j.$$

We usually find ourselves working problems with $n = 1, 2$ or 3 in this course. Many of the theorems known to us from calculus I apply equally well to vector-valued functions of a real variable. The key is that the differentiation concerns the domain whereas the range just rides along. If there was somehow a time-dependence for $\widehat{x}$, $\widehat{y}$, $\widehat{z}$ then the story would change, but we inist that $\widehat{x}$, $\widehat{y}$, $\widehat{z}$ are the unit-vectors of a fixed x, y, z-coodinate system[2].

[2] In physics one might consider moving coordinate systems and in such a context the rules are a bit more interesting.

Theorem 2.1.11. *fundamental theorems of calculus for space curves.*

$$(I.) \quad \frac{d}{dt}\int_a^t \vec{F}(\tau)d\tau = \vec{F}(t)$$

$$(II.) \quad \int_a^b \frac{d\vec{G}}{dt}dt = \vec{G}(b) - \vec{G}(a)$$

Proof: Apply the FTC part I componentwise as shown below:

$$\begin{aligned}
\frac{d}{dt}\int_a^t \vec{F}(\tau)d\tau = \frac{d}{dt}\int_a^t \left[\sum_{j=1}^n F_j(\tau)\,\widehat{x}_j\right]d\tau &= \frac{d}{dt}\sum_{j=1}^n \left[\int_a^t F_j(\tau)\,d\tau\right]\widehat{x}_j \\
&= \sum_{j=1}^n \left[\frac{d}{dt}\int_a^t F_j(\tau)\,d\tau\right]\widehat{x}_j \\
&= \sum_{j=1}^n F_j(t)\,\widehat{x}_j = \vec{F}(t).
\end{aligned}$$

thus (I.) holds true. Next apply FTC part II componentwise as shown below:

$$\begin{aligned}
\int_a^b \frac{d\vec{G}}{dt}dt = \int_a^b \frac{d}{dt}\left[\sum_{j=1}^n G_j(t)\,\widehat{x}_j\right]dt &= \int_a^b \left[\sum_{j=1}^n \frac{dG_j}{dt}\,\widehat{x}_j\right]dt \\
&= \sum_{j=1}^n \left[\int_a^b \frac{dG_j}{dt}dt\right]\widehat{x}_j \\
&= \sum_{j=1}^n \left[G_j(b) - G_j(a)\right]\widehat{x}_j \\
&= \sum_{j=1}^n G_j(b)\,\widehat{x}_j + \sum_{j=1}^n G_j(a)\,\widehat{x}_j \\
&= \vec{G}(b) - \vec{G}(a).
\end{aligned}$$

Therefore, part (II.) is true. $\square$

Example 2.1.12. .

Recall that $\int \frac{dt}{1+t^2} = \tan^{-1}(t) + C$ & $\int \frac{2t\,dt}{1+t^2} = \int \frac{du}{u} = \ln|1+t^2| + C$

thus we calculate,

$$\int_0^1 \left\langle 0, \frac{4}{1+t^2}, \frac{2t}{1+t^2} \right\rangle dt = \left\langle \int_0^1 0 \cdot dt, \int_0^1 \frac{4\,dt}{1+t^2}, \int_0^1 \frac{2t\,dt}{1+t^2} \right\rangle$$

$$= \left\langle 0, 4\tan^{-1}(t)\Big|_0^1, \ln|1+t^2|\Big|_0^1 \right\rangle$$

$$= \langle 0, 4\tan^{-1}(1), \ln(2) \rangle$$

$$= \boxed{\langle 0, \pi, \ln(2) \rangle}$$

Notice, $\tan(\pi/4) = 1$ $\therefore$ $\tan^{-1}(1) = \pi/4$.

Many other properties of differentiation and integration hold for vector-valued functions.

Theorem 2.1.13. *rules of calculus for space curves.*

Let $\vec{A}, \vec{B} : J \subseteq \mathbb{R} \to \mathbb{R}^n$ and $f : J \subseteq \mathbb{R} \to \mathbb{R}$ are differentiable functions and $c \in \mathbb{R}$,

(1.) $\displaystyle \frac{d}{dt}\left[\vec{A} + \vec{B}\right] = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$ (3.) $\displaystyle \int [\vec{A} + \vec{B}]\,dt = \int \vec{A}dt + \int \vec{B}dt$

(2.) $\displaystyle \frac{d}{dt}\left[c\vec{A}\right] = c\frac{d\vec{A}}{dt}$ (4.) $\displaystyle \int c\vec{A}dt = c\int \vec{A}dt$

Proof: The proof of the theorem above is easily derived by simply expanding what the vector notation means and borrowing the corresponding theorems from calculus I to simplify the component expressions. I might ask for this in homework so I'll not offer details here. $\Box$.

There are several types of products we can consider for vector-valued function. Each has a natural product rule.

Theorem 2.1.14. *product rules of calculus for space curves.*

Let $\vec{A}, \vec{B} : J \subseteq \mathbb{R} \to \mathbb{R}^n$ and $f : J \subseteq \mathbb{R} \to \mathbb{R}$ are differentiable functions and $c \in \mathbb{R}$,

$$(1.)\quad \frac{d}{dt}\Big[f\vec{A}\Big] = \frac{df}{dt}\vec{A} + f\frac{d\vec{A}}{dt}$$

$$(2.)\quad \frac{d}{dt}\Big[\vec{A}\bullet\vec{B}\Big] = \frac{d\vec{A}}{dt}\bullet\vec{B} + \vec{A}\bullet\frac{d\vec{B}}{dt}$$

$$(3.)\quad \text{for } n=3,\quad \frac{d}{dt}\Big[\vec{A}\times\vec{B}\Big] = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

Proof: let $\vec{A}$ and f be differentiable near t and suppose $\vec{A} = \sum_{j=1}^n A_j\,\widehat{x}_j$. Note $f\vec{A} = \sum_{j=1}^n fA_j\,\widehat{x}_j$ and calculate

$$\begin{aligned}
\frac{d}{dt}[\,f\vec{A}\,] &= \frac{d}{dt}\left[\sum_{j=1}^n fA_j\,\widehat{x}_j\right] \\
&= \sum_{j=1}^n \frac{d}{dt}[\,fA_j\,]\,\widehat{x}_j \\
&= \sum_{j=1}^n \left[\,\frac{df}{dt}A_j + f\frac{dA_j}{dt}\,\right]\widehat{x}_j \\
&= \frac{df}{dt}\sum_{j=1}^n A_j\,\widehat{x}_j + f\sum_{j=1}^n \frac{dA_j}{dt}\,\widehat{x}_j \\
&= \frac{df}{dt}\vec{A} + f\frac{d\vec{A}}{dt}.
\end{aligned}$$

The proof of (1.) is complete. Now consider the dot-product of $\vec{A}$ with $\vec{B}$,

$$\begin{aligned}
\frac{d}{dt}[\,\vec{A}\bullet\vec{B}\,] &= \frac{d}{dt}\left[\sum_{j=1}^n A_jB_j\right] \\
&= \sum_{j=1}^n \frac{d}{dt}[\,A_jB_j\,] \\
&= \sum_{j=1}^n \left[\frac{dA_j}{dt}B_j + A_j\frac{dB_j}{dt}\right] \\
&= \sum_{j=1}^n \frac{dA_j}{dt}B_j + \sum_{j=1}^n A_j\frac{dB_j}{dt} \\
&= \frac{d\vec{A}}{dt}\bullet\vec{B} + \vec{A}\bullet\frac{d\vec{B}}{dt}.
\end{aligned}$$

The proof of (2.) is complete. Only in $n = 3$ do we have a binary operation which is a cross-product, fortunately we have an easy notation so this will not be much harder than (2.).

$$\begin{aligned}
\frac{d}{dt}\Big[\ \vec{A} \times \vec{B}\ \Big] &= \frac{d}{dt}\Bigg[\ \sum_{i,j,k=1}^{n} A_i B_j \epsilon_{ijk}\, \widehat{x}_k\Bigg] \\
&= \sum_{k=1}^{n} \frac{d}{dt}\Bigg[\ \sum_{i,j=1}^{3} A_i B_j \epsilon_{ijk}\ \Bigg]\widehat{x}_k \\
&= \sum_{k=1}^{n}\Bigg[\ \sum_{i,j=1}^{3} \epsilon_{ijk} \frac{d}{dt}\big[\ A_i B_j\ \big]\ \Bigg]\widehat{x}_k \\
&= \sum_{k=1}^{n}\Bigg[\ \sum_{i,j=1}^{3} \epsilon_{ijk}\left[\frac{dA_i}{dt} B_j + A_i \frac{dB_j}{dt}\right]\ \Bigg]\widehat{x}_k \\
&= \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{dA_i}{dt} B_j\, \widehat{x}_k + \sum_{i,j,k=1}^{3} \epsilon_{ijk} A_i \frac{dB_j}{dt}\, \widehat{x}_k \\
&= \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}.
\end{aligned}$$

The proof of (3.) is complete. I know some students don't care for the use of summations in calculus , but I would encourage such students to work this out without sums and reconsider your thinking. In the proof above you would have 6 products which would yield 12 terms and then those have to be rearranged to see the cross-products. Not that its impossible, or even too difficult, it's just that the summation notation is much cleaner. $\square$

In the prime notation the product rules for vector products are

$$(\vec{A} \bullet \vec{B})' = \vec{A}' \bullet \vec{B} + \vec{A} \bullet \vec{B}' \qquad (\vec{A} \times \vec{B})' = \vec{A}' \times \vec{B} + \vec{A} \times \vec{B}'.$$

We use these in future section to help uncover the geometry of curves.

Example 2.1.15. *This short calculation shows that torque is the time-rate of change of the angular momentum.*

Angular Momentum $\vec{L}(t) = m\vec{r}(t) \times \vec{V}(t)$ and then Torque is $\vec{\tau}(t) = m\vec{r}(t) \times \vec{a}(t)$.

$$
\begin{aligned}
\frac{d\vec{L}}{dt} &= \frac{d}{dt}\left[m\vec{r}(t) \times \vec{V}(t)\right] \\
&= m\left[\frac{d\vec{r}}{dt} \times \vec{V}(t) + \vec{r}(t) \times \frac{d}{dt}\left(\vec{V}(t)\right)\right] \\
&= m\left[\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r}(t) \times \vec{a}(t)\right] \\
&= m\vec{r}(t) \times \vec{a}(t) \\
&= \boxed{\vec{\tau}(t) = \frac{d\vec{L}}{dt}}
\end{aligned}
$$

Example 2.1.16. *This short calculation shows that a time varying unit-vector perpendicular to it's tangent vector. (since it is a unit-vector the only thing that changes is it's direction)*

$$
\begin{aligned}
&|\vec{T}|^2 = \vec{T} \cdot \vec{T} = 1 \\
&\Rightarrow \underbrace{\frac{d\vec{T}}{dt} \cdot \vec{T} + \vec{T} \cdot \frac{d\vec{T}}{dt}} = \frac{d}{dt}(1) = 0 \\
&\qquad 2\frac{d\vec{T}}{dt} \cdot \vec{T} = 0 \\
&\qquad \therefore \boxed{\frac{d\vec{T}}{dt} \cdot \vec{T} = 0 \quad \text{thus } \vec{T} \text{ is } \perp \text{ to } \frac{d\vec{T}}{dt}}
\end{aligned}
$$

Theorem 2.1.17. *chain rule of calculus for space curves.*

Let $g : K \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable near t and $\vec{F} : J \subseteq \mathbb{R} \to \mathbb{R}^n$ be differentiable near $g(t)$ then near t we have

$$\frac{d}{dt}\big[\vec{F}(g(t))\big] = \frac{d\vec{F}}{dt}(g(t))\frac{dg}{dt}.$$

Proof: let $\vec{F} = \langle F_1, F_2, \dots, F_n \rangle$ and calculate,

$$\begin{aligned}
\frac{d}{dt}\big[\vec{F}(g(t))\big] &= \frac{d}{dt}\langle\ F_1(g(t)),\ F_2(g(t)),\ \dots,\ F_n(g(t))\ \rangle \\
&= \langle\ \frac{d}{dt}F_1(g(t)),\ \frac{d}{dt}F_2(g(t)),\ \dots,\ \frac{d}{dt}F_n(g(t))\ \rangle \\
&= \langle\ \frac{dF_1}{dg}\frac{dg}{dt},\ \frac{dF_2}{dg}\frac{dg}{dt},\ \dots,\ \frac{dF_n}{dg}\frac{dg}{dt}\ \rangle \\
&= \frac{dg}{dt}\langle\ \frac{dF_1}{dg},\ \frac{dF_2}{dg},\ \dots,\ \frac{dF_n}{dg}\ \rangle \\
&= \frac{dg}{dt}\frac{dF_1}{dt}(g(t)
\end{aligned}$$

where I have used the notation $\frac{dF_j}{dg} = \frac{dF_j}{dt}(g(t))$ which you might recall from calculus I. As usual, the proof amounts to sorting through a little notation and quoting the basic result from calculus I. $\square$

We can use the notation $\frac{d\vec{F}}{dg}$ in place of the clumsy, but more technically accurate, $\frac{d\vec{F}}{dt}(g(t))$. With this notation the chain-rule looks nice:

$$\boxed{\frac{d}{dt}\big[\vec{F}(g(t))\big] = \frac{d\vec{F}}{dg}\frac{dg}{dt}.}$$

Example 2.1.18. .

$\boxed{E29}$ Suppose $\vec{F}(t) = \langle t^3, t^2, t \rangle$ and $g(t) = \underline{\cosh(t)}$.

$\vec{F}'(t) = \langle 3t^2, 2t, 1 \rangle$

$\cosh(t) \equiv \frac{1}{2}(e^t + e^{-t})$
$\sinh(t) \equiv \frac{1}{2}(e^t - e^{-t})$

$$\begin{aligned}
\frac{d}{dt}\big[\vec{F}(g(t))\big] &= \vec{F}'(g(t)) \cdot g'(t) \\
&= \langle 3\cosh^2(t), 2\cosh(t), 1 \rangle \cdot \sinh(t) \\
&= \langle 3\sinh(t)\cosh^2(t), 2\sinh(t)\cosh(t), \sinh(t) \rangle
\end{aligned}$$

Of course you get the same answer if you first compose then differentiate.

$$\vec{F}(g(t)) = \langle \cosh^3 t, \cosh^2 t, \cosh t \rangle$$

$$\Rightarrow \frac{d}{dt}\big(\vec{F}(g(t))\big) = \langle 3\cosh^2(t)\sinh t, 2\cosh t \sinh t, \sinh t \rangle$$

2.2 geometry of smooth oriented curves

If the curve is assigned a sense of direction then we call it an **oriented curve**. A particular curve can be parametrized by many different paths. You can think of a parametrization of a curve as a process of pasting a flexible numberline onto the curve.

Definition 2.2.1.

Let $C \subseteq \mathbb{R}^n$ be an oriented curve which starts at P and ends at Q. We say that $\vec{\gamma} : [a, b] \to \mathbb{R}^n$ is a **smooth non-stop parametrization** of C if $\vec{\gamma}([a, b]) = C$, $\vec{\gamma}(a) = P$, $\vec{\gamma}(b) = Q$, and $\vec{\gamma}$ is smooth with $\vec{\gamma}\,'(t) \neq 0$ for all $t \in [a, b]$. We will typically call $\vec{\gamma}$ a **path** from P to Q which covers the curve C.

I have limited the definition to curves with endpoints however the definition for curves which go on without end is very similar. You can just drop one or both of the endpoint conditions.

2.2.1 arclength

Let's begin by analyzing the tangent vector to a path in three dimensional space. Denote $\vec{\gamma} = (x, y, z)$ where $x, y, z \in C^\infty([a, b], \mathbb{R})$ and calculate that

$$\vec{\gamma}\,'(t) = \tfrac{d\vec{\gamma}}{dt} = \langle \tfrac{dx}{dt}, \tfrac{dy}{dt}, \tfrac{dz}{dt} \rangle.$$

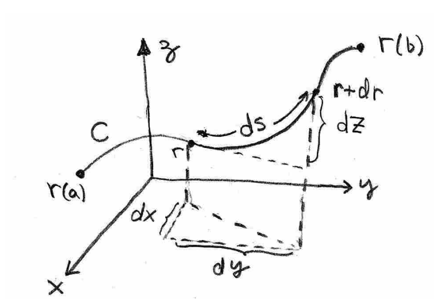
Multiplying by dt yields

$$\vec{\gamma}\,'(t)dt = \tfrac{d\vec{\gamma}}{dt}dt = \langle \tfrac{dx}{dt}, \tfrac{dy}{dt}, \tfrac{dz}{dt} \rangle dt.$$

The arclength ds subtended from time t to time $t + dt$ is simply the length of the vector $\vec{\gamma}\,'(t)dt$ which yields,

$$ds = ||\vec{\gamma}\,'(t)dt|| = \sqrt{\tfrac{dx}{dt}^2 + \tfrac{dy}{dt}^2 + \tfrac{dz}{dt}^2}\,dt$$

You can think of this as the length of a tiny bit of string that is laid out along the curve from the point $\vec{\gamma}(t)$ to the point $\vec{\gamma}(t + dt)$.



Of course this infinitesimal notation is just shorthand for an explicit limiting processes. If we sum together all the little bits of arclength we will arrive at the total arclength of the curve. In fact, this is how we define the arclength of a curve. The preceding discussion was in 3 dimensions but the formulas stated in terms of the norm generalizes naturally to $\mathbb{R}^n$.

Definition 2.2.2.

Let $\vec{\gamma} : [a,b] \to \mathbb{R}^n$ be a smooth, non-stop path which covers the oriented curve C. The **arclength function** of $\vec{\gamma}$ is a function $s_{\vec{\gamma}} : [a,b] \to \mathbb{R}$ where

$$s_{\vec{\gamma}} = \int_a^t ||\vec{\gamma}\,'(u)||\, du$$

for each $t \in [a,b]$. If $\tilde{\gamma}$ is a smooth non-stop path such that $||\tilde{\gamma}'(t)|| = 1$ then we say that $\tilde{\gamma}$ is a unit-speed curve. Moreover, we say $\tilde{\gamma}$ is parametrized with respect to arclength.

The examples below illustrate how we calculate arclength and also, for reasonable arclength functions, how we can explicitly reparametrize the path with respect to arclength. Sorry the notation in the examples below does not match the definition above. The connection is simple though, just think $\vec{r} = \vec{\gamma}$. This notational divide continues throughout my work, I sometimes use $\vec{r}$ for a path and sometimes $\vec{\gamma}$. Sometimes, I'll use another letter. Context is important and this is one of the reasons it is important to declare the domain and range for functions in this course. If we declare the domain and target spaces then the letter need not confuse us.

Example 2.2.3. .

$\boxed{E32}$ Consider $\vec{r}(t) = \langle R\cos t, R\sin t\rangle$ for $R > 0$. Notice that $\vec{r}\,'(t) = \langle -R\sin t, R\cos t\rangle$ thus $|\vec{r}\,'(t)| = \sqrt{R^2\sin^2 t + R^2\cos^2 t} = R$.
We want a circle so $0 \leq t \leq 2\pi$ then $a = 0$, $b = 2\pi$ so

$$s = \int_0^{2\pi} R\,dt = Rt\Big|_0^{2\pi} = \boxed{2\pi R}$$

the arclength of a circle is $2\pi R$, otherwise we'd be in trouble!

Example 2.2.4. .

$\boxed{E33}$ A helix with slope b is given by $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ for $0 \leq t \leq 2\pi$ (could let t keep going if you want the helix to continue onward.) anyway calculate, we assume a, b are constants.

$$\vec{r}\,'(t) = \langle -a\sin t, a\cos t, b\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

$$\therefore\ s = \int_0^{2\pi} \sqrt{a^2+b^2}\,dt = \left(\sqrt{a^2+b^2}\right)t\Big|_0^{2\pi} = \boxed{2\pi\sqrt{a^2+b^2}}$$

Of course when $b = 0$ we get a circle and we recover $2\pi a$ in that case (a good check of things here.)

Example 2.2.5. .

$\boxed{E34}$ The helix $\vec{r}(t) = \langle a\cos t, a\sin(t), bt\rangle$ $\quad 0 \le t \le 2\pi$
we found $|\vec{r}\,'(t)| = \sqrt{a^2+b^2}$ thus,

$$s(t) = \int_0^t \sqrt{a^2+b^2}\, d\tau = \left(\sqrt{a^2+b^2}\right)\tau\Big|_0^t = t\sqrt{a^2+b^2}$$

$$\Rightarrow t = \frac{s(t)}{\sqrt{a^2+b^2}} \quad \text{or changing notation,} \quad t(s) = \frac{s}{\sqrt{a^2+b^2}}$$

- it may be wise to suppress the of (t) and of (s) dependence here. Be careful not to confuse $s(t)$ with s times t.

We find that

$$\vec{r}(t(s)) = \boxed{\vec{r}(s) = \left\langle a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), b\frac{s}{\sqrt{a^2+b^2}}\right\rangle}$$

Explicit reparametrization of the curve below with respect to arclength is not a simple task. You'd likely need to break into cases.

Example 2.2.6. .

$\boxed{E35}$ Let $\vec{r}(t) = \langle t, \frac{\sqrt{2}}{2}t^2, \frac{1}{3}t^3\rangle$, $0 \le t \le 1$. Find the arclength function.

$$\vec{r}\,'(t) = \langle 1, \sqrt{2}t, t^2\rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{1+2t^2+t^4} = \sqrt{(1+t^2)^2}$$

Thus,

$$s = \int_0^t \sqrt{(1+\tau^2)^2}\, d\tau = \int_0^t (1+\tau^2)\,d\tau = \left(\tau + \tfrac{1}{3}\tau^3\right)\Big|_0^t = \boxed{t + \tfrac{1}{3}t^3 = s}$$

The arclength function has many special properties. Notice that item (1.) below is actually just the statement that the speed is the magnitude of the velocity vector.

Proposition 2.2.7.

Let $\vec{\gamma} : [a, b] \to \mathbb{R}^n$ be a smooth, non-stop path which covers the oriented curve C. The **arclength function** of $\vec{\gamma}$ denoted by $s_{\vec{\gamma}} : [a, b] \to \mathbb{R}$ has the following properties:

1. $\frac{d}{dt}(s_{\vec{\gamma}}(w)) = ||\vec{\gamma}\,'(w)|| \frac{dw}{dt}$,
2. $\frac{ds_{\vec{\gamma}}}{dt} > 0$ for all $t \in (a, b)$,
3. $s_{\vec{\gamma}}$ is a 1-1 function,
4. $s_{\vec{\gamma}}$ has inverse $s_{\vec{\gamma}}^{-1} : s_{\vec{\gamma}}([a, b]) \to [a, b]$.

Proof: We begin with (1.). We apply the fundamental theorem of calculus:

$$\frac{d}{dt}(s_{\vec{\gamma}}(w)) = \frac{d}{dt}\int_a^w ||\vec{\gamma}\,'(u)||\,du = ||\vec{\gamma}\,'(w)||\frac{dw}{dt}$$

for all $w \in (a,b)$. For (2.), set $w = t$ and recall that $||\vec{\gamma}\,'(t)|| = 0$ iff $\vec{\gamma}\,'(t) = 0$ however we were given that $\vec{\gamma}$ is non-stop so $\vec{\gamma}\,'(t) \neq 0$. We find $\frac{ds_{\vec{\gamma}}}{dt} > 0$ for all $t \in (a,b)$ and consequently the arclength function is an increasing function on (a,b). For (3.), suppose (towards a contradiction) that $s_{\vec{\gamma}}(x) = s_{\vec{\gamma}}(y)$ where $a < x < y < b$. Note that $\vec{\gamma}$ smooth implies $s_{\vec{\gamma}}$ is differentiable with continuous derivative on (a,b) therefore the mean value theorem applies and we can deduce that there is some point on $c \in (x,y)$ such that $s'_{\vec{\gamma}}(c) = 0$, which is impossible, therefore (3.) follows. If a function is 1-1 then we can construct the inverse pointwise by simply going backwards for each point mapped to in the range; $s_{\vec{\gamma}}^{-1}(x) = y$ iff $s_{\vec{\gamma}}(y) = x$. The fact that $s_{\vec{\gamma}}$ is single-valued follows from (3.). $\square$

If we are given a curve C covered by a path $\vec{\gamma}$ (which is smooth and non-stop but may not be unit-speed) then we can reparametrize the curve C with a unit-speed path $\tilde{\gamma}$ as follows:

$$\boxed{\tilde{\gamma}(s) = \vec{\gamma}(s_{\vec{\gamma}}^{-1}(s))}$$

where $s_{\vec{\gamma}}^{-1}$ is the inverse of the arclength function.

Proposition 2.2.8.

If $\vec{\gamma}$ is a smooth non-stop path then the path $\tilde{\gamma}$ defined by $\tilde{\gamma}(s) = \vec{\gamma}(s_{\vec{\gamma}}^{-1}(s))$ is unit-speed.

Proof: Differentiate $\tilde{\gamma}(t)$ with respect to t, we use the chain-rule,

$$\tilde{\gamma}'(t) = \tfrac{d}{dt}(\vec{\gamma}(s_{\vec{\gamma}}^{-1}(t))) = \vec{\gamma}\,'(s_{\vec{\gamma}}^{-1}(t))\tfrac{d}{dt}(s_{\vec{\gamma}}^{-1}(t)).$$

Hence $\tilde{\gamma}'(t) = \vec{\gamma}\,'(s_{\vec{\gamma}}^{-1}(t))\frac{d}{dt}(s_{\vec{\gamma}}^{-1}(t))$. Recall that if a function is increasing on an interval then its inverse is likewise increasing hence, by (2.) of the previous proposition, we can pull the positive constant $\frac{d}{dt}(s_{\vec{\gamma}}^{-1}(t))$ out of the norm. We find, using item (1.) in the previous proposition,

$$||\tilde{\gamma}'(t)|| = ||\vec{\gamma}\,'(s_{\vec{\gamma}}^{-1}(t))||\tfrac{d}{dt}(s_{\vec{\gamma}}^{-1}(t)) = \tfrac{d}{dt}(s_{\vec{\gamma}}(s_{\vec{\gamma}}^{-1}(t))) = \tfrac{d}{dt}(t) = 1.$$

Therefore, the curve $\tilde{\gamma}$ is unit-speed. We have $ds/dt = 1$ when $t = s$ (this last sentence is simply a summary of the careful argument we just concluded). $\square$

Remark 2.2.9.

While there are many paths which cover a particular oriented curve the unit-speed path is unique and we'll see that formulas for unit-speed curves are particularly simple.

Example 2.2.10. .

$$\gamma(t) = \vec{r}(t) = \langle R\cos t,\ 3,\ R\sin t\rangle \quad \text{for } t \geq 0,\ \underbrace{R > 0}_{\text{fixed constant}}.$$

$$\frac{d\vec{r}}{dt} = \langle -R\sin t,\ 0,\ R\cos t\rangle$$

$$\Rightarrow \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt} = R^2 \quad \& \quad \left\| \frac{d\vec{r}}{dt} \right\| = R$$

$$s(t) = \int_0^t \left\| \frac{d\vec{r}}{du} \right\| du = \int_0^t R\,du = Ru\Big|_0^t = \underline{Rt = s}.$$

For example, $s(2\pi) = 2\pi R$ (make sense ?)

Note $t = s/R$ hence we can reparametrize via s,

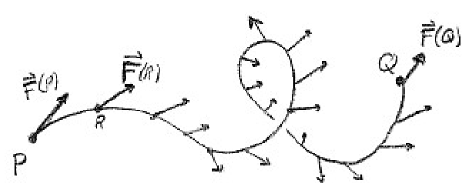
$$\underbrace{\tilde{\gamma}(s) = \vec{r}(s/R) = \langle R\cos(s/R),\ 3,\ R\sin(s/R)\rangle}_{\text{unit-speed parametrization of curve.}}$$

2.2.2 vector fields along a path

Definition 2.2.11.

Let $C \subseteq \mathbb{R}^3$ be an oriented curve which starts at P and ends at Q. A **vector field along the curve** C is a function which attaches a vector to each point on C.

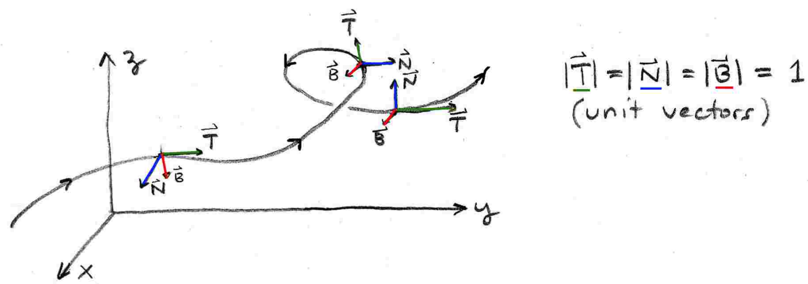
The tangent ($\vec{T}$), normal($\vec{N}$) and binormal ($\vec{B}$) vector fields defined below will allow us to identify when two oriented curves have the same shape.



vector field along curve C assigns vector at each point on C.

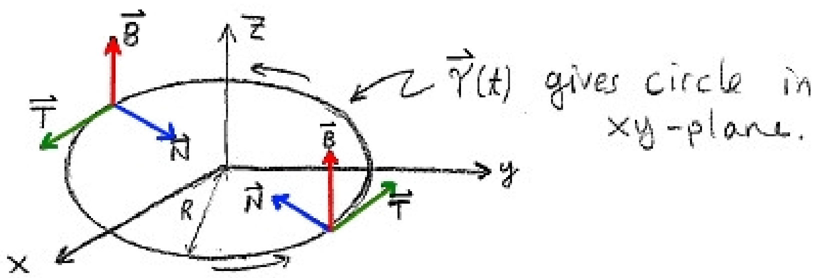
Definition 2.2.12.

Let $\vec{\gamma} : [a,b] \to \mathbb{R}^3$ be a path from P to Q in $\mathbb{R}^3$. The **tangent vector field** of $\vec{\gamma}$ is given by
$$\vec{T}(t) = \frac{1}{||\vec{\gamma}\,'(t)||}\vec{\gamma}\,'(t)$$
for each $t \in [a,b]$. Likewise, if $\vec{T}\,'(t) \neq 0$ for all $t \in [a,b]$ then the **normal vector field** of $\vec{\gamma}$ is defined by
$$\vec{N}(t) = \frac{1}{||\vec{T}\,'(t)||}\vec{T}\,'(t)$$
for each $t \in [a,b]$. Finally, if $\vec{T},'(t) \neq 0$ for all $t \in [a,b]$ then the **binormal vector field** of $\vec{\gamma}$ is defined by $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ for all $t \in [a,b]$



Example 2.2.13. *Let $R > 0$ and suppose $\vec{\gamma}(t) = (R\cos(t), R\sin(t), 0)$ for $0 \leq t \leq 2\pi$. We can calculate*
$$\vec{\gamma}\,'(t) = \langle -R\sin(t), R\cos(t), 0\rangle \quad \Rightarrow \quad ||\vec{\gamma}\,'(t)|| = R.$$
Hence $\vec{T}(t) = \langle -\sin(t), \cos(t), 0\rangle$ and we can calculate,
$$\vec{T}\,'(t) = \langle -\cos(t), -\sin(t), 0\rangle \quad \Rightarrow \quad ||\vec{T}\,'(t)|| = 1.$$
Thus $\vec{N}(t) = \langle -\cos(t), -\sin(t), 0\rangle$. Finally we calculate the binormal vector field,
$$\begin{aligned} \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \quad &= [-\sin(t)e_1 + \cos(t)e_2] \times [-\cos(t)e_1 - \sin(t)e_2] \\ &= [\sin^2(t)e_1 \times e_2 - \cos^2(t)e_2 \times e_1 \\ &= [\sin^2(t) + \cos^2(t)]e_1 \times e_2 \\ &= e_3 = \langle 0,0,1\rangle \end{aligned}$$

Notice that $\vec{T} \bullet \vec{N} = \vec{N} \bullet \vec{B} = \vec{T} \bullet \vec{B} = 0$. For a particular value of t the vectors $\{\vec{T}(t), \vec{N}(t), \vec{B}(t)\}$ give an orthogonal set of unit vectors, they provide a comoving frame for $\vec{\gamma}$. It can be shown that the tangent and normal vectors span the plane in which the path travels for times infintesimally close to t. This plane is called the **osculating plane**. *The binormal vector gives the normal to the osculating plane. The curve considered in this example has a rather boring osculating plane since $\vec{B}$ is constant. This curve is just a circle in the xy-plane which is traversed at constant speed.*



Example 2.2.14. *Notice that $s_{\vec{\gamma}}(t) = Rt$ in the preceding example. It follows that $\widetilde{\gamma}(s) = (R\cos(s/R), R\sin(s/R), 0)$ for $0 \leq s \leq 2\pi R$ is the unit-speed path for curve. We can calculate*

$$\widetilde{\gamma}'(s) = \langle -\sin(s/R), \cos(s/R), 0\rangle \quad \Rightarrow \quad ||\widetilde{\gamma}'(s)|| = 1.$$

Hence $\widetilde{T}(s) = \langle -\sin(s/R), \cos(s/R), 0\rangle$ and we can also calculate,

$$\widetilde{T}'(s) = \tfrac{1}{R}\langle -\cos(s/R), -\sin(s/R), 0\rangle \quad \Rightarrow \quad ||\widetilde{T}'(t)|| = 1/R.$$

Thus $\widetilde{N}(s) = \langle -\cos(s/R), -\sin(s/R), 0\rangle$. Note $\widetilde{B} = \widetilde{T} \times \widetilde{N} = \langle 0, 0, 1\rangle$ as before.

Example 2.2.15. *Let $m, R > 0$ and suppose $\vec{\gamma}(t) = (R\cos(t), R\sin(t), mt)$ for $0 \leq t \leq 2\pi$. We can calculate*

$$\vec{\gamma}\,'(t) = \langle -R\sin(t), R\cos(t), m\rangle \quad \Rightarrow \quad ||\vec{\gamma}\,'(t)|| = \sqrt{R^2 + m^2}.$$

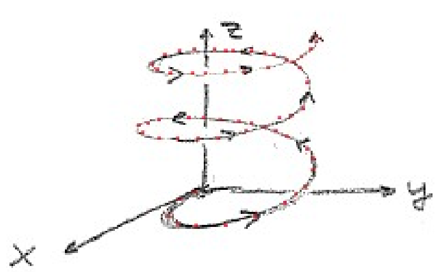
Hence $\vec{T}(t) = \frac{1}{\sqrt{R^2+m^2}}\langle -R\sin(t), R\cos(t), m\rangle$ and we can calculate,

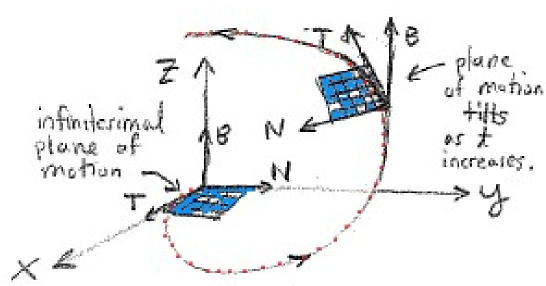
$$\vec{T}\,'(t) = \tfrac{1}{\sqrt{R^2+m^2}}\langle -R\cos(t), -R\sin(t), 0\rangle \quad \Rightarrow \quad ||\vec{T}\,'(t)|| = \tfrac{R}{\sqrt{R^2+m^2}}.$$

Thus $\vec{N}(t) = \langle -\cos(t), -\sin(t), 0\rangle$. Finally we calculate the binormal vector field,

$$\begin{aligned} \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) &= \tfrac{1}{\sqrt{R^2+m^2}}[-R\sin(t)e_1 + R\cos(t)e_2 + me_3] \times [-\cos(t)e_1 - \sin(t)e_2] \\ &= \tfrac{1}{\sqrt{R^2+m^2}}\langle m\sin(t), -m\cos(t), R\rangle \end{aligned}$$

We again observe that $\vec{T} \bullet \vec{N} = \vec{N} \bullet \vec{B} = \vec{T} \bullet \vec{B} = 0$. The **osculating plane** *is moving for this curve, note the t-dependence. This curve does not stay in a single plane, it is not a planar curve. In fact this is a circular helix with radius R and slope m.*





Example 2.2.16. *Lets reparametrize the helix as a unit-speed path. Notice that* $s_{\vec{\gamma}}(t) = t\sqrt{R^2+m^2}$ *thus we should replace* t *with* $s/\sqrt{R^2+m^2}$ *to obtain* $\widetilde{\gamma}(s)$. *Let* $a = 1/\sqrt{R^2+m^2}$ *and* $\widetilde{\gamma}(s) = (R\cos(as), R\sin(as), ams)$ *for* $0 \leq s \leq 2\pi\sqrt{R^2+m^2}$. *We can calculate*

$$\widetilde{\gamma}'(s) = \langle -Ra\sin(as), Ra\cos(as), am\rangle \quad \Rightarrow \quad ||\widetilde{\gamma}'(s)|| = a\sqrt{R^2+m^2} = 1.$$

Hence $\widetilde{T}(s) = a\langle -R\sin(as), R\cos(as), m\rangle$ *and we can calculate,*

$$\widetilde{T}'(s) = Ra^2\langle -\cos(as), -\sin(as), 0\rangle \quad \Rightarrow \quad ||\widetilde{T}'(s)|| = Ra^2 = \tfrac{R}{R^2+m^2}.$$

Thus $\widetilde{N}(s) = \langle -\cos(as), -\sin(as), 0\rangle$. *Next, calculate the binormal vector field,*

$$\begin{aligned} \widetilde{B}(s) = \widetilde{T}(s) \times \widetilde{N}(s) &= a\langle -R\sin(as), R\cos(as), m\rangle \times \langle -\cos(as), -\sin(as), 0\rangle \\ &= \tfrac{1}{\sqrt{R^2+m^2}}\langle m\sin(as), -m\cos(as), R\rangle \end{aligned}$$

Hopefully you can start to see that the unit-speed path shares the same $\vec{T}, \vec{N}, \vec{B}$ *frame at arclength* s *as the previous example with* $t = s/\sqrt{R^2+m^2}$.

2.2.3 Frenet Serret equations

We now prepare to prove the Frenet Serret formulas for the $\vec{T}, \vec{N}, \vec{B}$ frame fields. It turns out that for nonlinear curves the $\vec{T}, \vec{N}, \vec{B}$ vector fields always provide an orthonormal frame. Moreover, for nonlinear curves, we'll see that the **torsion** and **curvature** capture the geometry of the curve.

Proposition 2.2.17.

> If $\vec{\gamma}$ is a path with tangent, normal and binormal vector fields $\vec{T}, \vec{N}$ and $\vec{B}$ then $\{\vec{T}(t), \vec{N}(t), \vec{B}(t)\}$ is an orthonormal set of vectors for each $t \in dom(\vec{\gamma})$.

Proof: It is clear from $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ that $\vec{T}(t) \bullet \vec{B}(t) = \vec{N}(t) \bullet \vec{B}(t) = 0$. Furthermore, it is also clear that these vectors have length one due to their construction as unit vectors. In particular this means that $\vec{T}(t) \bullet \vec{T}(t) = 1$. We can differentiate this to obtain (by the product rule for dot-products)

$$\vec{T}\,'(t) \bullet \vec{T}(t) + \vec{T}(t) \bullet \vec{T}\,'(t) = 0 \quad \Rightarrow \quad 2\vec{T}(t) \bullet \vec{T}\,'(t) = 0$$

Divide by $||\vec{T}\,'(t)||$ to obtain $\vec{T}(t) \bullet \vec{N}(t) = 0$. $\square$

We omit the explicit t-dependence for the dicussion to follow here, also you should assume the vector fields are all derived from a particular path $\vec{\gamma}$. Since $\vec{T}, \vec{N}, \vec{B}$ are nonzero and point in three mutually distinct directions it follows that any other vector can be written as a linear combination of $\vec{T}, \vec{N}, \vec{B}$. This means[3] if $\vec{v}$ is a vector then there exist c_1, c_2, c_3 such that $v = c_1T + c_2N + c_3B$. The orthonormality is very nice because it tells us we can calculate the coefficients in terms of dot-products with $\vec{T}, \vec{N}$ and $\vec{B}$:

$$\boxed{\vec{v} = c_1\vec{T} + c_2\vec{N} + c_3\vec{B} \quad \Rightarrow \quad c_1 = \vec{v} \bullet \vec{T},\ c_2 = \vec{v} \bullet \vec{N},\ c_3 = \vec{v} \bullet \vec{B}}$$

[3] You might recognize $[v]_\beta = [c_1, c_2, c_3]^T$ as the coordinate vector with respect to the basis $\beta = \{\vec{T}, \vec{N}, \vec{B}\}$

We will make much use of the observations above in the calculations that follow. Suppose that

$$\begin{aligned}\vec{T}\,' &= c_{11}\vec{T} + c_{12}\vec{N} + c_{13}\vec{B}\\ \vec{N}\,' &= c_{21}\vec{T} + c_{22}\vec{N} + c_{23}\vec{B}\\ \vec{B}\,' &= c_{31}\vec{T} + c_{32}\vec{N} + c_{33}\vec{B}.\end{aligned}$$

We observed previously that $\vec{T}\,' \bullet \vec{T} = 0$ thus $c_{11} = 0$. It is easy to show $\vec{N}\,' \bullet \vec{N} = 0$ and $\vec{B}\,' \bullet \vec{B} = 0$ thus $c_{22} = 0$ and c_{33}. Furthermore, we defined $\vec{N} = \frac{1}{||\vec{T}\,'||}\vec{T}\,'$ hence $c_{13} = 0$. Note that

$$\vec{T}\,' = c_{12}\vec{N} = \tfrac{c_{12}}{||\vec{T}\,'||}\vec{T}\,' \quad \Rightarrow \quad c_{12} = ||\vec{T}\,'||.$$

To summarize what we've learned so far:

$$\begin{aligned}\vec{T}\,' &= c_{12}\vec{N}\\ \vec{N}\,' &= c_{21}\vec{T} + c_{23}\vec{B}\\ \vec{B}\,' &= c_{31}\vec{T} + c_{32}\vec{N}.\end{aligned}$$

We'd like to find some condition on the remaining coefficients. Consider that:

$$\begin{array}{lll} \vec{B} = \vec{T}\times\vec{N} & \Rightarrow\ \ \vec{B}\,' = \vec{T}\,'\times\vec{N} + \vec{T}\times\vec{N}\,' & \text{a product rule}\\ & \Rightarrow\ \ \vec{B}\,' = [c_{12}\vec{N}]\times\vec{N} + \vec{T}\times[c_{21}\vec{T} + c_{23}\vec{B}] & \text{using previous eqn.}\\ & \Rightarrow\ \ \vec{B}\,' = c_{23}\vec{T}\times\vec{B} & \text{noted } \vec{N}\times\vec{N} = \vec{T}\times\vec{T} = 0\\ & \Rightarrow\ \ \vec{B}\,' = -c_{23}\vec{N} & \text{you can show } \vec{N} = \vec{B}\times\vec{T}.\\ & \Rightarrow\ \ c_{31}\vec{T} + c_{32}\vec{N} = -c_{23}\vec{N} & \text{refer to previous eqn.}\\ & \Rightarrow\ \ c_{31} = 0 \text{ and } c_{32} = -c_{23}. & \text{using LI of } \{T, N\} \end{array}$$

The "LI" is linear independence. The fact that $\vec{T}, \vec{N}$ are LI follows from the fact that they form a nonzero and orthogonal set of vectors. We can equate coefficients of LI sums of vectors. This is the principle I'm using. Alternatively, you can just take the dot-product of the next to last equation with $\vec{N}$ and then $\vec{T}$ and use $\vec{T} \bullet \vec{N} = 0$ to obtain the final line. We have reduced the initial set of equations to the following:

$$\begin{aligned}\vec{T}\,' &= c_{12}\vec{N}\\ \vec{N}\,' &= c_{21}\vec{T} + c_{23}\vec{B}\\ \vec{B}\,' &= -c_{23}\vec{N}.\end{aligned}$$

The equations above encourage us to define the **curvature** and **torsion** as follows:

Definition 2.2.18.

Let C be a curve which is covered by the unit-speed path $\widetilde{\gamma}$ then we define the curvature κ and torsion τ as follows:

$$\boxed{\kappa(s) = \left|\left|\frac{d\widetilde{T}}{ds}\right|\right| \qquad \tau(s) = -\frac{d\widetilde{B}}{ds} \bullet \widetilde{N}(s)}$$

One of your homework questions is to show that $c_{21} = -c_{12}$. Given that result we find the famous **Frenet-Serret** equations:

$$\boxed{\frac{d\widetilde{T}}{ds} = \kappa\widetilde{N} \qquad \frac{d\widetilde{N}}{ds} = -\kappa\widetilde{T} + \tau\widetilde{B} \qquad \frac{d\widetilde{B}}{ds} = -\tau\widetilde{N}.}$$

We had to use the arclength parameterization to insure that the formulas above unambiguously define the curvature and the torsion. In fact, if we take a particular (unoriented) curve then there are two choices for orienting the curve. You can show that that the torsion and curvature are independent of the choice of orientation. Naturally the total arclength is also independent of the orientation of a given curve.

Example 2.2.19. .

$\boxed{E40}$ Consider the circle, $\vec{r}(t) = \langle a\cos t, a\sin t, 0\rangle$. We found in $\boxed{E37}$ that $\vec{T}(t) = \langle -\sin t, \cos t, 0\rangle$ and $\vec{T}'(t) = \langle -\cos t, -\sin t, 0\rangle$ so $|\vec{T}'(t)| = 1$ thus we find $\vec{N}(t) = \vec{T}'(t) = \langle -\cos t, -\sin t, 0\rangle$. Notice that

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \langle 0, 0, \sin^2 t + \cos^2 t\rangle = \langle 0, 0, 1\rangle.$$

Thus $\vec{B}$ is a constant vector which means we get $\boxed{\vec{B}(s) = \langle 0, 0, 1\rangle}$,

$$\frac{d\vec{B}}{ds} = 0 = -\tau\vec{N} \Rightarrow \boxed{\tau = 0}$$

This is good, we predicted that planar curves (like a circle) have zero torsion. We should note that usually we will need to reparametrize $\vec{B}(t)$ to $\vec{B}(s)$ to find τ, this case was special.

Example 2.2.20. .

$\boxed{E41}$ The helix $\vec{r}(t) = \langle a\cos t,\ a\sin t,\ bt\rangle$ (see $\boxed{E33}$, $\boxed{E34}$, $\boxed{E36}$, $\boxed{E39}$ to find $\vec{T}(t)$ and $\vec{T}'(t)$)

$$\vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin t,\ a\cos t,\ b\rangle$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \left(\frac{\cancel{\sqrt{a^2+b^2}}}{a}\right)\frac{1}{\cancel{\sqrt{a^2+b^2}}}\langle -a\cos t,\ -a\sin t,\ 0\rangle = \overbrace{\langle -\cos t, -\sin t, 0\rangle}^{(**)}$$

$$\vec{T}(t)\times\vec{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\alpha a\sin t & \alpha a\cos t & \alpha b \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

I let $\alpha \equiv \frac{1}{\sqrt{a^2+b^2}}$ because I'm tired of

$$= \langle \alpha b\sin t,\ -\alpha b\cos t,\ \alpha a\sin^2 t + \alpha a\cos^2 t\rangle$$

$$= \alpha\cdot\langle b\sin t,\ -b\cos t,\ a\rangle = \vec{B}(t)$$

We found in $\boxed{E34}$ that $t = s/\sqrt{a^2+b^2} = s\alpha$ thus,

$$\vec{B}(s) = \alpha\langle b\sin(\alpha s),\ -b\cos(\alpha s),\ a\rangle$$

$$\therefore \frac{d\vec{B}}{ds} = \alpha^2\langle b\cos(\alpha s),\ b\sin(\alpha s),\ 0\rangle$$

$$= -b\alpha^2\langle -\cos(\alpha s),\ -\sin(\alpha s),\ 0\rangle$$: this is $(**)$ reparam. with $t = \alpha s$. This is $\vec{N}(s)$.

$$= -\tau\vec{N}$$

Comparing we find $\tau = b\alpha^2 = \boxed{\frac{b}{a^2+b^2} = \tau}$

Notice that as $b \to 0$ the torsion goes to zero, which is in agreement with $\boxed{E40}$.

Curvature, torsion can also be calculated in terms of a path which is not unit speed. We simply replace s with the arclength function $s_{\vec{\gamma}}(t)$ and make use of the chain rule. Notice that $d\vec{F}/dt = (ds/dt)(d\widetilde{F}/ds)$ hence,

$$\frac{d\vec{T}}{dt} = \frac{ds}{dt}\frac{d\widetilde{T}}{ds}, \quad \frac{d\vec{N}}{dt} = \frac{ds}{dt}\frac{d\widetilde{N}}{ds}, \quad \frac{d\vec{B}}{dt} = \frac{ds}{dt}\frac{d\widetilde{B}}{ds}$$

Or if you prefer, use the dot-notation $ds/dt = \dot{s}$ to write:

$$\frac{1}{\dot{s}}\frac{d\vec{T}}{dt} = \frac{d\widetilde{T}}{ds}, \quad \frac{1}{\dot{s}}\frac{d\vec{N}}{dt} = \frac{d\widetilde{N}}{ds}, \quad \frac{1}{\dot{s}}\frac{d\vec{B}}{dt} = \frac{d\widetilde{B}}{ds}$$

Substituting these into the unit-speed Frenet Serret formulas yield:

$$\boxed{\frac{d\vec{T}}{dt} = \dot{s}\kappa\vec{N} \qquad \frac{d\vec{N}}{dt} = -\dot{s}\kappa\vec{T} + \dot{s}\tau\vec{B} \qquad \frac{d\vec{B}}{dt} = -\dot{s}\tau\vec{N}.}$$

where $\widetilde{T}(s_{\vec{\gamma}}(t)) = \vec{T}(t), \widetilde{N}(s_{\vec{\gamma}}(t)) = \vec{N}(t)$ and $\widetilde{B}(s_{\vec{\gamma}}(t)) = \vec{B}(t)$. Likewise deduce[4] that

$$\kappa(t) = \frac{1}{\dot{s}}\left|\left|\frac{d\vec{T}}{dt}\right|\right| \qquad \tau(t) = -\frac{1}{\dot{s}}\left(\frac{d\vec{B}}{dt}\bullet\vec{N}(t)\right)$$

Let's see how these formulas are useful in an example or two.

Example 2.2.21. .

$\boxed{E42}$ Consider $\vec{r}(t) = \langle e^t\cos t,\ e^t\sin t,\ e^t\rangle$. Calculate the $\vec{T}, \vec{N}, \vec{B}$ frame and the curvature and torsion. We begin,

$$\vec{r}\,'(t) = \langle e^t(\cos t - \sin t),\ e^t(\sin t + \cos t),\ e^t\rangle$$

$$\begin{aligned}
\vec{r}\,'(t)\bullet\vec{r}\,'(t) &= (e^t)^2(\cos t - \sin t)^2 + (e^t)^2(\sin t + \cos t)^2 + (e^t)^2 \\
&= e^{2t}\left[\cos^2 t - \cancel{2\sin t\cos t} + \sin^2 t + \sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + 1\right] \\
&= 3e^{2t} \quad\Rightarrow\quad |\vec{r}\,'(t)| = \sqrt{3}\,e^t
\end{aligned}$$

Thus the unit tangent vector $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$ is

$$\boxed{\vec{T}(t) = \frac{1}{\sqrt{3}}\langle \cos t - \sin t,\ \sin t + \cos t,\ 1\rangle}$$

Recall that the chain rule says $\frac{d\vec{T}}{dt} = \frac{ds}{dt}\frac{d\vec{T}}{ds} \therefore \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt}$

But we know $\frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{3}e^t$ thus we find $\frac{d\vec{T}}{ds}$ as a function of t,

$$\frac{d\vec{T}}{ds} = \frac{1}{3e^t}\langle -\sin t - \cos t,\ \cos t - \sin t,\ 0\rangle$$

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{3e^t}\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2} = \boxed{\frac{\sqrt{2}}{3e^t} = \kappa(t)}$$

[4] I'm using the somewhat ambiguous notation $\kappa(t) = \kappa(s_\gamma(t))$ and $\tau(t) = \tau(s_\gamma(t))$. We do this often in applications of calculus. Ask me if you'd like further clarification on this point.

I calculate the Normal vector $\vec{N}$ using an indirect method,

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{3e^t}{\sqrt{2}}\frac{1}{3e^t}\langle -\sin t - \cos t,\ \cos t - \sin t,\ 0\rangle$$

$$\therefore \boxed{\vec{N} = \frac{1}{\sqrt{2}}\langle -\sin t - \cos t,\ \cos t - \sin t,\ 0\rangle}$$

And now the binormal follows from straightforward computation of $\vec{B} = \vec{T} \times \vec{N}$ which yields, (I leave it for you)

$$\boxed{\vec{B} = \frac{1}{\sqrt{6}}\langle \sin t - \cos t,\ -\sin t - \cos t,\ 2\rangle}$$

Now we may deduce the torsion, again use the chain rule trick,

$$\frac{d\vec{B}}{ds} = \frac{1}{ds/dt}\frac{d\vec{B}}{dt} = \frac{1}{\sqrt{3}e^t}\frac{1}{\sqrt{6}}\langle \cos t + \sin t,\ -\cos t + \sin t,\ 0\rangle$$

$$= \underbrace{\left(\frac{-1}{3e^t}\right)}_{-\tau}\underbrace{\frac{1}{\sqrt{2}}\langle -\sin t - \cos t,\ \cos t - \sin t,\ 0\rangle}_{\vec{N}}$$

$$\boxed{\tau = \frac{1}{3e^t}}$$

<u>Remark</u>: the chainrule has saved us the trouble of computing the arclength!

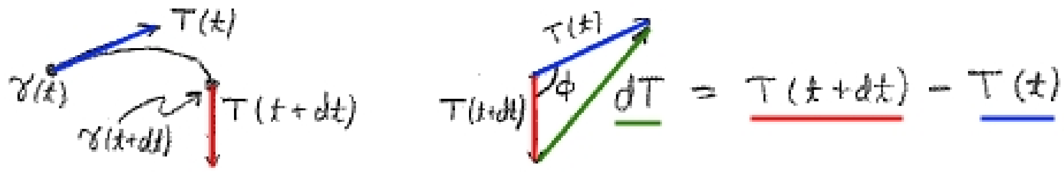
We've seen in this section how calculus and vector algebra encourage us to define curvature and torsion. It remains to examine the geometric significance of those definitions. We pursue that geometry in the remainder of this section.

2.2.4 curvature

Let use begin with the curvature. Assume $\vec{\gamma}$ is a non-stop smooth path,

$$\kappa = \frac{1}{\dot{s}}\left|\left|\frac{d\vec{T}}{dt}\right|\right|$$

Infinitesimally this equation gives $||d\vec{T}|| = \kappa\dot{s}dt = \kappa\frac{ds}{dt}dt = \kappa\, ds$. But this is a strange equation since $||\vec{T}|| = 1$. So what does this mean? Perhaps we should add some more detail to resolve this puzzle; let $d\vec{T} = \vec{T}(t+dt) - \vec{T}(t)$.



Notice that

$$\begin{aligned}
||d\vec{T}||^2 &= [\vec{T}(t+dt) - \vec{T}(t)] \bullet [\vec{T}(t+dt) - \vec{T}(t)] \\
&= \vec{T}(t+dt) \bullet \vec{T}(t+dt) + \vec{T}(t) \bullet \vec{T}(t) - 2\vec{T}(t) \bullet \vec{T}(t+dt) \\
&= \vec{T}(t+dt) \bullet \vec{T}(t+dt) + \vec{T}(t) \bullet \vec{T}(t) - 2\vec{T}(t) \bullet \vec{T}(t+dt) \\
&= 2(1 - \cos(\phi)))
\end{aligned}$$

where we define ϕ to be the angle between $\vec{T}(t)$ and $\vec{T}(t+dt)$. This angle measures the change in direction of the tangent vector at t goes to $t + dt$. Since this is a small change in time it is reasonable to expect the angle ϕ is small thus $\cos(\phi) \approx 1 - \frac{1}{2}\phi^2$ and we find that

$$||d\vec{T}|| = \sqrt{2(1-\cos(\phi)} = \sqrt{2(1-1+\tfrac{1}{2}\phi^2)} = \sqrt{\phi^2} = |\phi|$$

Therefore, $||d\vec{T}|| = \kappa\, ds = |\phi|$ and we find $\boxed{\kappa = \pm\frac{ds}{d\phi}}$.

Example 2.2.22. .

$\boxed{E37}$ The circle $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$, $0 \le t \le 2\pi$, assume $a > 0$,

$\vec{r}\,'(t) = \langle -a\sin t, a\cos t\rangle \quad \therefore \quad |\vec{r}\,'(t)| = \frac{ds}{dt} = \sqrt{a^2} = a$

Thus the unit tangent,

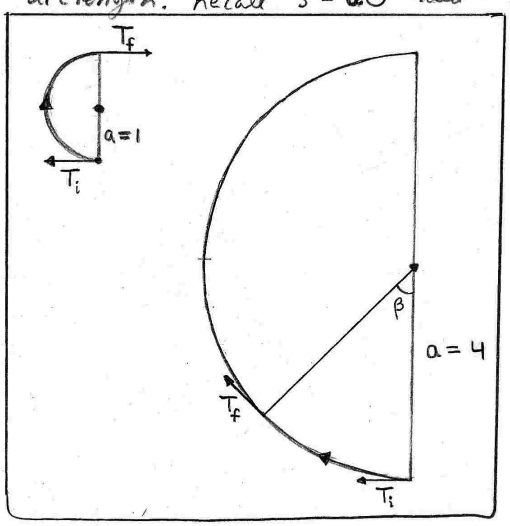
$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin t, \cos t\rangle$$

$$\vec{T}'(t) = \langle -\cos t, -\sin t\rangle \quad \therefore \quad |\vec{T}'(t)| = 1.$$

$$\therefore \ \kappa(t) = \frac{|\vec{T}'(t)|}{ds/dt} = \boxed{\frac{1}{a} = \kappa}$$

: the curvature of a circle is inversely proportional to the radius of the circle.

I'll endeavor to sketch this, its clear that smaller circles force the $\vec{T}$-vector to turn all the way around quicker for a given arclength. Recall $s = a\theta$ need $4\beta = \pi \Rightarrow \beta = \pi/4$ on big circle gives same arclength.



You can clearly see that the Tangent vector completely reverses direction for the circle of $R = 1$. whereas for the larger circle of radius $R = 4$ the Tangent vector only changes direction by $\pi/4$ relative to it's initial state.

Example 2.2.23. .

$\boxed{E38}$ Consider the line $\vec{r}(t) = \vec{a}t + \vec{b}$ for $\vec{a}, \vec{b}$ fixed vectors. Then $\vec{r}\,'(t) = \vec{a}$ thus $\vec{T}(t) = \frac{\vec{a}}{|\vec{a}|}$ hence $\vec{T}'(t) = 0$. Consequently

$$\kappa(t) = \frac{|\vec{T}'(t)|}{ds/dt} = \boxed{0 = \kappa}$$

lines have no curvature. One might also anticipate this result from $\boxed{E37}$ taking the radius $a \longrightarrow \infty \Rightarrow \frac{1}{a} \longrightarrow 0$.

Example 2.2.24. .

$\boxed{E39}$ The helix, $\vec{r}(t) = \langle a\cos t,\ a\sin t,\ bt\rangle$. Recall we found in $\boxed{E36}$ that

$$\vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin t,\ a\cos t,\ b\rangle \quad \text{and} \quad |\vec{r}\,'(t)| = \frac{ds}{dt} = \sqrt{a^2+b^2}$$

Calculate then,

$$\vec{T}\,'(t) = \frac{1}{\sqrt{a^2+b^2}}\langle -a\cos t,\ -a\sin t,\ 0\rangle$$

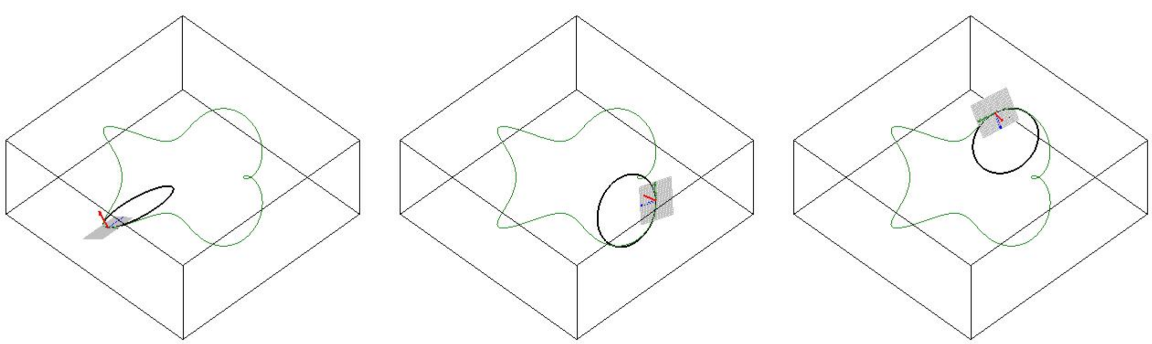
$$|\vec{T}\,'(t)| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + a^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\therefore\ \kappa(t) = \frac{|\vec{T}\,'(t)|}{ds/dt} = \frac{a/\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = \boxed{\frac{a}{a^2+b^2} = \kappa}$$

Again when $b = 0$ we find $\kappa = a/a^2 = 1/a$ which is consistent with what we found for the circle in $\boxed{E37}$.

Remark 2.2.25.

The curvature measures the infinitesimal change in the direction of the unit-tangent vector to the curve. We say the the reciprocal of the curvature is the **radius of curvature** $r = \frac{1}{\kappa}$. This makes sense as $ds = |1/\kappa| d\phi$ suggests that a circle of radius $1/\kappa$ fits snuggly against the path at time t. We form the **osculating circle at each point along the path by placing a circle of radius $1/\kappa$ tangent to the unit-tangent vector in the plane with normal $\vec{B}(t)$**. Here's a picture, the red-vector is the tangent, the blue the binormal, the green the normal and the black circle is in the grey osculating plane. I have an animated version on my webpage, go take a look.



2.2.5 osculating plane and circle

It was claimed that the "infinitesimal" motion of the path resides in a plane with normal $\vec{B}$. Suppose that at some time t_o the path reaches the point $\vec{\gamma}(t_o) = P_o$. Infinitesimally the tangent line matches the path and we can write the parametric equation for the tangent line as follows:

$$\vec{l}(t) = \vec{\gamma}(t_o) + t\vec{\gamma}\,'(t_o) = P_o + tv_o\vec{T}_o$$

where we used that $\vec{\gamma}\,'(t) = \dot{s}T(t)$ and we evaluated at $t = t_o$ to define $\dot{s}(t_o) = v_o$ and $\vec{T}(t_o) = \vec{T}_o$. The normal line through P_o has parametric equations (using $\vec{N}_o = \vec{N}(t_o)$):

$$\vec{n}(\lambda) = P_o + \lambda\vec{N}_o$$

We learned in the last section that the path bends away from the tangent line along a circle whose radius is $1/\kappa_o$. We find the infinitesimal motion resides in the plane spanned by $\vec{T}_o$ and $\vec{N}_o$ which has normal $\vec{T}_o \times \vec{N}_o = \vec{B}(t_o)$. The tangent line and the normal line are perpendicular and could be thought of as a xy-coordinate axes in the osculating plane. The osculating circle is found with its center on the normal line a distance of $1/\kappa_o$ from P_o. Thus the center of the circle is at:

$$Q_o = P_o - \tfrac{1}{\kappa_o}\vec{N}_o$$

I'll think of constructing x, y, z coordinates based at P_o with respect to the $\vec{T}_o, \vec{N}_o, \vec{B}_o$ frame. We suppose $\vec{r}$ be a point on the osculating circle and x, y, z to be the coefficients in $\vec{r} = P_o + x\vec{T}_o + y\vec{N}_o + z\vec{B}_o$. Since the circle is in the plane based at P_o with normal $\vec{B}_o$ we should set $z = 0$ for our circle thus $\vec{r} = x\vec{T} + y\vec{N}$.

$$||\vec{r} - Q_o||^2 = \tfrac{1}{\kappa_o^2} \quad \Rightarrow \quad ||x\vec{T}_o + (y + \tfrac{1}{\kappa_o})\vec{N}_o]||^2 = \tfrac{1}{\kappa_o^2}.$$

Therefore, by the pythagorean theorem for orthogonal vectors, the x, y, z equations for the osculating circle are simply[5] :

$$\boxed{x^2 + (y + \tfrac{1}{\kappa_o})^2 = \tfrac{1}{\kappa_o^2}, \quad z = 0.}$$

[5] Of course if we already use x, y, z in a different context then we should use other symbols for the equation of the osculating circle.

Example 2.2.26. .

Find the eq$^{\underline{n}}$'s of the normal plane and osculating plane of the curve $\vec{r}(t) = \langle 2\sin(3t), t, 2\cos(3t)\rangle$ at $(0, \pi, -2)$.

$\vec{r}\,'(t) = \langle 6\cos(3t), 1, -6\sin(3t)\rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{37}$

$\vec{r}\,''(t) = \langle -18\sin(3t), 0, -18\cos(3t)\rangle \Rightarrow |\vec{r}\,''(t)| = 18$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{1}{\sqrt{37}}\langle 6\cos(3t), 1, -6\sin(3t)\rangle$

$\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \dfrac{\cancel{\frac{1}{\sqrt{37}}}\,\vec{r}\,''(t)}{\cancel{\frac{1}{\sqrt{37}}}\,|\vec{r}\,''(t)|} = \dfrac{1}{18}\langle -18\sin 3t, 0, -18\cos 3t\rangle$

we could continue calculating for arbitrary t but $\underline{t=\pi}$ gives the point $(0, \pi, -2)$. Notice

$\vec{T}(\pi) = \frac{1}{\sqrt{37}}\langle -6, 1, 0\rangle$

$\vec{N}(\pi) = \frac{1}{18}\langle 0, 0, 18\rangle = \hat{k}$

$\vec{B}(\pi) \equiv \vec{T}(\pi) \times \vec{N}(\pi) = \frac{1}{\sqrt{37}}(-6\hat{i} + \hat{j}) \times \hat{k} = \frac{+6}{\sqrt{37}}\hat{j} + \frac{1}{\sqrt{37}}\hat{i}$

$\vec{B}(\pi) = \frac{1}{\sqrt{37}}\langle 1, 6, 0\rangle$

Plane determined by $\vec{N}$ and $\vec{B}$ is the "Normal Plane", it has normal $\vec{N} \times \vec{B} = \vec{T}$ so we can use $\langle -6, 1, 0\rangle$ and the point $(0, \pi, -2)$ to give

$\boxed{-6x + y - \pi = 0 \quad \text{Normal Plane}}$

The osculating plane has normal along $\vec{B} = \frac{1}{\sqrt{37}}\langle 1, 6, 0\rangle$ dropping the ugly $1/\sqrt{37}$ still gives same direction and,

$\boxed{x + 6(y - \pi) = 0 \quad \text{Osculating Plane}}$

Finally, notice that if the torsion is zero then the Frenet Serret formulas simplify to:

$$\boxed{\frac{d\vec{T}}{dt} = \dot{s}\kappa\vec{N} \qquad \frac{d\vec{N}}{dt} = -\dot{s}\kappa\vec{T} \qquad \frac{d\vec{B}}{dt} = 0.}$$

we see that $\vec{B}$ is a constant vector field and motion will remain in the osculating plane. The change in the normal vector causes a change in the tangent vector and vice-versa however the binormal vector is not coupled to $\vec{T}$ or $\vec{N}$.

Remark 2.2.27.

The torsion measures the infinitesimal change in the direction of the binormal vector relative to the normal vector of the curve. Because the normal vector is in the plane of infinitesimal motion and the binormal is perpendicular to that plane we can say that the torsion measures how the path lifts or twists up off the plane of infinitesimal motion. Furthermore, we can expect path which is trapped in a particular plane (these are called **planar** curves) will have torsion which is identically zero. We should also expect that the torsion for something like a helix will be nonzero everywhere since the motion is always twisting up off the plane of infinitesimal motion. It is probable you will explore these questions in your homework.

Finally, I quote a theorem from Colley's *Vector Calculus.*

Thm/ Let C_1 and C_2 be smooth curves in $\mathbb{R}^3$ both with strictly positive curvatures κ_1 and κ_2. Then if $\kappa_1(s) = \kappa_2(s)$ and $\tau_1(s) = \tau_2(s)$ $\forall s$ then the curves C_1 & C_2 are congruent in the sense of highschool geometry ($C_1 = C_2 + \vec{b}$ ← some fixed vector) Moreover the converse is true, given a positive arclength function and torsion one can uniquely reconstruct a curve upto translations.

If you are really interested in digging deeper then my suggestion would be to take linear algebra (Math 321) and then pursue an independent study in elementary differential geometry. There are many nice books suitable for self-study of the topic. For example, Opreas' *Differential Geometry and its Applications.* I will probably survey a swath of the subject in the *Advanced Calculus* course, but my focus is more towards calculations and the structure of calculus. A course focused on differential geometry would dig much deeper. It is a rich subject full of interesting history and calculation.

2.3 physics of motion

In this section we study kinematics. That is, we study how position, velocity and acceleration are related for physical motions. We do not ask where the force comes from, that is a question for physics. Our starting point is the equation of motion $\vec{F} = m\vec{A}$ which is called Newton's Second Law. Given the force and some initial conditions we can in principle integrate the equations of motion and derive the resulting kinematics. We have already, in the LU calculus sequence, twice studied kinematics. In calculus I for one-dimensional motion, in calculus II for two-dimensional motion. I recycle some examples for our current discussion. However, some comments are added since we now have the proper machinery to break-down vectors along a physical path. Let's see how the preceding section is useful in the analysis of the motion of physical objects. The solution of Newton's equation $\vec{F} = m\vec{A}$ is a path $t \mapsto \vec{r}(t)$. It follows we can analyze the velocity and acceleration of the physical path in terms of the **Frenet Frame** $\{T, N, B\}$. To keep it interesting we'll assume the motion is non-stop and smooth so that the analysis of the last section applies.

In this section the notations $\vec{r}$, $\vec{v}$ and $\vec{a}$ are special and set-apart. I don't use these as abstract variables here with no set meaning. Instead, these are connected as is described in the definition that follows:

Definition 2.3.1. *position, velocity and acceleration.*

The position, velocity and acceleration of an object are vector-valued functions of time and we define them as follows:

1. $\vec{r}(t)$ is the position at time t. (we insist physical paths are parametrized by time)
2. $\vec{v}(t) = \frac{d\vec{r}}{dt}$ is the velocity at time t.
3. $\vec{a}(t) = \frac{d\vec{v}}{dt}$ is the acceleration at time t.

We also define the tangential and normal accelerations of the motion by

$$\vec{a}_T = \vec{a} \bullet \vec{T} \qquad \vec{a}_N = \vec{a} \bullet \vec{N} \qquad \text{note:} \qquad \vec{a} = \vec{a}_T\vec{T} + \vec{a}_N\vec{N}.$$

We know from our study of the geometry of curves that the binormal component of the acceleration is trivial. Acceleration must lie in the osculating plane and as such is perpendicular to the binormal vector which is the normal to the osculating plane. If you're curious, the position vector itself can have nontrivial components in each direction of the Frenet frame whereas the velocity vector clearly has only a tangential component; $\vec{v} = v\vec{T}$.

If we are given the position vector as a function of time then we need only differentiate to find the velocity and acceleration. On the other hand, if we are given the acceleration then we need to integrate and apply initial conditions to obtain the velocity and position.

Example 2.3.2. *Suppose R and ω are positive constants and the motion of an object is observed to follow the path $\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle = R\langle\cos(\omega t), \sin(\omega t)\rangle$. We wish to calculate the velocity and acceleration as functions of time.*

Differentiate to obtain the velocity

$$\boxed{\vec{v}(t) = R\omega\langle -\sin(\omega t), \cos(\omega t)\rangle.}$$

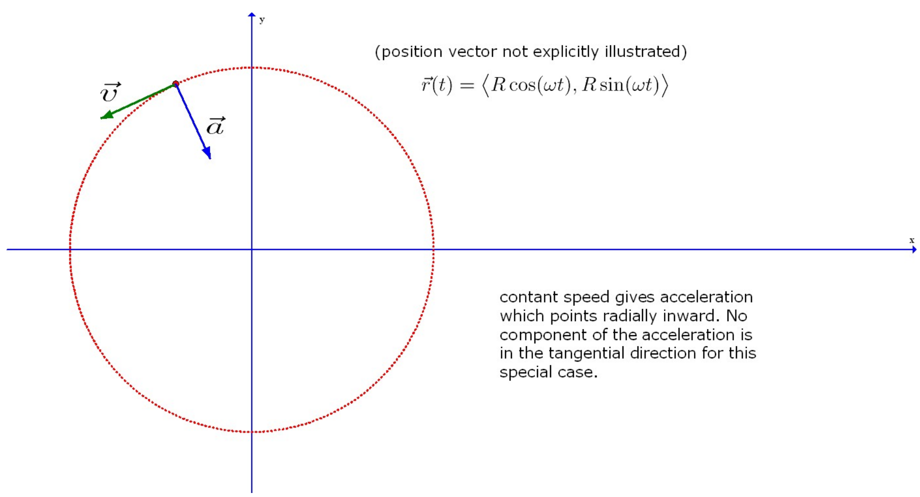
Differentiate once more to obtain the acceleration:

$$\vec{a}(t) = R\omega\langle -\omega\cos(\omega t), -\omega\sin(\omega t)\rangle = \boxed{-R\omega^2\langle\cos(\omega t), \sin(\omega t)\rangle.}$$

Notice we can write that $\vec{a}(t) = -\omega^2\vec{r}(t) = R\omega^2\vec{N}$ in this very special example. This means the acceleration is opposite the direction of the position and it is purely normal. Furthermore, we can calculate

$$r = R, \qquad v = R\omega, \qquad a = R\omega^2$$

Thus the magnitudes of the position, velocity and acceleration are all constant. However, their directions are always changing. Perhaps you recognize these equations as the foundational equations describing constant speed circular motion. This acceleration is called the **centripetal** *or center-seeking acceleration since it points towards the center. Here we imagine attaching the acceleration vector to the object which is traveling in the circle.*



Incidentally, you might wonder how the binormal should be thought of in the example above. We should adjoin a zero to make the vectors three-dimensional and then the cross-product of $\vec{T}\times\vec{N}$ points in the direction given by the right-hand-rule for circles. Curl your right hand around the circle following the motion and your thumb will point in the binormal direction. You can calculate that the binormal is constant:

$$\vec{B} = \vec{T}\times\vec{N} = \langle -\sin(\omega t), \cos(\omega t)\rangle \times \langle -\cos(\omega t), -\sin(\omega t)\rangle = \langle 0, 0, 1\rangle$$

Often when we consider planar motion we omit the third dimension in the vectors since those components are zero throughout the whole discussion. That said, if we wish to properly employ the Frenet Frame analysis then we must think in three dimensions. The next example is also two-dimensional[6].

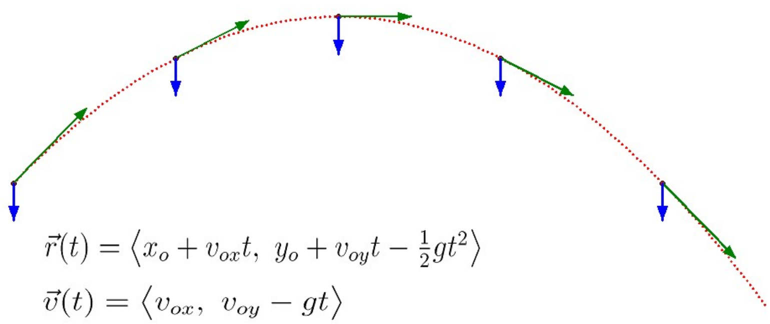
Example 2.3.3. *Suppose that the acceleration of an object is known to be $\vec{a} = \langle 0, -g\rangle$ where g is a positive constant. Furthermore, suppose that initially the object is at $\vec{r}_o$ and has velocity $\vec{v}_o$. We wish to calculate the position and velocity as functions of time.*

Integrate the acceleration from 0 *to* t*,*

$$\int_0^t \frac{d\vec{v}}{d\tau}d\tau = \int_0^t a(\tau)d\tau \quad \Rightarrow \quad \vec{v}(t) - \vec{v}(0) = \int_0^t \langle 0, -g\rangle d\tau \quad \Rightarrow \quad \boxed{\vec{v}(t) = \vec{v}_o + \langle 0, -gt\rangle}$$

Integrate the velocity from 0 *to* t*,*

$$\int_0^t \frac{d\vec{r}}{d\tau}d\tau = \int_0^t v(\tau)d\tau \quad \Rightarrow \quad \vec{r}(t) - \vec{r}(0) = \int_0^t \big(\vec{v}_o + \langle 0, -gt\rangle\big)d\tau \quad \Rightarrow \quad \boxed{\vec{r}(t) = \vec{r}_o + t\vec{v}_o + \langle 0, -\frac{1}{2}gt^2\rangle}$$



$$\vec{r}(t) = \left\langle x_o + v_{ox}t,\ y_o + v_{oy}t - \tfrac{1}{2}gt^2\right\rangle$$
$$\vec{v}(t) = \left\langle v_{ox},\ v_{oy} - gt\right\rangle$$
$$\vec{a}(t) = \left\langle 0, -g\right\rangle$$

The acceleration is constant for this parabolic trajectory.
The velocity is changing in the vertical direction, but is constant in the x-direction.

I'm curious how the decomposition of the acceleration into normal and tangential components works out for the example above. Maybe I'll make it a homework.

The best understanding of Newtonian Mechanics is given by a combination of both vectors and calculus. We need vectors to phrase the geometry of force addition whereas we need calculus to understand how the position, velocity and acceleration variables change in concert.

[6] all motion generated from Newtonian gravity alone is planar. A more general result states all central force motion lies in a plane, probably a homework of yours

2.3.1 position vs. displacement vs. distance traveled

The position of an object is simply the (x, y, z) coordinates of the object. Usually it is convenient to think of the position as a vector-valued function of time which we denote $\vec{r}(t)$. The displacement is also a vector, however it compares two possibly distinct positions:

Definition 2.3.4. *displacement and distance traveled.*

Suppose $\vec{r}(t)$ is the position at time t of some object.

1. The **displacement** from position $\vec{r}_1$ to position $\vec{r}_2$ is the vector $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$.
2. The **distance travelled** during the interval $[t_1, t_2]$ along the curve $t \mapsto \vec{r}(t)$ is given by
$$s_{12} = \int_{t_1}^{t_2} v(t)dt = \int_{t_1}^{t_2} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt$$
where $v(t) = ||d\vec{r}/dt||$.

Note that the position is the displacement from the origin. Distance travelled is a **scalar** quantity which means it is just a number or if we think of an endpoint as variable it could be a function.

Definition 2.3.5. *arclength function and speed.*

We define
$$s(t) = \int_{t_1}^{t} v(\tau)d\tau = \int_{t_1}^{t} \sqrt{\frac{dx}{d\tau}^2 + \frac{dy}{d\tau}^2 + \frac{dz}{d\tau}^2}\, d\tau$$
to be the arclength travelled from time t_1 to t along the parametrized curve $t \mapsto \vec{r}(t)$. Futhermore, we define the **speed** to be the instantaneous rate of change in the arclength; speed is ds/dt.

Notice it is simple to show that the speed is also equal to the magnitude of the velocity; $ds/dt = v$. Also, note that we drop the z-terms for a typical two-dimensional problem. If you insist on being three dimensional you can just adjoin a bunch of zeros in the examples below. There are unavoidably three dimensional examples a little later in the section.

Example 2.3.6. *Let $\omega, R > 0$. Suppose $\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle$ for $t \geq 0$. We can calculated that*

$$\frac{d\vec{r}}{dt} = \langle -R\omega\sin(\omega t), R\omega\cos(\omega t)\rangle \quad \Rightarrow \quad v(t) = \sqrt{(-R\omega\sin(\omega t))^2 + (R\omega\cos(\omega t))^2} = \sqrt{R^2\omega^2} = R\omega.$$

Now use this to help calculate the distance travelled during the interval $[0, t]$

$$s(t) = \int_0^t v(\tau)d\tau = \int_{t_1}^t R\omega d\tau = R\omega\tau\Big|_0^t = R\omega t.$$

In other words, $\Delta s = R\omega\Delta t$. On a circle the arclength subtended Δs divided by the radius R is defined to be the radian measure of that arc which we typically denote $\Delta\theta$. We find that $\Delta\theta = \omega\Delta t$ or if you prefer $\omega = \Delta\theta/\Delta t$.

Circular motion which is not at a constant speed can be obtained mathematically by replacing the constant ω with a function of time. Let's examine such an example.

Example 2.3.7. *Suppose $\vec{r}(t) = \langle R\cos(\theta), R\sin(\theta)\rangle$ for $t \geq 0$ where $\theta_o, \omega_o, \alpha$ are constants and $\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$. To calculate the distance travelled it helps to first calculate the velocity:*

$$\frac{d\vec{r}}{dt} = \langle -R(\omega_o + \alpha t)\sin(\theta), R(\omega_o + \alpha t)\cos(\theta)\rangle$$

Next, the speed is the length of the velocity vector,

$$v = \sqrt{[-R(\omega_o + \alpha t)\sin(\theta)]^2 + [R(\omega_o + \alpha t)\cos(\theta)]^2} = R\sqrt{(\omega_o + \alpha t)^2} = R|\omega_o + \alpha t|.$$

Therefore, the distance travelled is given by the integral below:

$$s(t) = \int_0^t R|\omega_o + \alpha\tau| d\tau$$

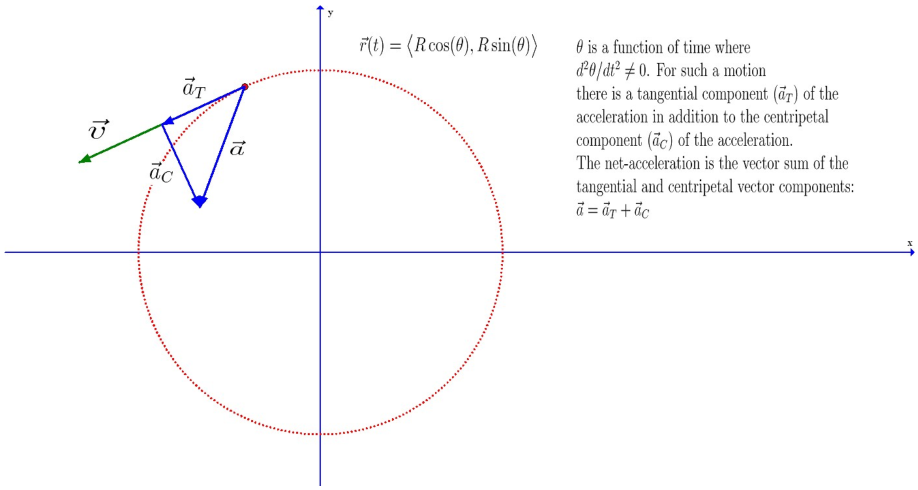
To keep things simple, let's suppose that ω_o, α are given such that $\omega_o + \alpha t \geq 0$ hence $v = R\omega_o + R\alpha t$. To suppose otherwise would indicate the motion came to a stopping point and reversed direction, which is interesting, just not to us here.

$$s(t) = R\int_0^t (\omega_o + \alpha\tau) d\tau = R\omega_o t + \frac{1}{2}R\alpha t^2.$$

Observe that $\theta(t) - \theta_o = (s(t) - s(0))/R$ thus we find that $\Delta\theta = \omega_o t + \frac{1}{2}\alpha t^2$ which is the formula for the angle subtended due to motion at a constant **angular acceleration** *α. In invite the reader to differentiate the position twice and show that*

$$\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = -\underbrace{R\omega^2\Big\langle \cos(\theta(t)), \sin(\theta(t))\Big\rangle}_{centripetal} + \underbrace{R\alpha\Big\langle -\sin(\theta(t)), \cos(\theta(t))\Big\rangle}_{tangential}$$

where $\omega = \omega_o + \alpha t$.



Distance travelled is not always something we can calculate in closed form. Sometimes we need to relegate the calculation of the arclength integral to a numerical method. However, the example that follows is still calculable without numerical assistance. It did require some thought.

Example 2.3.8. *We found that $\vec{a} = \langle 0, -g\rangle$ twice integrated yields a position of $\vec{r}(t) = \vec{r}_o + t\vec{v}_o + \langle 0, -\frac{1}{2}gt^2\rangle$ for some constant vectors $\vec{r}_o = \langle x_o, y_o\rangle$ and $\vec{v}_o = \langle v_{ox}, v_{oy}\rangle$. Thus,*

$$\vec{r}(t) = \left\langle x_o + v_{ox}t, y_o + v_{oy}t - \frac{1}{2}gt^2\right\rangle$$

From which we can differentiate to derive the velocity,

$$\vec{v}(t) = \left\langle v_{ox}, v_{oy} - gt\right\rangle.$$

If you've had any course in physics, or just a proper science education, you should be happy to observe that the zero-acceleration in the x-direction gives rise to constant-velocity motion in the x-direction whereas the gravitational acceleration in the y-direction makes the object fall back down as a consequence of gravity. If you think about $v_{oy} - gt$ it will be negative for some $t\rangle 0$ whatever the initial velocity v_{oy} happens to be, this point where $v_{oy} - gt = 0$ is the turning point in the flight of the object and it gives the top of the parabolic[7] trajectory which is parametrized by $t \to \vec{r}(t)$. Suppose $x_o = y_o = 0$ and calculate the distance travelled from time $t = 0$ to time $t_1 = v_{oy}/g$. Additionally, let us assume $v_{ox}, v_{oy} \geq 0$.

$$\begin{aligned} s = \int_0^{t_1} v(t)dt &= \int_0^{t_1} \sqrt{(v_{ox})^2 + (v_{oy} - gt)^2}dt \\ &= \int_{v_{oy}}^{0} \sqrt{(v_{ox})^2 + (u)^2}\left(\frac{du}{-g}\right) \qquad u = v_{oy} - gt \\ &= \frac{1}{g}\int_0^{v_{oy}} \sqrt{(v_{ox})^2 + (u)^2}du \end{aligned}$$

[7]no, we have not shown this is a parabola, I invite the reader to verify this claim. That is find A, B, C such that the graph $y = Ax^2 + Bx + C$ is the same set of points as $\vec{r}(\mathbb{R})$.

Recall that a nice substitution for an integral such as this is provided by the $\sinh(z)$ *since* $1+\sinh^2(z)=\cosh^2(z)$ *hence a* $u=v_{ox}\sinh(z)$ *subsitution will give*

$$(v_{ox})^2+(u)^2=(v_{ox})^2+(v_{ox}\sinh(z))^2=v_{ox}^2\cosh^2(z)$$

and $du=v_{ox}\cosh(z)dz$ *thus,* $\int\sqrt{(v_{ox})^2+(u)^2}du=\int\sqrt{v_{ox}^2\cosh^2(z)}v_{ox}\cosh(z)dz=\int v_{ox}^2\cosh^2(z)dz$. *Furthermore,* $\cosh^2(z)=\frac{1}{2}(1+\cosh(2z))$ *hence*

$$\int\sqrt{(v_{ox})^2+(u)^2}du=\frac{v_{ox}^2}{2}\left[z+\frac{1}{2}\sinh(2z)\right]+c=\frac{v_{ox}^2}{2}\left[z+\sinh(z)\cosh(z)\right]+c$$

Note $u=v_{ox}\sinh(z)$ *and* $v_{ox}cosh(z)=\sqrt{(v_{ox})^2+(u)^2}$ *hence substituting,*

$$\int\sqrt{(v_{ox})^2+(u)^2}du=\frac{1}{2}\left[v_{ox}^2\sinh^{-1}\left(\frac{u}{v_{ox}}\right)+u\sqrt{v_{ox}^2+u^2}\right]+c$$

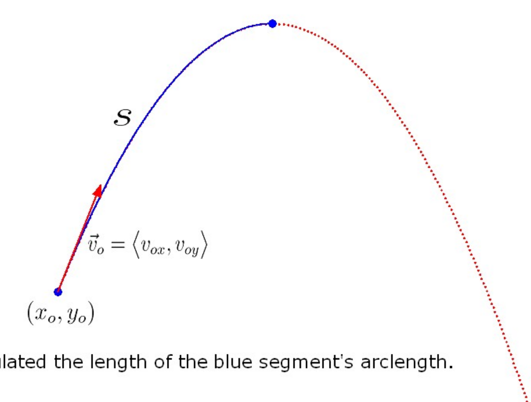
Well, I didn't think that was actually solvable, but there it is. Returning to the definite integral to calculate s *we can use the antiderivative just calculated together with FTC part II to conclude: (provided* $v_{ox}\neq 0$*)*

$$\boxed{s=\frac{1}{2g}\left[v_{ox}^2\sinh^{-1}\left(\frac{v_{oy}}{v_{ox}}\right)+v_{oy}\sqrt{v_{ox}^2+v_{oy}^2}\right]}$$

If $v_{ox}=0$ *then the problem is much easier since* $v(t)=|v_{oy}-gt|=v_{oy}-gt$ *for* $0\leq t\leq t_1=v_{oy}/g$ *hence*

$$s=\int_0^{t_1}v(t)dt=\int_0^{t_1}(v_{oy}-gt)dt=\left[v_{oy}t-\frac{1}{2}gt^2\right]\Bigg|_0^{v_{oy}/g}=\boxed{\frac{v_{oy}^2}{2g}}$$

Interestingly, this is the formula for the height of the parabola even if $v_{ox}\neq 0$. *The initial* x*-velocity simply determines the horizontal displacement as the object is accelerated vertically by gravity.*



We calculated the length of the blue segment's arclength.

Example 2.3.9. .

Suppose that $\vec{v}(t) =< e^t, t^2, \cos(t) >$ represents the velocity of the Dwight. If the initial position of Dwight is the origin then find the position of Dwight at time t. Also find the acceleration of Dwight at time t.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \boxed{\langle e^t, 2t, -\sin(t)\rangle = \vec{a}(t)}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \Rightarrow \int_0^t \vec{v}(u)\,du = \int_0^t \frac{d\vec{r}}{du}\,du = \vec{r}(t) - \vec{r}(0)$$

But, $\vec{r}(0) = \vec{0}$ hence,

$$\vec{r}(t) = \int_0^t \vec{v}(u)du = \int_0^t \langle e^u, u^2, \cos(u)\rangle du$$

$$= \left\langle e^u\Big|_0^t, \tfrac{1}{3}u^3\Big|_0^t, \sin(u)\Big|_0^t \right\rangle$$

$$= \boxed{\langle e^t - 1, \tfrac{1}{3}t^3, \sin(t)\rangle = \vec{r}(t)}$$

Example 2.3.10. .

Let the position be $\vec{r}(t) = \langle t^2+1, t^3, t^2-1\rangle$ assume $t \geq 0$,

velocity $= \vec{r}\,'(t) = \boxed{\vec{v}(t) = \langle 2t, 3t^2, 2t\rangle}$

acceleration $= \vec{r}\,''(t) = \boxed{\vec{a}(t) = \langle 2, 6t, 2\rangle}$

Speed $= |\vec{r}\,'(t)| = |\vec{v}(t)| = \sqrt{4t^2 + 9t^4 + 4t^2} = \boxed{t\sqrt{8+9t^2}}$

Example 2.3.11. .

Find $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{r}(t)$ given $\vec{a}(t) = \langle 2, 6t, 12t^2\rangle$ and the initial conditions $\vec{v}(0) = \langle 1, 0, 0\rangle$ and $\vec{r}(0) = \langle 0, 1, -1\rangle$

$$\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow \int_{\vec{v}(0)}^{\vec{v}(t)} d\vec{v} = \int_0^t \vec{a}\,d\tilde{t} \Rightarrow \vec{v}(t) - \vec{v}(0) = \int_0^t \langle 2, 6\tilde{t}, 12\tilde{t}^2\rangle d\tilde{t}$$

Thus $\vec{v}(t) = \langle 1, 0, 0\rangle + \langle 2t, 3t^2, 4t^3\rangle = \boxed{\langle 2t+1, 3t^2, 4t^3\rangle = \vec{v}}$

$$\vec{v} = \frac{d\vec{r}}{dt} \Rightarrow \int_{\vec{r}(0)}^{\vec{r}(t)} d\vec{r} = \int_0^t \vec{v}\,d\tilde{t} = \int_0^t \langle 2\tilde{t}+1, 3\tilde{t}^2, 4\tilde{t}^3\rangle d\tilde{t}$$

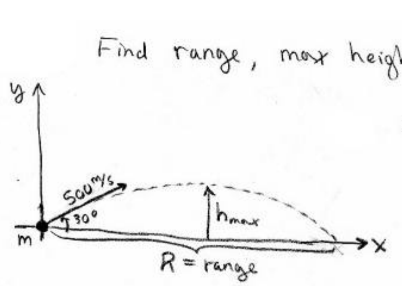
$$\Rightarrow \vec{r}(t) = \langle 0, 1, -1\rangle + \langle t^2+t, t^3, t^4\rangle$$

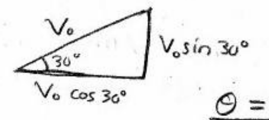
$$\boxed{\vec{r}(t) = \langle t^2+t, t^3+1, t^4-1\rangle}$$

Example 2.3.12. .

Find range, max height $\qquad g \equiv 9.8\,m/s^2$.

$$\vec{V}_o = \langle 500\cos 30^\circ, 500\sin 30^\circ\rangle$$





$\underline{\theta = 30^\circ}$

Newton's 2^{nd} law says that $\vec{F} = m\vec{a}$ meaning,

$$m\langle 0, -g\rangle = m\langle x'', y''\rangle$$

m cancels, ($m \neq 0$ we assume). Integrate both sides

$$\int_0^t \langle 0, -g\rangle\, d\tilde{t} = \int_0^t \langle x'', y''\rangle\, dt$$

$$\Rightarrow \langle 0, -gt\rangle = \left\langle \int_0^t \frac{d}{dt}\left(\frac{dx}{dt}\right)dt, \int_0^t \frac{d}{dt}\left(\frac{dy}{dt}\right)dt \right\rangle$$

$$= \left\langle \frac{dx}{dt} - \frac{dx}{dt}\Big|_{t=0}, \frac{dy}{dt} - \frac{dy}{dt}\Big|_{t=0} \right\rangle$$

$$= \langle V_x(t) - V_o\cos\theta, V_y(t) - V_o\sin\theta\rangle$$

Thus $\vec{V}(t) = \langle V_o\cos\theta, V_o\sin\theta - gt\rangle = \dfrac{d\vec{r}}{dt}$.

$$\int_{\vec{r}(0)}^{\vec{r}(t)} d\vec{r} = \vec{r}(t) - \vec{r}(0) = \int_0^t \langle V_o\cos\theta, V_o\sin\theta - g\tilde{t}\rangle\, d\tilde{t}$$

$$\Rightarrow \underline{\vec{r}(t) = \langle V_o\cos\theta\, t, V_o\sin\theta\, t - \tfrac{1}{2}gt^2\rangle}$$

$y = 0$ when $V_o\sin\theta\, t - \frac{1}{2}gt^2 = t\,(V_o\sin\theta - \frac{1}{2}gt) = 0$
which gives 2 sol^{n}'s namely $t_1 = 0$ and $t_2 = 2V_o\sin\theta/g$.

$$x(t_2) = V_o\cos\theta \cdot 2V_o\sin\theta/g = \boxed{\frac{V_o^2\sin(2\theta)}{g} = R}$$

Max height $\dfrac{dy}{dt} = 0 = V_o\sin\theta - gt \;\therefore\; t_{max} = \dfrac{V_o\sin\theta}{g}$

note $y'' = -g < 0 \;\therefore$ is max by 2^{nd} derivative test

thus $h_{max} = V_o\sin\theta \cdot \dfrac{V_o\sin\theta}{g} - \dfrac{g}{2}\left(\dfrac{V_o\sin\theta}{g}\right)^2 = \boxed{\dfrac{V_o^2\sin^2\theta}{2g} = h_{max}}$

Example 2.3.13. .

find tangential & normal components of $\vec{a}(t)$ for

$$\vec{r}(t) = \langle 3t - t^3,\ 3t^2,\ 0 \rangle.$$

$$\vec{r}\,'(t) = \langle 3 - 3t^2, 6t, 0 \rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{9 - 18t^2 + 9t^4 + 36t^2}$$

$$\vec{r}\,''(t) = \langle -6t, 6, 0 \rangle \qquad = \sqrt{9 + 9t^4 + 18t^2}$$

$$= 3\sqrt{t^4 + 2t^2 + 1}$$

$$= \boxed{3(t^2+1) = |\vec{r}\,'(t)|}$$

Thus we find

$$\vec{T}(t) = \frac{1}{3(t^2+1)} \langle 3(1-t^2), 6t, 0 \rangle$$

$$\vec{T}(t) = \left\langle \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, 0 \right\rangle$$

$$\vec{T}\,'(t) = \left\langle \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2}, \frac{2(1+t^2) - 2t(2t)}{(1+t^2)^2}, 0 \right\rangle$$

$$\vec{T}\,'(t) = \frac{1}{(1+t^2)^2} \langle -4t, 2 - 2t^2, 0 \rangle$$

$$|\vec{T}\,'(t)| = \frac{1}{(1+t^2)^2}\sqrt{16t^2 + 4(1 - 2t^2 + t^4)} = \frac{2}{(1+t^2)^2}\sqrt{\underbrace{t^4 + 2t^2 + 1}_{(t^2+1)^2}}$$

$$|\vec{T}\,'(t)| = 2/(1+t^2)$$

$$\vec{T} = \frac{1}{1+t^2}\langle 1 - t^2, 2t, 0 \rangle$$

$$\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|} = \left(\frac{1+t^2}{2}\right) \cdot \frac{1}{(1+t^2)^2} \langle -4t, 2(1-t^2), 0 \rangle$$

$$\vec{N} = \frac{1}{1+t^2} \langle -2t, 1 - t^2, 0 \rangle$$

Now we may find a_T and a_N for $a = 6\langle -t, 1, 0 \rangle$

$$a_T = \vec{a} \cdot \vec{T} = \frac{6}{1+t^2} \langle -t, 1, 0 \rangle \cdot \langle 1 - t^2, 2t, 0 \rangle$$

$$a_T = \frac{6}{1+t^2}(t^3 - t + 2t) = \frac{6t(t^2+1)}{1+t^2} = \boxed{6t = a_T}$$

$$a_N = \vec{a} \cdot \vec{N} = \frac{6}{1+t^2} \langle -t, 1, 0 \rangle \cdot \langle -2t, 1 - t^2, 0 \rangle$$

$$a_N = \frac{6}{1+t^2}(2t^2 + 1 - t^2) = \frac{6(1+t^2)}{1+t^2} = \boxed{6 = a_N}$$