Chapter 6

integration

MULTIVARIATE INTEGRATION

To begin we will study double & triple integrals over boxlike or rectangular regions, these are as easy as integrations from calc. I & II. Then we find how to integrate over type I & II regions in the xy-plane. and general volumes in xyz-space, this is not as easy but once you understand the importance of graphing and set-up it becomes clear. After exhausting topics in Cartesian Coordinates we'll study the Jacobian, this will give us a derivation of how to integrate in sphericals or cylindricals or polars or whatever system of coordinates you might invent. Then we apply our Jacobian theory to do integrals in polars, cylindricals and sphericals. We scatter select applications throughout our discussion.

Definitions: integrals are defined as the limit of a weighted sum over f,

$$\int_a^b f(x)\,dx \equiv \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad : \quad \Delta x = \frac{b-a}{n}$$

$$\iint_R f(x,y)\,dA \equiv \lim_{n\to\infty}\lim_{k\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{k} f(x_i^*, y_j^*)\,\Delta x\,\Delta y$$

$$R = [a,b]\times[c,d] \quad \text{and} \quad \Delta x = \frac{b-a}{n} \quad \text{while} \quad \Delta y = \frac{d-c}{k}$$

$$\iiint_B f(x,y,z)\,dV \equiv \lim_{m\to\infty}\lim_{n\to\infty}\lim_{k\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{k} f(x_i^*, y_j^*, z_l^*)\,\Delta x\,\Delta y\,\Delta z$$

$$B = [a,b]\times[c,d]\times[p,q] \quad \text{and} \quad \Delta x = \frac{b-a}{n}, \ \Delta y = \frac{d-c}{n}, \ \Delta z = \frac{q-p}{k}$$

We note that in Cartesian coordinates $dA = dx\,dy$ = infinitesimal area element in xy-plane. $dV = dx\,dy\,dz$ = infinitesimal volume element. As in calc I and II. the sample points are chosen randomly, but it doesn't matter in the limit. In practice the limit is rarely seen, instead the F.T.C. or evaluation rule and here the Fubini Th$^{\underline{m}}$ will keep us from ever using the def$^{\underline{n}}$ directly (THANKFULLY!)

Several Properties of the integral follow directly from the properties of the limit itself: Let R be a rectangular region

$$\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

$$\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA$$

$$f(x,y) \geq g(x,y) \ \ \forall (x,y) \in R \Rightarrow \iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

Likewise for $f(x,y,z)$ and $g(x,y,z)$ over a boxlike region. We assume that f, g are continuous most everywhere. Meaning we can integrate $f(x,y)$ if it has a finite number of curve discontinuities, or $f(x,y,z)$ if it has a finite number of planar discontinuities. We just chop the integral into a finite # of regions on which f is continuous.

FUBINI'S THM (WEAK FORM): known to Cauchy for continuous f in early 19^{th} century. Let $R = [a,b] \times [c,d]$ and let f be a mostly continuous fnct. of (x,y)

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

where the expressions on the RHS are "iterated integrals" which you work inside out, treating the outside variable as a constant to begin.

$\boxed{E87}$ Let $R = [0,\pi] \times [0,2]$ that is $0 \leq x \leq \pi$ and $0 \leq y \leq 2$. Integrate $f(x,y) = \sin(x) + y$ over R.

$$\iint_R f(x,y)\,dA = \int_0^{\pi}\left(\int_0^2 [\sin(x) + y]\,dy\right)dx$$

: parentheses added to emphasize the order of operations here.

$$= \int_0^{\pi} [y\sin(x) + \tfrac{1}{2}y^2 \,\big|_0^2\,dx$$

: note $\sin(x)$ is regarded as a constant in the dy integration.

$$= \int_0^{\pi} [2\sin(x) + 2]\,dx$$

$$= -2\cos(x)\Big|_0^{\pi} + 2x\Big|_0^{\pi}$$

$$= -2\cos\pi + 2\cos(0) + 2\pi$$

$$= \boxed{4 + 2\pi}$$

Exercise: compute $\int_0^2\int_0^{\pi}(\sin(x) + y)\,dx\,dy$ you should get same answer.

E88 Let $R = \{(x,y) \mid 0 \leq x \leq \pi/2,\ 0 \leq y \leq 2\}$.

$$\iint_R y\cos(xy)\,dA = \int_0^2 \left(\int_0^{\pi/2} y\cos(xy)\,dx \right) dy \quad : \int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$= \int_0^2 \left(\frac{y}{y}\sin(xy)\Big|_0^{\pi/2} \right) dy$$

$$= \int_0^2 \left(\sin\left(\frac{\pi y}{2}\right) - \sin(0) \right) dy$$

$$= \frac{-2}{\pi}\cos\left(\frac{\pi y}{2}\right)\Big|_0^2 = \frac{-2}{\pi}\left(\cos(\pi) - \cos(0)\right) = \boxed{\frac{4}{\pi}}$$

Remark: the order of $\int$ here is easier than if we reversed to $dy\,dx$

GEOMETRY: the double integral of $f(x,y)$ over R is the volume of the solid bounded by $z = f(x,y)$, $z = 0$ and $(x,y) \in R$. The $Th^{\underline{m}}$ of Fubini can be seen as merely saying you can slice up the volume along x or y crossections. Infinitesimally

$$dV = \underbrace{(z_{TOP} - z_{bottom})}_{\text{height of the box}} \underbrace{dx\,dy}_{\text{area of the box}}$$

So if $z_{bottom} = 0$ and $z_{TOP} \geq 0$ then we get the volume, however as in $\int f(x)\,dx$ we count volume below the xy-plane as negative so the integral calculates the "signed" volume. I'm not even going to pretend I can draw these things... I'll let Maple do the artistry. I do hope we can all see these things in our "minds eye" in the end.

E89 Let $B = \{(x,y,z) \mid 0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq z \leq c\}$

$$\iiint dV = \int_0^c \int_0^b \int_0^a dx\,dy\,dz$$

$$= \int_0^c \int_0^b x\Big|_0^a\,dy\,dz$$

$$= \int_0^c \left(\int_0^b a\,dy \right) dz$$

$$= \int_0^c ab\,dz$$

$$= \boxed{abc = V}$$

- If we integrate 1 over B we find the volume of B. Likewise if we had integrated 1 over a rectangle $R \subset \mathbb{R}^2$ we would have found the area.

E90 Let $B = [0,1]\times[0,2]\times[0,3]$. Let $\rho = \frac{dm}{dV} = xyz$. Consider,

$$\iiint_B xyz\, dV = \int_0^3\int_0^2\int_0^1 xyz\, dx\, dy\, dz$$

$$= \int_0^3 z\,dz \int_0^2 y\,dy \int_0^1 x\,dx$$

: only allowed if our functions factors into functions of x, y, z
$f(x,y,z) = f_1(x) f_2(y) f_3(z)$

$$= \frac{1}{2}z^2\Big|_0^3 \frac{1}{2}y^2\Big|_0^2 \frac{1}{2}x^2\Big|_0^1$$

$$= \frac{1}{8}(3)^2(2)^2$$

$$= \boxed{27/2}$$

What is the meaning of such an integration? Well if ρ = mass density = dm/dV then $\rho\, dV = dm$ thus

$$m = \int_B dm = \iiint_B \rho\, dV = \boxed{27/2}$$ = mass of object B with density $\rho = xyz$.

Or you could interpret it as $\rho = dq/dV$ = charge/volume so $q = \int \rho\, dV = 27/2$ = charge of object B. I'm sure you could imagine other densities.

E91 Another interpretation of $\iint_R f(x,y)\, dA$ is that $f(x,y)$ represents an area density. So say $f(x,y) = \sigma(x,y)$

$$\sigma(x,y) = \frac{dq}{dA} \Rightarrow q = \iint_R \sigma(x,y)\, dA$$ = charge on the planar region R

$$\sigma(x,y) = \frac{dm}{dA} \Rightarrow m = \iint_R \sigma(x,y)\, dA$$ = mass of the rectangle R.

E92 Another interpretation of $\int_a^b f(x)dx$ is that $f(x)$ represents a linear density. So say $f(x) = \lambda(x)$ and,

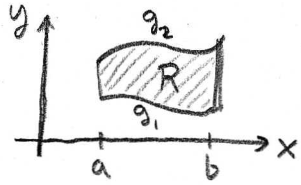
$$\lambda(x) = \frac{dq}{dx} \Rightarrow q = \int_a^b \lambda(x)\,dx, \quad \lambda = \frac{dm}{dx} \Rightarrow m = \int_a^b \lambda(x)dx$$

<u>Remark</u>: linear density is more exciting once we know about line-integrals (which are actually along curves generally). Note "density" is multifaceted.

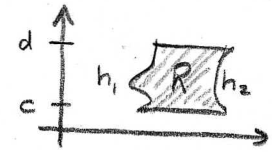
Double Integrals over General Regions:

Given an arbitrary connected region in the xy-plane there are two primary descriptions of the region, say R (not necessarily a rectangle any more). Your text classifies them as,

TYPE I: $\begin{cases} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{cases}$



TYPE II: $\begin{cases} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{cases}$



Of course, you can imagine regions which don't conveniently fit either TYPE. And on the other hand a rectangle is both TYPES at once, $g_1(x) = c$, $g_2(x) = d$ to get TYPE I, $h_1(y) = a$, $h_2(y) = b$ to get TYPE II.

Th^{m} (Fubini, Strong Version): Suppose f is mostly continuous.
Given $R_I = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ a TYPE I region,

$$\iint_{R_I} f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Given $R_{II} = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$ a TYPE II region,

$$\iint_{R_{II}} f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

E93 Let $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$.

$$\iint_R e^{x^2}dA = \int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 \left(e^{x^2} y \Big|_0^x\right)dx = \int_0^1 x e^{x^2}dx$$

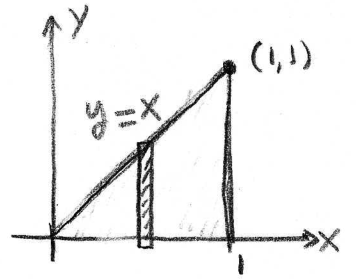
$$= \frac{1}{2}e^{x^2}\Big|_0^1$$

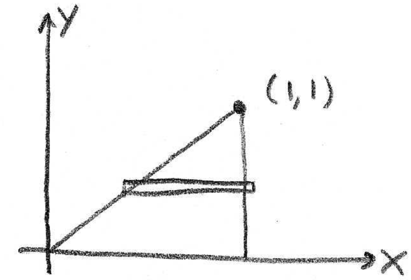
$$= \frac{1}{2}(e^1 - e^0)$$

$$= \boxed{\frac{1}{2}(e-1)}$$

Remark: we could just as well describe R as a TYPE II region. However, then we'd be faced with $\int e^{x^2}dx$. This is not an elementary integral.

$\boxed{E94}$ Using R from $\boxed{E93}$ calc. $\iint_R e^{y^2}\,dA$. Since treating R as TYPE I leads us to $\int e^{y^2}dy$ we need to make dx me first, so convert R to a TYPE II region. A picture helps,





$y_{TOP} = x$
$y_{bottom} = 0$

$x_{LEFT} = y$
$x_{RIGHT} = 1$

$$\underbrace{\{(x,y)\,|\,0\le x\le 1,\ 0\le y\le x\}}_{\text{TYPE I description}} = R = \underbrace{\{(x,y)\,|\,0\le y\le 1,\ y\le x\le 1\}}_{\text{TYPE II description}}$$

$$\iint_R e^{y^2}dA = \int_0^1\int_y^1 e^{y^2}\,dx\,dy \quad :$$

$$= \int_0^1 (1-y)e^{y^2}\,dy \quad :$$

the integral of the constant e^{y^2} is the product of the integration region length $(1-y)$ and the constant.

$$= \int_0^1 e^{y^2}dy - \int_0^1 ye^{y^2}dy$$

$$= \int_0^1 e^{y^2}dy - \tfrac{1}{2}(e-1) \quad : \text{ curses, I had hoped for better.}$$

$$= 1.463 - \tfrac{1}{2}(e-1) \quad : \int_0^1 e^{y^2}dy \text{ req's numerical sol}^{n}.$$

__Remark:__ not all integrals result in pretty sums & products, if we just make up some example on a hunch then it can get ugly. Incidentally while __indefinite__ integrals of e^{x^2} are not known in terms of elementary functions, there are __improper__ integrals of e^{-x^2} which do come out quite nicely. See §16.4 #36, we need a few toys to make it easier.

$\boxed{E95}$ Another application of double integrals is finding the area of a region. For example, $S = \{(x,y) \mid 0 \le x \le R,\ 0 \le y \le \sqrt{R^2-x^2}\}$

$$A(R) = \iint_S dA = \int_0^R \int_0^{\sqrt{R^2-x^2}} dy\,dx$$

$$= \int_0^R \sqrt{R^2-x^2}\,dx$$: use trig-substitution $x = R\cos\theta$ so $dx = -R\sin\theta\,d\theta$ and $\sqrt{R^2-x^2} = \sqrt{R^2\sin^2\theta} = R\sin\theta$ and the bounds change to $\pi/2 \to 0$

$$= \int_{\pi/2}^0 -R^2\sin^2\theta\,d\theta$$

$$= R^2\int_0^{\pi/2} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$: $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= \frac{R^2}{2}\left(\theta - \tfrac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{R^2}{2}\left(\frac{\pi}{2}\right) = \boxed{\pi R^2/4.}$$: since S is a quarter-circle this makes good sense.

Remark: this will be much easier in polar coordinates, see $\boxed{E106}$ on 344.

$\boxed{E96}$ We may also define the average of a function over R as

$$\boxed{f_{avg}^R = \frac{1}{A(R)}\iint_R f(x,y)\,dA}$$

Consider $f(x,y) = xy$. If $R = [0,1]\times[0,1]$ and S = quarter circle with $R=1$ from E95, do you think $f_{avg}^R > f_{avg}^S$ or vice-versa?

$$\iint_R xy\,dA = \int_0^1\int_0^1 xy\,dx\,dy = \int_0^1 x\,dx \int_0^1 y\,dy = \frac{1}{2}\frac{1}{2} = \frac{1}{4}.$$

$$\iint_S xy\,dA = \int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \int_0^1\left(\tfrac{1}{2}xy^2\Big|_0^{\sqrt{1-x^2}}\right)dx = \int_0^1 \tfrac{1}{2}(x-x^3) = \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{1}{4}\right) = \tfrac{1}{8}$$

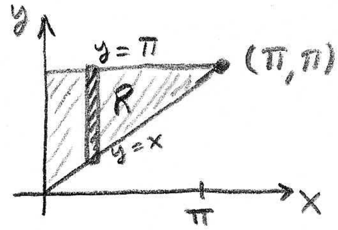
Thus

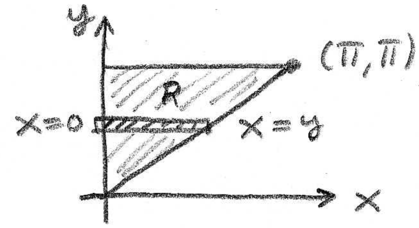
$$f_{avg}^R = \frac{1/4}{A(R)} = \frac{1/4}{1} = \frac{1}{4} \quad \text{whereas} \quad f_{avg}^S = \frac{1/8}{A(S)} = \frac{1/8}{\pi/4} = \frac{1}{2\pi}$$

The average of xy is larger on the unit-square since $\frac{1}{4} > \frac{1}{2\pi}$.

Remark: Notice $\iint_S xy\,dA$ was considerably easier than $\iint_S dA$.

E97 Calculate $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$. Notice we need to reverse the order of integration to do a TYPE II integration (has $dx\,dy$ instead). Our given integral suggests $0 \le x \le \pi$ and $x \le y \le \pi$,





$\underline{0 \le y \le \pi,\ \ 0 \le x \le y}$

$$\iint_R \frac{\sin y}{y}\, dA = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \sin y\, dy = -\cos y\Big|_0^{\pi} = \boxed{2}.$$

E98 Calculate,

$$\int_0^{3/2}\int_0^{9-4x^2} 16x\, dy\, dx = \int_0^{3/2} 16x(9-4x^2)\, dx$$

: $16x$ is a constant wrt dy integration, simply multiply by int. reg. length.

$$= \int_0^{3/2} (144x - 64x^3)\, dx$$

$$= 72x^2\Big|_0^{3/2} - 16x^4\Big|_0^{3/2}$$

$$= 72(3/2)^2 - 16(3/2)^4$$

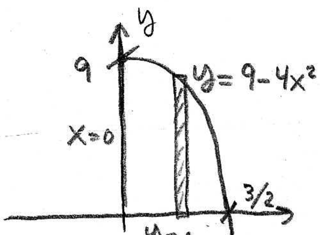
$$= 162 - 81 = \boxed{81}$$

Lets reverse the order of integration for fun. Note $9-4x^2=0 \Rightarrow x^2 = \frac{9}{4}$

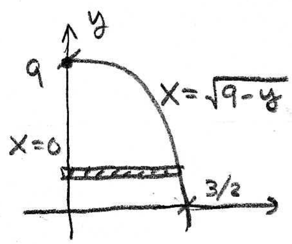
$y = 9-4x^2$ is a parabola with x-intercepts $x = \pm 3/2$ and y-intercept 9. Solve for x and keep positive root,

$$x^2 = \frac{1}{4}(9-y)$$

$$x = \frac{1}{2}\sqrt{9-y}$$



TYPE I: find y bounds in terms of x



TYPE II: find x bounds in terms of y.

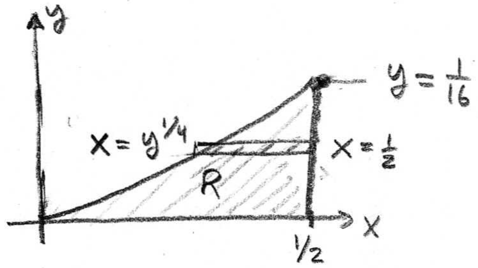
$$\iint_R 16x\, dA = \int_0^9\int_0^{\frac{1}{2}\sqrt{9-y}} 16x\, dx\, dy = \int_0^9 \left(8x^2\Big|_0^{\frac{1}{2}\sqrt{9-y}}\right) dy = \int_0^9 2(9-y)\, dy$$

$$= (18y - y^2\Big|_0^9$$

$$= 18(9) - 81$$

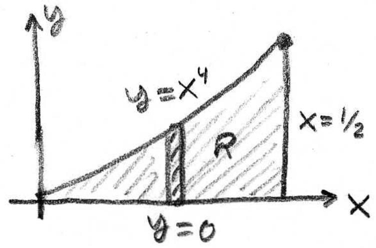
$$= \boxed{81}$$

E99 Calculate $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\,dx\,dy$. Seems changing bounds may be helpful here. To begin $0 \le y \le 1/16$ and $y^{1/4} \le x \le \frac{1}{2}$ which is type II, lets graph to guide our conversion to TYPE I,



$\frac{1}{2} = y^{1/4} \Rightarrow y = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

thus $x = 1/2$ and $x = y^{1/4}$ intersect at the point $(1/2, 1/16)$ as graphed.



$0 \le x \le 1/2$

$0 \le y \le x^4$

well isn't that convenient.

$$\iint_R \cos(16\pi x^5)\,dA = \int_0^{1/2}\int_0^{x^4} \cos(16\pi x^5)\,dy\,dx$$: notice $\cos(16\pi x^5)$ is constant in the dy-integration.

$$= \int_0^{1/2} x^4 \cos(16\pi x^5)\,dx$$: let $u = 16\pi x^5$,

$$= \frac{1}{80\pi}\sin(16\pi x^5)\Big|_0^{1/2}$$

$$= \frac{1}{80\pi}\left(\sin\left(\frac{16\pi}{32}\right) - \sin(0)\right)$$

$$= \boxed{\frac{1}{80\pi}}$$

__Remark__: these arguments should be familiar from Calc. II, see p. 132 – 139d. Our methods for finding area were more specialized, now we add $f(x,y)$ into the integration but the essential idea of viewing the graph as TYPE I or TYPE II was there as well. I'd say type II regions needed horizontal slicing whereas type I were vertically sliced. We can see the formula on 132 as type I

$$A = \int_a^b\int_{g(x)}^{f(x)} dy\,dx = \int_a^b (f(x) - g(x))\,dx = \int_a^b (y_{Top} - y_{bottom})\,dx$$

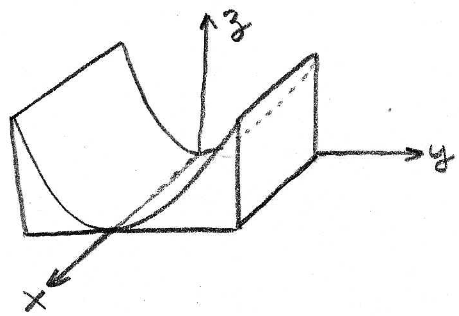
or for TYPE II : $$A = \int_c^d\int_{x_L(y)}^{x_R(y)} dx\,dy = \int_c^d (x_R - x_L)\,dy$$

these formulas are special cases of our double integrals.

TRIPLE INTEGRALS OVER GENERAL BOUNDED REGIONS IN $\mathbb{R}^3$

Rather than explicitly stating the 3-d Fubini Th$^{\underline{m}}$ I will simply illustrate with a few examples. Usually we can bound z in terms of x & y then we can bound y in terms of x or vice-versa, that gives two orders of integration. Then other problems allow x to be bound in terms of y & z or possibly y in terms of x & z, in total there are 6 ways to write a particular integral over a volume. In §12.7 #31 I explicitly show 6 ways to write a particular integral. I don't give general advice on how to rewrite and switch bounds, its a subtle buisness and I would advocate double checking with Mathematica. Generalities aside, lets do a few typical problems.

E100 Let us find the volume of the region between $z = y^2$ and the xy-plane bounded by $x=0$, $x=1$, $y=1$ and $y=-1$. Notice $dV = dx\,dy\,dz$ so integrating dV gives volume V,



$0 \leq z \leq y^2$
$0 \leq x \leq 1$
$-1 \leq y \leq 1$

- we must integrate w.r.t. z either first or second.

$$V = \int_{-1}^{1}\int_{0}^{1}\int_{0}^{y^2} dz\,dx\,dy \quad : \text{ we work inside out as usual.}$$

$$= \int_{-1}^{1}\int_{0}^{1} y^2\,dx\,dy \quad : \text{ back to 2-d integrals.}$$

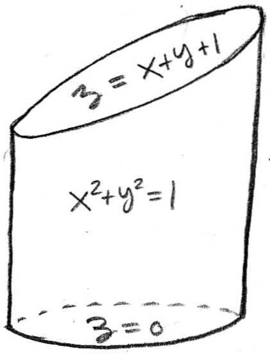
$$= \int_{-1}^{1} y^2\,dy \quad : \text{ back to 1-d integral.}$$

$$= \frac{1}{3}y^3\Big|_{-1}^{1}$$

$$= \frac{1}{3}\left(1-(-1)^3\right)$$

$$= \boxed{2/3}$$

$\boxed{E101}$ Consider the cylinder $x^2+y^2=1$, let $z=0$ bound it from below and let $z=x+y+1$ bound it above, Call this solid B. A sketch of B reveals the inequalities to the right of it



$$0 \le z \le x+y+1$$

$$0 \le x^2+y^2 \le 1 \quad \begin{array}{l} \to -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ \to -1 \le x \le 1 \end{array}$$

Calculate then,

$$\iiint_B x\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{x+y+1} x\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+xy+x)\,dy\,dx$$

$$= \int_{-1}^{1} \left[(x^2+x)y + \tfrac{1}{2}xy^2\right|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1}\Big(\underbrace{2x^2\sqrt{1-x^2}}_{\text{I need to think about this one.}} + \overbrace{\cancel{2x\sqrt{1-x^2}}}^{\text{zero, odd function over even}}\Big)dx$$

trig substitution.

$$\boxed{\begin{array}{ll} x = \sin\theta & dx = \cos\theta\,d\theta \\ 1-x^2 = \cos^2\theta & \sqrt{1-x^2} = \cos\theta \end{array}}$$

$$\int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta\cos\theta\cos\theta\,d\theta$$

$$= \int(\sin^2\theta - \sin^4\theta)\,d\theta$$

$$= \int\left[\tfrac{1}{2}(1-\cos(2\theta)) - \tfrac{1}{4}(1-2\cos(2\theta)+\cos^2(2\theta))\right]d\theta$$

$$= \int\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta) - \tfrac{1}{4} + \tfrac{1}{2}\cos(2\theta) - \tfrac{1}{4}\cos^2(2\theta)\right)d\theta$$

$$= \int\left[\tfrac{1}{4} - \tfrac{1}{8}(1-\cos(4\theta))\right]d\theta$$

$$= \theta/8 + \sin(4\theta)/32$$

Change bounds on x from $-1 = x \to 1 = x \Rightarrow -\frac{\pi}{2} = \theta \to \frac{\pi}{2} = \theta$.

$$\iiint_B x\,dV = \left[\frac{\theta}{8} + \frac{1}{32}\sin(4\theta)\right|_{-\pi/2}^{\pi/2} = \boxed{\frac{\pi}{4}}$$

E102 Let B be bounded by coordinate planes and the plane passing through $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$. We find the eq$^{\underline{n}}$ of this plane to begin, note $\vec{V}$ & $\vec{W}$ are on the plane

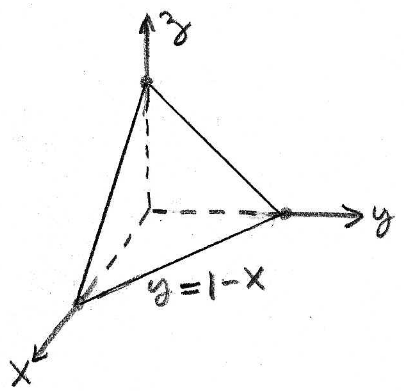
$$\vec{V} = (0,0,1) - (0,1,0) = \langle 0,-1,1 \rangle$$
$$\vec{W} = (0,0,1) - (1,0,0) = \langle -1,0,1 \rangle$$

$$\vec{V} \times \vec{W} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle -1,-1,-1 \rangle = \langle a,b,c \rangle \text{ the normal.}$$

Choose $r_o = (0,0,1)$ to base the plane eq$^{\underline{n}}$,

$$-x - y - (z-1) = 0 \Rightarrow \underline{z = 1 - x - y}$$

Lets plot it, note $z = 1-x-y$ intersects $z=0$ on the line $y = 1-x$ in the xy-plane.



$$\left.\begin{array}{l} 0 \leq z \leq 1-x-y \\ 0 \leq y \leq 1-x \\ 0 \leq x \leq 1 \end{array}\right\} \text{this is a useful description of } B.$$

Lets find the volume of B,

$$V = \iiint_B dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$
$$= \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx$$
$$= \int_0^1 \Big[(1-x)y - \tfrac{1}{2}y^2 \Big|_0^{1-x} \Big] dx$$
$$= \int_0^1 \Big[(1-x)^2 - \tfrac{1}{2}(1-x)^2 \Big] dx$$
$$= \int_0^1 \tfrac{1}{2}(1 - 2x + x^2)\, dx$$
$$= \tfrac{1}{2}\left(1 - \tfrac{2}{2} + \tfrac{1}{3}\right) = \boxed{1/6}$$

Remark: these problems are pretty much easy once you figure out how to describe the solid.

$\boxed{E103}$ find the average value of $f(x,y,z) = x$ on the solid region from $\boxed{E102}$. The average is defined to be

$$\boxed{f^{B}_{avg} \equiv \frac{1}{vol(B)} \iiint_B f(x,y,z)\, dV}$$

we just found $vol(B) = 1/6$, lets focus on the $\iiint_B f\,dV$,

$$\iiint_B f\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, dz\, dy\, dx$$

$$= \int_0^1 x \int_0^{1-x} (1-x-y)\, dy\, dx$$

$$= \int_0^1 x \left[(1-x)y - \tfrac{1}{2}y^2 \Big|_0^{1-x} \right] dx$$

$$= \int_0^1 \tfrac{1}{2} x(1-x)^2\, dx$$

$$= \int_0^1 \tfrac{1}{2}(x - 2x^2 + x^3)\, dx$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}\right)$$

$$= \tfrac{1}{2}\left(\tfrac{3}{4} - \tfrac{2}{3}\right)$$

$$= \frac{1}{24} \Rightarrow f^{B}_{avg} = \frac{1/24}{1/6} = \boxed{\frac{1}{4}}$$

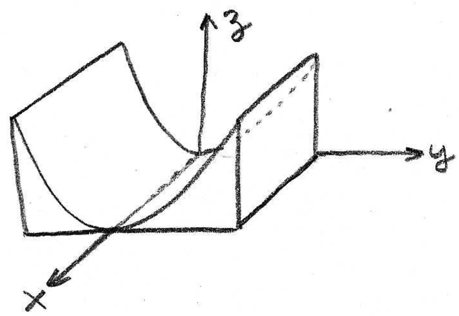
that seems pretty reasonable from the picture in $\boxed{E102}$.

<u>Remark</u>: We have studied how to integrate in Cartesian Coordinates in some detail. It turns out that this is quite limiting. To do many interesting problems with better efficiency it pays to employ cylindrical or spherical coordinates. Before getting to those special choices we consider a general coordinate change briefly and in the process derive what we later use for the cylindrical & spherical coordinates.

Triple Integrals over General Bounded Regions in $\mathbb{R}^3$

Rather than explicitly stating the 3-d Fubini Th$^{\underline{m}}$ I will simply illustrate with a few examples. Usually we can bound z in terms of x & y then we can bound y in terms of x or vice-versa, that gives two orders of integration. Then other problems allow x to be bound in terms of y & z or possibly y in terms of x & z, in total there are 6 ways to write a particular integral over a volume. In §12.7 #31 I explicitly show 6 ways to write a particular integral. I don't give general advice on how to rewrite and switch bounds, its a subtle buisness and I would advocate double checking with Mathematica. Generalities aside, lets do a few typical problems.

E100 Let us find the volume of the region between $z = y^2$ and the xy-plane bounded by $x=0$, $x=1$, $y=1$ and $y=-1$. Notice $dV = dx\,dy\,dz$ so integrating dV gives volume V,



$$0 \le z \le y^2$$
$$0 \le x \le 1$$
$$-1 \le y \le 1$$

- we must integrate w.r.t. z either first or second.

$$V = \int_{-1}^{1}\int_{0}^{1}\int_{0}^{y^2} dz\,dx\,dy \quad : \text{ we work inside out as usual.}$$

$$= \int_{-1}^{1}\int_{0}^{1} y^2\,dx\,dy \quad : \text{ back to 2-d integrals.}$$

$$= \int_{-1}^{1} y^2\,dy \quad : \text{ back to 1-d integral.}$$

$$= \frac{1}{3}y^3\Big|_{-1}^{1}$$

$$= \frac{1}{3}\left(1-(-1)^3\right)$$

$$= \boxed{2/3}$$