Homework 7, Calculus III

§16.2#6

$$\int_{\pi/6}^{\pi/2}\int_{-1}^{5} \cos(y)\,dx\,dy = \int_{\pi/6}^{\pi/2}\left(\cos(y)\, x\Big|_{-1=x}^{5=x}\right)dy$$

$$= \int_{\pi/6}^{\pi/2} 6\cos(y)\,dy$$

$$= 6\sin(y)\Big|_{\pi/6}^{\pi/2}$$

$$= 6\sin(\pi/2) - 6\sin(\pi/6)$$

$$= 6 - 3$$

$$= \boxed{3}.$$

§16.2#10

$$\int_0^1\int_0^3 e^{x+3y}\,dx\,dy = \int_0^1\int_0^3 e^x e^{3y}\,dx\,dy$$

$$= \int_0^1 e^{3y}\int_0^3 e^x\,dx\,dy$$

Only can do this When bounds are constants

$$= \int_0^1 e^{3y}\left(e^x\Big|_{0=x}^{3=x}\right)dy$$

$$= \int_0^1 (e^3-1)e^{3y}\,dy$$

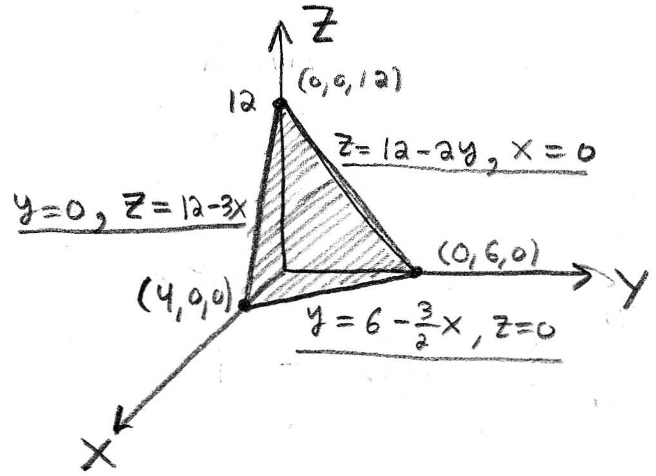
$$= \left[\frac{e^3-1}{3}\right]e^{3y}\Big|_0^1$$

$$= \left(\frac{e^3-1}{3}\right)(e^3-1)$$

$$= \boxed{\frac{1}{3}(e^3-1)^2}$$

§16.2 # 25 Find volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle
$$R = \{(x,y) \mid 0 \le x \le 1,\ -2 \le y \le 3\}.$$

We ought to integrate $z = 12 - 3x - 2y \equiv f(x,y)$ on R. This gives the sum of volumes with height z. Well, let's be careful, it gives the <u>signed</u> volume hopefully $f(x,y) \geq 0$ for $(x,y) \in R$.



Graph $z = 12 - 3x - 2y$ to be safe.

$x = 0,\ z = 12 - 2y$

$y = 0,\ z = 12 - 3x$

$z = 0,\ 0 = 12 - 3x - 2y$

$y = 6 - \frac{3}{2}x$

As you can see the graph $z = 12 - 3x - 2y$ is entirely above the xy-plane for the region describe, $0 \le x \le 1$ with $-2 \le y \le 3$ puts $z = f(x,y) > 0$.
You can argue that the analysis above is not needed because the problem states the volume lies above R and below $z = 12 - 3x - 2y$. That's ok, you didn't have to do this graph, <u>BUT</u> it would be reasonable to ask you to do such an argument. Can you follow it?

§16.2 #25 The preceding page showed this is a reasonable calculation to find the volume of the solid.

$$V = \iint_R f \, dA$$

$$= \int_{-2}^{3} \int_0^1 (12 - 3x - 2y) \, dx \, dy$$

$$= \int_{-2}^{3} \left[12x - \frac{3}{2}x^2 - 2yx\right]\Big|_0^1 dy$$

$$= \int_{-2}^{3} \left[12 - \frac{3}{2} - 2y\right] dy$$

$$= \int_{-2}^{3} \left(\frac{21}{2} - 2y\right) dy$$

$$= \left(\frac{1}{2}21y - y^2\right)\Big|_{-2}^{3}$$

$$= \left(\frac{1}{2}(21)(3) - 9\right) - \left(-\frac{42}{2} - 4\right)$$

$$= \left(63/2 - 18/2\right) - (-21 - 4)$$

$$= 45/2 + 25$$

$$= 45/2 + 50/2$$

$$= \boxed{95/2}$$

§16.6 #3

$$\int_0^1\int_0^z\int_0^{x+z} 6xz\,dy\,dx\,dz = \int_0^1\int_0^z \left(6xyz\Big|_{0=y}^{x+z=y}\right)dx\,dz$$

$$= \int_0^1\int_0^z 6x(x+z)z\,dx\,dz$$

$$= \int_0^1\int_0^z \left(6x^2z + 6xz^2\right)dx\,dz$$

$$= \int_0^1 \left(2x^3z + 3x^2z^2\Big|_{x=0}^{x=z}\right)dz$$

$$= \int_0^1 \left(2z^4 + 3z^2\right)dz$$

$$= \int_0^1 5z^4\,dz$$

$$= z^5\Big|_0^1$$

$$= \boxed{1}$$

§16.6 #10 Let $E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x,\ x \le z \le 2x\}$

$$\iiint_E yz\cos(x^5)\,dV = \int_0^1\left(\int_x^{2x}\left(\int_0^x yz\cos(x^5)\,dy\right)dz\right)dx$$

$$= \int_0^1\int_x^{2x}\left(z\cos(x^5)\frac{y^2}{2}\Big|_0^x\right)dz\,dx$$

$$= \int_0^1\int_x^{2x}\left(\tfrac{1}{2}x^2 z\cos(x^5)\right)dz\,dx$$

$$= \int_0^1\left(\tfrac{1}{2}x^2\cos(x^5)\,\frac{z^2}{2}\Big|_x^{2x}\right)dx$$

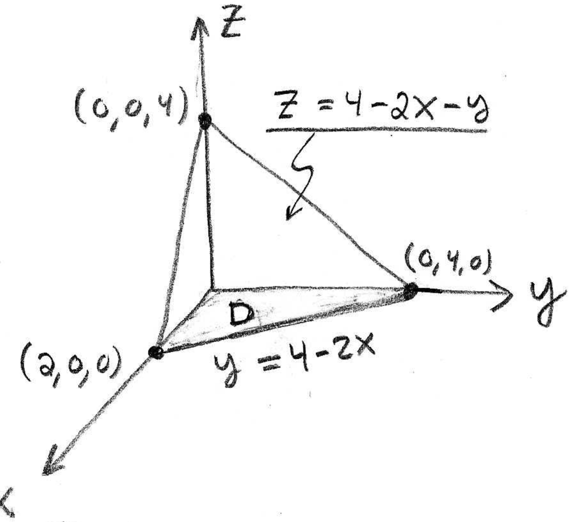
$$= \int_0^1\left(\tfrac{1}{2}x^2\cos(x^5)\,\tfrac{1}{2}(4x^2 - x^2)\right)dx$$

$$= \int_0^1 \tfrac{3}{4}x^4\cos(x^5)\,dx \qquad \boxed{\begin{matrix} u = x^5 & du = 5x^4dx \\ u(1) = 1, & u(0) = 0\end{matrix}}$$

$$= \int_0^1 \tfrac{3}{20}\cos(u)\,du$$

$$= \tfrac{3}{20}\left(\sin(1) - \sin(0)\right) = \boxed{\frac{3\sin(1)}{20}}$$

§16.6 #19 Tetrahedron enclosed by coordinate planes $2x + y + z = 4$. Look at intersection with coord. planes



Gather information,

$x = 0, \quad z = 4 - y$

$y = 0, \quad z = 4 - 2x$

$z = 0, \quad y = 4 - 2x$

$x = y = 0, \quad (0,0,4)$

$x = z = 0, \quad (0,4,0)$

$y = z = 0, \quad (2,0,0)$

the facts above help make the graph.

Observe from graph,

$$D = \{(x,y) \mid 0 \le x \le 2, \ 0 \le y \le 4-2x\}$$

Thus

$$V = \iint_D (4-2x-y)\,dA$$

$$= \int_0^2 \int_0^{4-2x} (4-2x-y)\,dy\,dx$$

$$= \int_0^2 \left(\left[(4-2x)y - \tfrac{1}{2}y^2 \right|_0^{4-2x} \right) dx$$

$$= \int_0^2 \left((4-2x)(4-2x) - \tfrac{1}{2}(4-2x)^2 \right) dx$$

$$= \int_0^2 \tfrac{1}{2}(4-2x)^2\,dx$$

$$= \int_0^2 \tfrac{1}{2}(16 - 16x + 4x^2)\,dx$$

$$= \int_0^2 (8 - 8x + 2x^2)\,dx$$

$$= \left(8x - 4x^2 + \tfrac{2}{3}x^3 \right|_0^2 = 16 - 16 + \left(\tfrac{2}{3}\right)8 = \boxed{16/3}$$

HOMEWORK 8, CALCULUS III

§16.9 #1 Find Jacobian of the following transformation

$x = 5u - v \Rightarrow x_u = 5, \quad x_v = -1$

$y = u + 3v \Rightarrow y_u = 1, \quad y_v = 3.$

$$\frac{\partial(x,y)}{\partial(u,v)} \equiv \det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$

← "Jacobian" is determinant of Jacobian matrix.

$$= \det\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} = 15 + 1 = \boxed{16.}$$

> Remark: $F(u,v) = (x(u,v), y(u,v))$ then $DF = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$ so $\frac{\partial(x,y)}{\partial(u,v)} = \det(DF)$

§16.9 #5 Let $x = u/v$, $y = v/w$, $z = w/u$ then

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

$$= \det\begin{bmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1/w & -v/w^2 \\ -w/u^2 & 0 & 1/u \end{bmatrix}$$

$$= \frac{1}{v}\left(\frac{1}{w}\frac{1}{u} + \frac{v}{w^2}(0)\right) + \frac{u}{v^2}\left(0\left(\frac{1}{u}\right) - \frac{v}{w^2}\left(\frac{w}{u^2}\right)\right)$$

$$= \frac{1}{uvw} - \frac{uvw}{u^2v^2w^2}$$

$$= \boxed{0.}$$

Remark: This transformation would not be allowed if we wish to use the change of variables integration Th$^{\underline{m}}$. That Th$^{\underline{m}}$ insists that Jacobian $\neq 0$.

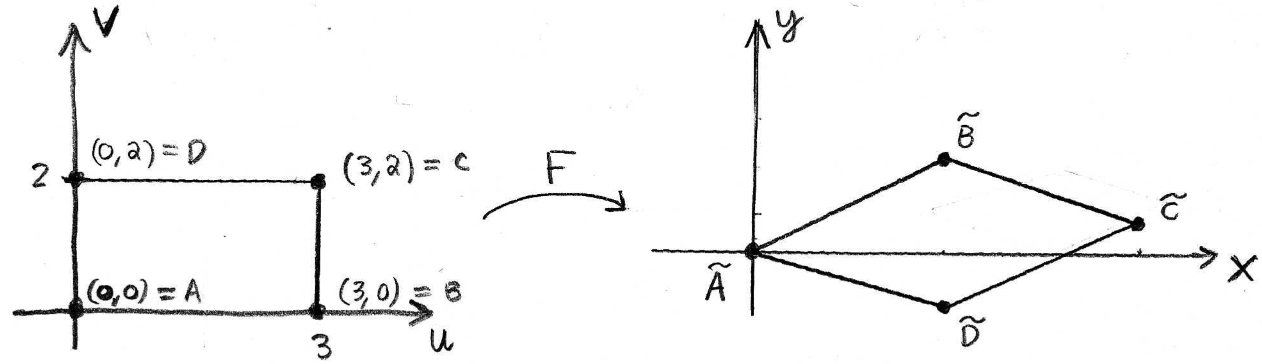
§16.9 #7 (You are not responsible for the proof/Remark in Calc III). ②

Let $S = \{(u,v) \mid 0 \le u \le 3,\ 0 \le v \le 2\}$

Suppose $F(u,v) = (x(u,v), y(u,v))$ where

$$x(u,v) = 2u + 3v$$
$$y(u,v) = u - v$$

Find the image of S under the transformation F.



We can prove that a nonzero linear transformation will map lines to lines thus if we figure out where the corners map to then we can just connect the dots.

$$F(0,0) = (0,\ 0) = \tilde{A}$$
$$F(3,0) = (6,\ 3) = \tilde{B}$$
$$F(3,2) = (6+6,\ 3-2) = (12,1) = \tilde{C}$$
$$F(0,2) = (6,\ -2) = \tilde{D}$$

Remark: to make transformation maintain # of distinct corners we need $\det(A) \neq 0$

Proof: Suppose $F(u,v) = (au + bv,\ cu + dv)$ this is a linear transformation of $\mathbb{R}^2 \to \mathbb{R}^2$. Using matrix notation,

$$F(u,v) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Or we may simply write $F(\vec{u}) = A\vec{u}$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$. Take a parametrically described line in uv-space it will be $\vec{u}(t) = \vec{u}_o + t\vec{v}_o$ generically. Observe

$$F(\vec{u}(t)) = A(\vec{u}_o + t\vec{v}_o) = (A\vec{u}_o) + t(A\vec{v}_o)$$

This is a line $\vec{x}(t) = \vec{x}_o + t\vec{y}_o$ where $\vec{x}_o = A\vec{u}_o$ and $\vec{y}_o = A\vec{v}_o$.
We need to assume $\det(A) \neq 0$ keep F one-one mapping.

§16.9 #13 Let $R = \{(x,y) \mid 9x^2+4y^2 \le 36\}$

Let $x = 2u$ and $y = 3v$ thus we'll change the ellipse to

$$36 = 9x^2+4y^2 = 9(2u)^2+4(3v)^2$$
$$= 36u^2+36v^2 \longrightarrow \boxed{u^2+v^2=1}$$

Go from ellipse in xy-plane to unit circle in uv-space

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6.$$

Thus calculate, let $S = \{(u,v) \mid u^2+v^2 \le 1\}$

$$\iint_R x^2\,dA = \iint_S (2u)^2 \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

$$= \iint_S 24u^2\,du\,dv$$

let $u = r\cos\theta$
$v = r\sin\theta$
then S becomes
$0 \le \theta \le 2\pi$ and
$0 \le r \le 1$ in
$r\theta$ - space.

$$= \int_0^1 \int_0^{2\pi} 24r^2\cos^2\theta\, r\,d\theta\,dr$$

$$= \int_0^1 24r^3\,dr \int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \left(6r^4\Big|_0^1\right)\left(\frac{1}{2}\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi}\right)$$

$$= (6-0)\left[\frac{1}{2}\left(2\pi+\underbrace{\frac{\sin(4\pi)}{2}}_{0}\right) - \frac{1}{2}\left(0+\frac{1}{2}\underbrace{\sin(0)}_{0}\right)\right]$$

$$= \boxed{6\pi}$$

§16.9 #17a Let $E = \{(x,y,z) \mid x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1\}$ calculate the volume of E by making the change of coordinates

$$x = au$$
$$y = bv$$
$$z = cw$$

under this change of coordinates E morphs from an ellipsoid to a ball B;

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = (au)^2/a^2 + (bv)^2/b^2 + (cw)^2/c^2$$
$$= u^2 + v^2 + w^2$$

That is $B = \{(u,v,w) \mid u^2+v^2+w^2 \le 1\}$

$$\iiint_E dV = \iiint_E dx\,dy\,dz$$

$$= \iiint_B \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \quad : \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

$$= \iiint_B abc\, du\,dv\,dw$$

$$= abc \underbrace{\iiint_B du\,dv\,dw}_{\text{volume of unit sphere } = \frac{4}{3}\pi.}$$

$$= \boxed{\frac{4\pi abc}{3}}$$

Remark: the volume of the unit sphere is calculated as follows,

$$\iiint_B du\,dv\,dw = \int_0^1 \int_0^\pi \int_0^{2\pi} \left|\frac{\partial(u,v,w)}{\partial(\rho,\theta,\phi)}\right| d\theta\, d\phi\, d\rho \quad : \quad \begin{aligned} u &= \rho\cos\theta\sin\phi \\ v &= \rho\sin\theta\sin\phi \\ w &= \rho\cos\phi \end{aligned}$$

Calculate the Jacobian,

$$\frac{\partial(u,v,w)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} u_\rho & u_\theta & u_\phi \\ v_\rho & v_\theta & v_\phi \\ w_\rho & w_\theta & w_\phi \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= -\rho^2\cos^2\theta\sin^3\phi + \rho^2\sin\theta\sin\phi\left[-\sin^2\phi\sin\theta - \cos^2\phi\sin\theta\right] - \rho^2\cos^2\theta\cos^2\phi\sin\phi$$
$$= -\rho^2\cos^2\theta\sin\phi - \rho^2\sin^2\theta\sin\phi$$
$$= -\rho^2\sin\phi$$

Thus, $\displaystyle\iiint_B du\,dv\,dw = \int_0^1\int_0^\pi\int_0^{2\pi} \rho^2\sin\phi\, d\theta\, d\phi\, d\rho = 2\pi\left(\frac{\rho^3}{3}\Big|_0^1\right)\left(-\cos\phi\Big|_0^\pi\right) = \frac{4\pi}{3}.$

Homework 10, Calculus III

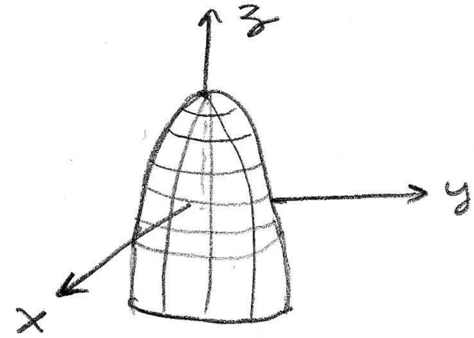
§16.7 #7

$z = 4 - r^2 = 4 - x^2 - y^2$

When $x = 0$ we have $z = 4 - y^2$
When $y = 0$ we have $z = 4 - x^2$
This is a paraboloid that opens down and has top point $(0,0,4)$. For any constant $z = z_0 < 4$ we have

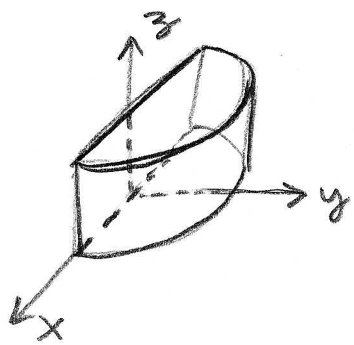
$$4 - z_0 = r^2$$

a circle.



§16.7 #11

$0 \le r \le 2$
$-\pi/2 \le \theta \le \pi/2$
$0 \le z \le 1$



half-cylinder with height 1 resting on the xy-plane.

§16.7 #17 E with $x^2 + y^2 \le 16$ and $-5 \le z \le 4$

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_{-5}^{4}\int_0^4 r^2\,dr\,dz\,d\theta$$

$\boxed{dV = r\,dr\,dz\,d\theta}$ (since $\left|\frac{\partial(x,y,z)}{\partial(r,z,\theta)}\right| = r$)

$$= \int_0^{2\pi} d\theta \int_{-5}^{4} dz \int_0^4 r^2\,dr$$

$$= (2\pi)(9)(16/3)$$

$$= \boxed{384\pi/3}$$

§16.7 #20 $E = \{ (x,y,z) \mid 0 \le z \le x+y+5 \ , \ 4 \le x^2+y^2 \le 9 \}$

We can convert E to cylindrical coordinates via $x = r\cos\theta, y = r\sin\theta$ and of course $z = z$. Observe $x^2+y^2 = r^2$, E becomes,

$$0 \le \theta \le 2\pi$$
$$4 \le r^2 \le 9 \ \Rightarrow \ 2 \le r \le 3$$
$$0 \le z \le r\cos\theta + r\sin\theta + 5$$

Thus motivating,

$$\iiint_E x\, dV = \int_0^{2\pi}\int_2^3\int_0^{r(\cos\theta+\sin\theta)+5} (\cos\theta\, r^2) dz\, dr\, d\theta \quad , \quad \underline{dV = r\,dz\,dr\,d\theta}$$

$$= \int_0^{2\pi}\int_2^3 \left(z\Big|_0^{r(\cos\theta+\sin\theta)+5} \right)\cos\theta\, r^2\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_2^3 \left[r^3(\cos^2\theta + \sin\theta\cos\theta) + 5r^2\cos\theta \right] dr\, d\theta$$

$$= \int_0^{2\pi} \left[\left(\frac{r^4}{4}\Big|_2^3 \right)(\cos^2\theta + \sin\theta\cos\theta) + \left(\frac{5}{3} r^3\Big|_2^3 \right)\cos\theta \right] d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{1}{4}(81-16)(\cos^2\theta + \sin\theta\cos\theta) + \tfrac{5}{3}(27-8)\cos\theta \right] d\theta$$

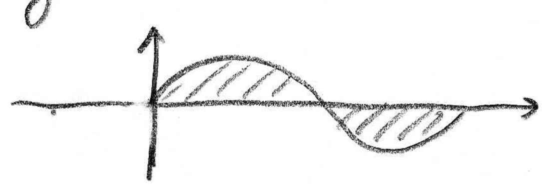
$$= \int_0^{2\pi} \left[\frac{65}{4}\left(\frac{1}{2} + \underbrace{\frac{1}{2}\cos(2\theta) + \frac{1}{2}\sin(2\theta) \Big) + 25\cos\theta}_{} \right] d\theta$$

$$= \int_0^{2\pi} \frac{65}{8}\, d\theta$$

$$= \frac{65}{8}\theta\Big|_0^{2\pi}$$

$$= \boxed{\frac{65\pi}{4}}$$

these are periodic functions which are about to be integrated over a whole period. By symmetry these integrals are zero.



(this can be a very useful labor saving observation)

§16.7 #26) Find mass m of a ball B which is the set of all points (x, y, z) such that $x^2+y^2+z^2 \leq a^2$ where a is some constant and the mass-density

$$\rho = k\sqrt{x^2+y^2} = kr \quad \left(\text{proportionality constant is } k\right)$$

Observe that if $\rho = \frac{dm}{dV}$ then we can add up all the little masses $dm = \rho\, dV$ by integrating,

$$m = \int dm = \iiint_B \rho\, dV \longrightarrow$$

notice
$$x^2+y^2+z^2 \leq a^2$$
$$\Rightarrow r^2 + z^2 \leq a^2$$
$$\Rightarrow z^2 \leq a^2 - r^2$$
$$\Rightarrow |z| \leq \sqrt{a^2-r^2}$$
$$\Rightarrow -\sqrt{a^2-r^2} \leq z \leq \sqrt{a^2-r^2}$$

$$= \int_0^{2\pi}\int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} kr^2\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^a 2kr^2\sqrt{a^2-r^2}\, dr\, d\theta$$

$$= \int_0^a 4\pi k \underbrace{r^2\sqrt{a^2-r^2}\, dr}_{\text{not an obvious integral, prepare for math battle.}}$$

: note the integrand is constant in θ so we just have the $d\theta$ integration yield $\theta\big|_0^{2\pi} = 2\pi$.

$$\int x^2\sqrt{a^2-x^2}\, dx = \int (a^2\sin^2\theta)(a\cos\theta)a\cos\theta\, d\theta$$

Let $x = a\sin\theta$
then $dx = a\cos\theta\, d\theta$
$\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2\theta)}$
$= \sqrt{a^2\cos^2\theta}$
$= a\cos\theta.$

$$= a^4\int \sin^2\theta\cos^2\theta\, d\theta$$

$$= a^4\int \frac{-1}{4}(e^{i\theta}-e^{-i\theta})^2\frac{1}{4}(e^{i\theta}+e^{-i\theta})^2 d\theta$$

$$= \frac{-a^4}{16}\int (e^{2i\theta} - 2 + e^{-2i\theta})(e^{2i\theta} + 2 + e^{-2i\theta})\, d\theta$$

$$= \frac{-a^4}{16}\int \left(e^{4i\theta} + \cancel{2e^{2i\theta}} + 1 - \cancel{2e^{2i\theta}} - 4 - \cancel{2e^{-2i\theta}} + 1 + \cancel{2e^{-2i\theta}} + e^{-4i\theta}\right) d\theta$$

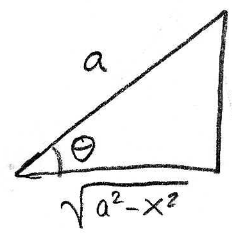
§ 16.7 #26

$$\int x^2\sqrt{a^2-x^2}\,dx = \frac{-a^4}{16}\int \left(2\cos(4\theta) - 2\right)d\theta$$

$$= \frac{-a^4}{32}\sin(4\theta) + \frac{a^4}{8}\theta$$

$$= \frac{a^4}{32}\sin\left(4\sin^{-1}\left(\frac{x}{a}\right)\right) + \frac{a^4}{8}\sin^{-1}\left(\frac{x}{a}\right)$$

$x = a\sin\theta$



$\theta = \sin^{-1}\left(\frac{x}{a}\right)$

Returning to the original integral,

$$\int_a^a 4\pi k r^2\sqrt{a^2-r^2}\,dr = 4\pi k\int_0^{\pi/2}\left[2\cos(4\theta)-2\right]d\theta$$

$0 \le r \le a$
$\Rightarrow r = a\sin\theta$
bounded by
$0 \le \theta \le \pi/2$

$$= 4\pi k\left(\frac{-a^4}{32}\sin(4\theta) + \frac{a^4}{8}\theta\right)\Bigg|_0^{\pi/2}$$

$$= 4\pi k\left\{\frac{-a^4}{32}\sin(2\pi) + \frac{a^4}{8}\frac{\pi}{2} + \frac{a^4}{32}\sin(0) + \frac{a^4}{8}(0)\right\}$$

$$= \frac{\pi^2 k a^4}{4}$$

$$\boxed{m = \frac{\pi^2}{4}ka^4}$$

<u>Remark</u>: you were given permission to use Mathematica to do this integral.

§16.7 #26 | Use spherical coordinates instead

$$m = \iiint_B \rho \, dV$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{a} k\sqrt{x^2+y^2} \, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{a} k\sqrt{\rho^2 \cos^2\theta \sin^2\phi + \rho^2 \sin^2\theta \sin^2\phi} \, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{a} k \rho^3 \sin^2\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} 2\pi k \left(\frac{\rho^4}{4} \Big|_0^a \right) \frac{1}{2}(1 - \cos(2\phi)) \, d\phi$$

$$= \frac{\pi a^4 k}{4} \left(\theta - \frac{1}{2} \sin(2\phi) \Big|_0^{\pi} \right.$$

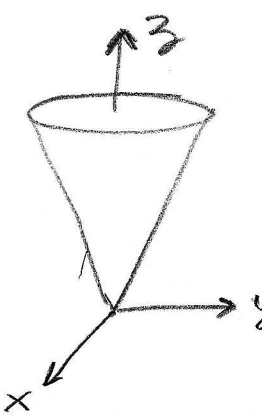
$$= \boxed{\frac{1}{4} k \pi^2 a^4}$$

Coordinate choice matters.

This problem has both cylindrical and spherical symmetries associated to it. Often the symmetry of the bounds is the better one to take advantage from. (the integrand suggests cylindricals)

Homework 11, Calculus III

§16.8 #5



$\phi = \pi/3$ is a cone centered about the z-axis.

$$z = \rho\cos\phi = \rho\cos\left(\tfrac{\pi}{3}\right) = \frac{2\rho}{\sqrt{3}}.$$

$$\left.\begin{aligned} x &= \rho\cos\theta\sin\phi = \tfrac{1}{2}\rho\cos\theta \\ y &= \rho\sin\theta\sin\phi = \tfrac{1}{2}\rho\sin\theta \end{aligned}\right\} \text{ since } \sin\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2}.$$

Thus $x^2 + y^2 = \frac{1}{4}\rho^2(\cos^2\theta + \sin^2\theta)$

$$4x^2 + 4y^2 = x^2 + y^2 + z^2$$

$$\boxed{z^2 = 3x^2 + 3y^2}$$

§16.8 #9 Write the equation $z^2 = x^2 + y^2$ in sphericals,

$$(\rho\cos\varphi)^2 = (\rho\cos\theta\sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2$$

$$\rho^2\cos^2\varphi = \rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi$$

$$= \rho^2(\cos^2\theta + \sin^2\theta)\sin^2\varphi$$

$$= \rho^2\sin^2\varphi$$

$$\Rightarrow \cos^2\varphi = \sin^2\varphi$$

$$\Rightarrow \tan^2\varphi = 1$$

$$\tan\varphi = \pm 1$$

$$\therefore \boxed{\varphi = \frac{\pi}{4} \text{ or } \frac{-\pi}{4}}$$

§16.8 #21) Let B : (x,y,z) with $x^2+y^2+z^2 \le 25$. Recall $dV = \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$

$$\iiint_B (x^2+y^2+z^2)^2 \, dV = \int_0^5 \int_0^\pi \int_0^{2\pi} \rho^6 \sin\phi \, d\theta \, d\phi \, d\rho :$$

θ-integration gives $\theta\Big|_0^{2\pi}$ a.k.a 2π.

$$= \int_0^5 \int_0^\pi 2\pi \rho^6 \sin\phi \, d\phi \, d\rho$$

$$= \int_0^5 2\pi\rho^6 \left(-\cos\phi \Big|_0^\pi\right) d\rho$$

$$= \int_0^5 2\pi\rho^6 \left(-\cos\pi + \cos(0)\right) d\rho$$

$$= \frac{4\pi\rho^7}{7}\Big|_0^5$$

$$= \boxed{\frac{312{,}500\pi}{7}}$$

§16.8 #23) E is $1 \le \rho \le 2$, and $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$

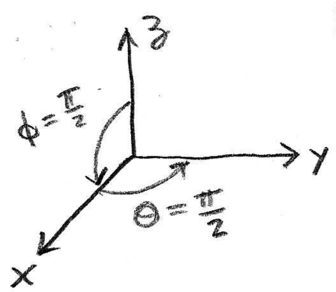
$$\iiint_E z \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 (\rho\cos\phi)(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta)$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi \cos\phi \, d\phi \int_1^2 \rho^3 \, d\rho$$

$$= \left(\theta\Big|_0^{\pi/2}\right)\left(\frac{1}{2}\sin^2\phi \Big|_0^{\pi/2}\right)\left(\frac{1}{4}\rho^4\Big|_1^2\right)$$

$$= \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{16-1}{4}\right)$$

$$= \boxed{\frac{15\pi}{16}}$$



§16.8 # 27) Find volume of part of ball $\rho \leq a$ that lies between $\phi = \pi/6$ and $\phi = \pi/3$.

$$V = \iiint_{\text{part of ball}} dV = \int_0^a \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_0^a \rho^2 d\rho \int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/3} \sin\phi \, d\phi$$

why can we do this here?

$$= \left(\frac{1}{3}\rho^3 \Big|_0^a\right)\left(\theta \Big|_0^{2\pi}\right)\left(-\cos\phi \Big|_{\pi/6}^{\pi/3}\right)$$

$$= \left(\frac{1}{3}a^3\right)(2\pi)\left(-\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{2\pi a^3}{3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{\pi a^3}{3}\left(\sqrt{3} - 1\right)}$$

Remark: You probably need to do more basic multiple integration problems. I doubt that the assigned problems are sufficient to build the requisite skill. Also I hope you understand why its ok to break up

$$\int_4^5 \int_2^3 \int_0^1 f(x)g(y)h(z)\, dx\,dy\,dz = \int_0^1 f(x)dx \int_2^3 g(y)dy \int_4^5 h(z)dz$$

we could not do that if instead

$$\int_4^5 \underbrace{\int_z^3 \int_{y+z}^1}_{\substack{\text{variable bounds require certain orders}\\ \text{of operations, in contrast constant bounds are easy.}}} f(x)g(y)h(z)\, dx\,dy\,dz$$

§12.2 #6 Calculate the double integral,

$$\int_1^4\int_0^2 (x+\sqrt{y})\,dx\,dy = \int_1^4 \left\{ \tfrac{1}{2}x^2\Big|_0^2 + x\sqrt{y}\Big|_0^2 \right\} dy$$

$$= \int_1^4 \left\{ 2 + 2\sqrt{y} \right\} dy$$

$$= \left(2y + \frac{4y^{3/2}}{3}\right)\Big|_1^4$$

$$= \left[2(4) + \tfrac{4}{3}(\sqrt{4})^3\right] - \left[2 + \tfrac{4}{3}\right]$$

$$= \left[8 + \tfrac{32}{3}\right] - \left[\tfrac{10}{3}\right] = \frac{24+32-10}{3} = \boxed{\frac{46}{3}}$$

§12.2 #10 Integrate.

$$\int_1^2\int_0^1 \frac{1}{(x+y)^2}\,dx\,dy = \int_1^2 \frac{-1}{(x+y)}\Big|_0^1 dy$$

$$= \int_1^2 \left[\frac{-1}{(y+1)} + \frac{1}{y}\right] dy$$

$$= \left[-\ln|y+1| + \ln|y|\right]\Big|_1^2$$

$$= \left[-\ln(3) + \ln(2)\right] - \left[-\ln(2) + \ln(1)\right]$$

$$= -\ln(3) + 2\ln(2)$$

$$= \ln(4) - \ln(3) = \boxed{\ln(4/3)}$$

§12.2 #12 Integrate.

$$\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx = \int_0^1 \left\{ \int_{x^2}^{x^2-1} \sqrt{u}\cdot\frac{x\,du}{2} \right\} dx$$

u-substitution

$u = x^2+y^2$, x-fixed.
$y=0 \Rightarrow u = x^2$
$y=1 \Rightarrow u = x^2+1$
$du = 2y\,dy$
$\frac{1}{2}x\,du = xy\,dy$

$$= \int_0^1 x\left(\tfrac{1}{2}\tfrac{2}{3}u^{3/2}\Big|_{x^2}^{x^2-1}\right) dx$$

Remark: You may find it easier to go off to the side and calculate difficult integrals indefinitely. Otherwise need to change bounds as I have here.

$$= \int_0^1 x\left[\tfrac{1}{3}[x^2+1]^{3/2} - \tfrac{1}{3}[x^2]^{3/2}\right] dx$$

$$= \int_0^1 \left(\tfrac{1}{3}x\,(x^2+1)^{3/2} - \tfrac{1}{3}x^4\right) dx$$

$W = x^2 - 1$
$dW = 2x\,dx$

$$= \tfrac{1}{3}\left[\tfrac{1}{5}(x^2+1)^{5/2} - \tfrac{1}{5}x^5\right]\Big|_0^1$$

$$= \tfrac{1}{3}\left[\left(\tfrac{1}{5}(\sqrt{2})^5 - \tfrac{1}{5}\right) - \left(\tfrac{1}{5}\right)\right] = \boxed{\tfrac{1}{15}\left(4\sqrt{2} - 2\right)}$$

$\boxed{\S 12.2\#14}$ Let $R = \{(x,y) \mid 0 \le x \le \pi,\ 0 \le y \le \pi/2\}$. Calculate

$$\iint_R \cos(x+2y)\,dA = \int_0^{\pi}\left(\int_0^{\pi/2} \cos(x+2y)\,dy\right)dx$$

$$= \int_0^{\pi}\left(\frac{1}{2}\sin(x+2y)\Big|_{0=y}^{\pi/2=y}\right)dx$$

$$= \int_0^{\pi} \frac{1}{2}(\sin(x+\pi) - \sin(x))\,dx$$

$$= \frac{1}{2}(-\cos(x+\pi) + \cos(x)\Big|_0^{\pi}$$

$$= \frac{1}{2}(-\cos(2\pi) + \cos(\pi)) - \frac{1}{2}(-\cos(\pi) + \cos(0))$$

$$= \frac{1}{2}(-1-1) - \frac{1}{2}(1+1) = \boxed{-2}$$

$\boxed{\S 12.2\#16}$ $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$

$$\iint_R \frac{1+x^2}{1+y^2}\,dA = \int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx$$

$$= \int_0^1 (1+x^2)\,dx \int_0^1 \frac{1}{1+y^2}\,dy$$

$$= (x + \tfrac{1}{3}x^3\Big|_0^1 (\tan^{-1}(y)\Big|_0^1$$

$$= \left(\frac{4}{3}\right)[\tan^{-1}(1) - \tan^{-1}(0)] = \boxed{\frac{\pi}{3}} \quad \left(\xrightarrow{\text{used}} \begin{array}{l} \tan(0) = 0 \Rightarrow \tan^{-1}(0) = 0 \\ \tan(\pi/4) = 1 \Rightarrow \tan^{-1}(1) = \frac{\pi}{4} \end{array}\right)$$

$\boxed{\S 12.2\#23}$ Find volume of solid bounded by $z = 1 - x^2/4 - y^2/9$ (the top) and $R = [-1,1]\times[-2,2]$ (the bottom) (the sides are $x=-1$, $x=1$, $y=2$, $y=-2$) we view R as a subset of (xy)-plane. In this special case the volume is found by integrating $z = 1 - x^2/4 + y^2/9$,

$$V = \iint_R \left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right)dA = \int_{-1}^{1}\int_{-2}^{2}\left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right)dy\,dx$$

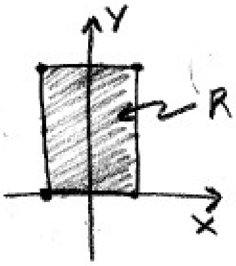
$$= \int_{-1}^{1}\left[y\left(1-\tfrac{1}{4}x^2\right) - \tfrac{1}{27}y^3\Big|_{-2}^{2}\,dx\right.$$

$$= \int_{-1}^{1}[4 - x^2 - 16/27]\,dx \qquad 4 - 16/27 = \frac{108-16}{27} = \frac{92}{27}$$

$$= 2\left(\frac{92}{27}\right) - \frac{1}{3}x^3\Big|_{-1}^{1}$$

$$= 2\left(\frac{92}{27}\right) - \frac{1}{3}(1-(-1)) = \frac{184-18}{27} = \boxed{\frac{166}{27}}$$

§12.2 #31 The average of $f(x,y) = x^2y$ over some region R is defined to be the $\iint_R f(x,y)\,dA$ divided by the area of $R = \iint_R dA = A(R)$. Let R be region with vertices $(-1,0), (-1,5), (1,5), (1,0)$



$R = [-1,1]\times[0,5]$

$A(R) = \text{Area}(R) = 2(5) = 10.$

$$f_{avg} = \frac{1}{10}\int_0^5\int_{-1}^1 x^2y\,dx\,dy = \frac{1}{10}\int_0^5 \frac{1}{3}x^3y\Big|_{-1}^1 dy = \frac{1}{10}\int_0^5 \frac{2}{3}y\,dy = \frac{2}{30}\,\frac{y^2}{2}\Big|_0^5$$

$$\therefore \boxed{f_{avg} = \frac{25}{30} = \frac{5}{6}}$$

Remark: So §12.2 integrals aren't any more difficult than the integrals we saw in Calc I or II. The new features really start in the next section. There we will find graphing a needed ally.

§12.3 #3 Integrate,

$$\int_0^1\int_y^{e^y}\sqrt{x}\,dx\,dy = \int_0^1\left(\frac{2}{3}x^{3/2}\Big|_y^{e^y}\right)dy$$

$$= \frac{2}{3}\int_0^1\left(e^{3y/2} - y^{3/2}\right)dy$$

$$= \frac{2}{3}\left[\frac{2}{3}e^{3y/2} - \frac{2}{5}y^{5/2}\Big|_0^1\right.$$

$$= \frac{4}{3}\left[\left(\frac{1}{3}e^{3/2} - \frac{1}{5}\right) - \left(\frac{2}{3} - 0\right)\right] \qquad \frac{1}{5} + \frac{2}{3} = \frac{3+10}{15} = \frac{13}{15}.$$

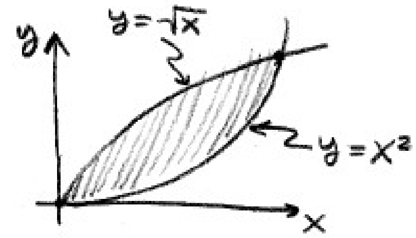
$$= \frac{4}{3}\left(\frac{1}{3}e^{3/2} - \frac{13}{15}\right) = \boxed{\frac{4}{9}e^{3/2} - \frac{13}{45}}$$

§12.3 #10

$$\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1\left(xe^{y^2}\Big|_0^y\right)dy = \int_0^1 ye^{y^2}\,dy = \frac{1}{2}e^{y^2}\Big|_0^1 = \boxed{\frac{1}{2}(e-1)}$$

- Notice that if we had tried to integrate with respect to y first we would have been stuck since $\int e^{y^2}dy$ is <u>not</u> an elementary integral. Sometimes reversing the order of integration makes the problem easier.

§12.3 #12 Let $D = \{(x,y) \mid \text{bounded by } y=\sqrt{x} \text{ and } y = x^2\}$



points of intersection have

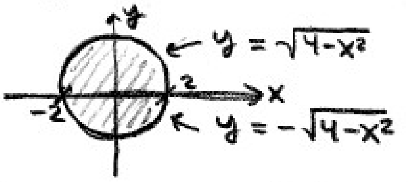
$$\sqrt{x} = x^2$$
$$x = x^4$$
$$x(x^3-1) = 0 \Rightarrow \underline{x=0 \text{ or } x=1}$$

Thus the region $D = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$. Now we can do the integration, we must do it in the order below

unless we do some extra work to convert to the complementary inequalities
$0 \le y \le 1$
$y^2 \le x \le \sqrt{y}$

$$\iint_D (x+y)\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx$$
$$= \int_0^1 \left(xy + \tfrac{1}{2}y^2\right|_{x^2}^{\sqrt{x}} dx$$
$$= \int_0^1 \left[\left(x^{3/2} + \tfrac{1}{2}x\right) - \left(x^3 + \tfrac{1}{2}x^4\right)\right] dx$$
$$= \left(\tfrac{2}{5}x^{5/2} + \tfrac{1}{4}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{10}x^5\right|_0^1$$
$$= \tfrac{2}{5} + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{4}} - \tfrac{1}{10} = \tfrac{4}{10} - \tfrac{1}{10} = \boxed{\tfrac{3}{10}}$$

§12.3 #15 $D = \{(x,y) \mid x^2+y^2 \le 4\}$



$$-2 \le x \le 2$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$

Now we can integrate,

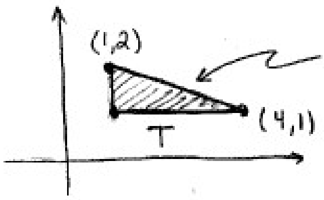
$$\iint_D (2x-y)\,dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\,dy\,dx$$
$$= \int_{-2}^{2} \left[2xy - \tfrac{1}{2}y^2\right|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$
$$= \int_{-2}^{2} \left[4x\sqrt{4-x^2} - \cancel{\tfrac{1}{2}(4-x^2)} + \cancel{\tfrac{1}{2}(4-x^2)}\right] dx$$

could have argued this is zero since $f(-x) = -f(x)$ $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$ (odd fnct. trick)

$$= \frac{-4}{3}(4-x^2)^{3/2}\Big|_{-2}^{2}$$
$$= -\tfrac{4}{3}(0-0) = \boxed{0}$$

H67

§12.3#19 find volume under $z = xy$ and above the triangle with vertices $(1,1)$, $(4,1)$ and $(1,2)$.



$y = 2 + \frac{-1}{3}(x-1)$ (used two-point formula. You can check that $(1,2)$ & $(4,1)$ lie on the line)

$y = -\frac{1}{3}x + \frac{7}{3} \longrightarrow x = -3y + 7$ for later

The triangle $T = \{(x,y) \mid 1 \le x \le 4 \,,\ 1 \le y \le \frac{1}{3}(7-x)\}$

$$V = \int_1^4 \int_1^{\frac{1}{3}(7-x)} xy \, dy \, dx$$

$$= \int_1^4 \left(\tfrac{1}{2} x y^2 \Big|_1^{\frac{1}{3}(7-x)} \right) dx$$

$$= \int_1^4 \left[\tfrac{1}{2} x \left(\tfrac{1}{3}(7-x)\right)^2 - \tfrac{1}{2} x \right] dx$$

this is somewhat messy, I'll let you finish it. Instead I'll go another way. Notice that we can also write T as

$$T = \{(x,y) \mid 1 \le y \le 2 \,,\ 1 \le x \le 7 - 3y\}$$

then we can integrate x first then y.

$$V = \int_1^2 \int_1^{7-3y} xy \, dx \, dy$$

$$= \int_1^2 \tfrac{1}{2} y x^2 \Big|_1^{7-3y} dy$$

$$= \int_1^2 \tfrac{1}{2} y \left((7-3y)^2 - 1\right) dy$$

$$= \int_1^2 \tfrac{1}{2} y \left(48 - 42y + 9y^2\right) dy$$

$$= \tfrac{1}{2} \int_1^2 \left(48y - 42y^2 + 9y^3\right) dy$$

$$= \tfrac{1}{2} \left[24y^2 - \tfrac{42}{3} y^3 + \tfrac{9}{4} y^4 \Big|_1^2 \right.$$

$$= \tfrac{1}{2} \left[24(4-1) - 14(8-1) + \tfrac{9}{4}(16-1) \right]$$

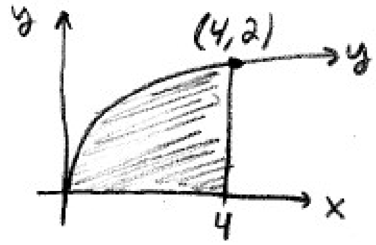
$$= \tfrac{1}{2} \left[72 - 98 + \tfrac{135}{4} \right] = \tfrac{1}{2}\left[-26 + \tfrac{135}{4} \right] = \tfrac{1}{2} \left[\tfrac{-104 + 135}{4} \right] = \boxed{\tfrac{31}{8}}$$

- Ok, its messy anyway you slice it.

§12.3 # 33 Consider the following integral,

$$\int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$$

this indicates the integral is over $0 \le y \le \sqrt{x}$ and $0 \le x \le 4$.



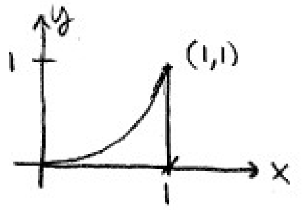
$y = \sqrt{x} \rightarrow x = y^2$

Equivalently we could say $y^2 \le x \le 4$ and $0 \le y \le 2$

$$\int_0^2 \int_{y^2}^4 f(x,y)\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$$

for these problems you just have to draw the picture and sort it out.

§12.3 # 40 Notice that $0 \le y \le 1$, $\sqrt{y} \le x \le 1$ is the graph below



$\sqrt{y} = x$

$y = x^2 \longrightarrow$ $0 \le x \le 1$

$0 \le y \le x^2$

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx$$

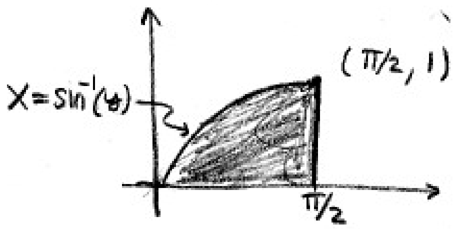
$$= \int_0^1 y\sqrt{x^3+1}\Big|_0^{x^2} dx$$

$$= \int_0^1 x^2\sqrt{x^3+1}\,dx$$

$$= \frac{2}{9}(x^3+1)^{3/2}\Big|_0^1$$

$$= \frac{2}{9}\left[(2)^{3/2} - 1\right] = \boxed{\frac{2}{9}\left[2\sqrt{2} - 1\right]}$$

§12.3#43 Consider the region $0 \le y \le 1$ and $\sin^{-1}(y) \le x \le \pi/2$



$\sin^{-1}(y) = x$

$\Rightarrow y = \sin(x)$

$\Rightarrow \boxed{\begin{cases} 0 \le x \le \pi/2 \\ 0 \le y \le \sin(x) \end{cases}}$

Then

$$\int_0^1 \int_{\sin^{-1}(y)}^{\pi/2} \cos(x)\sqrt{1+\cos^2 x}\, dx\, dy = \int_0^{\pi/2}\int_0^{\sin(x)} \cos(x)\sqrt{1+\cos^2 x}\, dy\, dx =$$

$$= \int_0^{\pi/2} \sin(x)\cos(x)\sqrt{1+\cos^2 x}\, dx$$

$\boxed{\begin{aligned} u &= 1+\cos^2 x \\ du &= -2\cos x \sin x\, dx \end{aligned}}$

$$= \frac{-1}{2}\,\frac{2}{3}\left(1+\cos^2 x\right)^{3/2}\Big|_0^{\pi/2}$$

$$= \frac{-1}{3}\left(1 - 2^{3/2}\right)$$

$$= \boxed{\frac{1}{3}\left(2\sqrt{2}-1\right)}$$

§12.7#2 evaluate the integral three different ways. The region of integration is $E = \{(x,y,z) \mid -1 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 1\}$

$$\iiint_E (xz - y^3)\,dV = \int_{-1}^1\int_0^2\int_0^1 (xz - y^3)\,dz\,dy\,dx$$

$$= \int_{-1}^1 \int_0^2 \left(\tfrac{1}{2}x - y^3\right) dy\, dx$$

$$= \int_{-1}^1 \left(x - \tfrac{1}{4}(16)\right)dx = -4(2) = \boxed{-8}$$

$$\iiint_E (xz - y^3)\,dV = \int_0^1\int_0^2\int_{-1}^1 (\overset{0}{\cancel{xz}} - y^3)\,dx\,dy\,dz$$

here xz is an odd-function integrated over symmetric interval about zero $\Rightarrow$ vanishes.

$$= \int_0^1\int_0^2 \left(-xy^3\right)\Big|_{-1}^1\, dy\, dz$$

$$= \int_0^1\int_0^2 -2y^3\, dy\, dz$$

$$= \int_0^1 \frac{-1}{4}(2)^4\, dz = -8z\Big|_0^1 = \boxed{-8}$$

There are four other ways to iterate the integral. Each will yield -8. This is Fubini's Th$^{\text{m}}$ for $\iiint$ in action.

§12.7 #5

$$\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} z e^y \, dx\, dz\, dy = \int_0^3\int_0^1 z e^y x\Big|_0^{\sqrt{1-z^2}} dz\, dy$$

$$= \int_0^3\int_0^1 z\sqrt{1-z^2}\, e^y \, dz\, dy$$

$$= \int_0^3 \left(\frac{-1}{3}(1-z^2)^{3/2} e^y \Big|_0^1 \right) dy$$

$$= \int_0^3 \frac{1}{3} e^y \, dy = \boxed{\frac{1}{3}(e^3-1)}$$

§12.7 #6

$$\int_0^1\int_0^z\int_0^y z e^{-y^2} dx\, dy\, dz = \int_0^1\int_0^z \left(z e^{-y^2} x\Big|_0^y \right) dy\, dz$$

$$= \int_0^1\int_0^z z e^{-y^2} y \, dy\, dz$$

$$= \int_0^1 \left(\frac{-1}{2} z e^{-y^2} \Big|_{0=y}^{z=y} \right) dz$$

$$= \int_0^1 \left(-\frac{1}{2} z e^{-z^2} + \frac{1}{2} z \right) dz$$

$$= \left(\frac{1}{4} e^{-z^2} + \frac{1}{4} z^2 \right|_0^1$$

$$= \frac{1}{4}\left(e^{-1} + 1 - 1\right) = \boxed{\frac{1}{4e}}$$

§12.7 #8

$$E = \{ (x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x,\ x \le z \le 2x \}$$

We must integrate dz then dy then dx. That is the natural order here.

$$\iiint_E yz\cos(x^5)\, dV = \int_0^1\int_0^x\int_x^{2x} yz\cos(x^5)\, dz\, dy\, dx$$

$$= \int_0^1 \cos(x^5) \left(\int_0^x\int_x^{2x} yz\, dz\, dy \right) dx$$

$$= \int_0^1 \cos(x^5) \left(\int_0^x \frac{1}{2} y\left[(2x)^2 - x^2\right] dy \right) dx$$

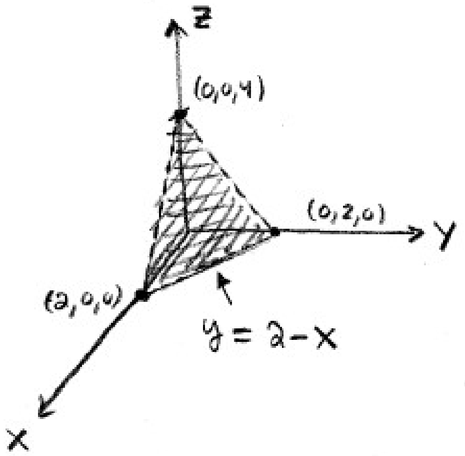
$$= \int_0^1 \frac{3}{2} x^2 \cos(x^5) \left(\int_0^x y\, dy \right) dx$$

$$= \int_0^1 \frac{3}{4} x^4 \cos(x^5)\, dx = \frac{3}{20}\sin(x^5)\Big|_0^1 = \boxed{\frac{3}{20}\sin(1)}$$

§12.7#10 Let $E \subset \mathbb{R}^3$ bounded by $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$
Notice that this plane passes through the first octant.

(xz)-plane $(y=0)$: $2x+z=4$ $\therefore$ $z=4-2x$
(xy)-plane $(z=0)$: $2x+2y=4$ $\therefore$ $y=2-x$
(yz)-plane $(x=0)$: $2y+z=4$ $\therefore$ $z=4-2y$

these details are not strictly speaking necessary but sometime it helps to get some additional details to help insure graph is correct.



So we can describe the region of integration as

$$0 \le z \le 4-2x-2y$$

but what about x & y? Note on (xy)-plane we have

$$0 \le y \le 2-x$$

And finally

$$0 \le x \le 2$$

Now integrate,

$$\iiint_E y\,dV = \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} y\,dz\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} y(4-2x-2y)\,dy\,dx$$

$$= \int_0^2 \left[y^2(2-x) - \tfrac{2}{3}y^3 \right|_0^{2-x} dx$$

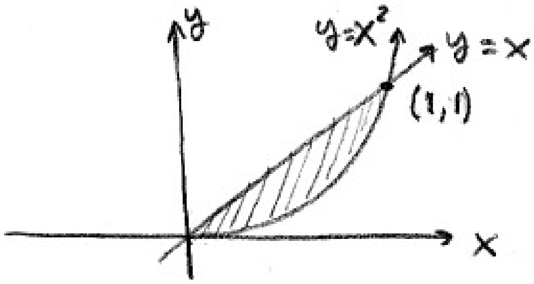
$$= \int_0^2 \left[(2-x)^3 - \tfrac{2}{3}(2-x)^3 \right] dx$$

$$= \int_0^2 \tfrac{1}{3}(2-x)^3\,dx$$

$$= \frac{-1}{12}(2-x)^4 \Big|_0^2$$

$$= \frac{-1}{12}(0-16) = \frac{16}{12} = \boxed{4/3}$$

§12.7 #14 $E \subset \mathbb{R}^3$ bounded by the parabolic cylinder $y = x^2$ and the planes $x = z$, $x = y$ and $z = 0$. We can bound z to begin, $0 \le z \le x$. Then a 2-dim'l picture will do,

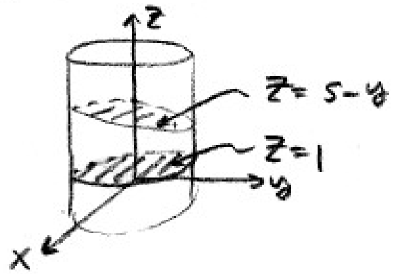


$$x^2 \le y \le x$$
$$0 \le x \le 1$$

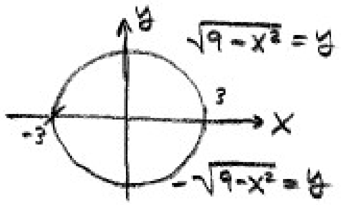
Now integrate

$$\iiint_E (x+2y)\,dV = \int_0^1 \int_{x^2}^{x} \int_0^{x} (x+2y)\,dz\,dy\,dx$$
$$= \int_0^1 \int_{x^2}^{x} (x^2 + 2yx)\,dy\,dx$$
$$= \int_0^1 \left(x^2 y + x y^2 \Big|_{x^2}^{x} dx\right.$$
$$= \int_0^1 \left[x^3 + x^3 - x^4 - x^5\right] dx$$
$$= \left(\tfrac{2}{4}x^4 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6\Big|_0^1\right. = \tfrac{1}{2} - \tfrac{1}{5} - \tfrac{1}{6} = \tfrac{15-6-5}{30} = \boxed{\tfrac{2}{15}}$$

§12.7 #19 Find volume enclosed by $x^2 + y^2 = 9$ and $y + z = 5$ and $z = 1$
$z = 5 - y$



$$1 \le z \le 5 - y$$
$$-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$$
$$-3 \le x \le 3$$



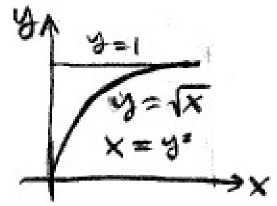
$$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} dz\,dy\,dx =$$
$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - \cancel{y})\,dy\,dx \quad \text{0 (odd fnct about even interval)}$$
$$= \int_{-3}^{3} 8\sqrt{9-x^2}\,dx \qquad \boxed{\begin{array}{l} x = 3\sin\theta \\ 9 - x^2 = 9\cos^2\theta \\ dx = 3\cos\theta\,d\theta \end{array}}$$
$$= \int_{-\pi/2}^{\pi/2} 24 \cdot 3\cos^2\theta\,d\theta$$
$$= 72 \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta = \boxed{36\pi}$$

§12.7 #31 Consider $\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$. Find five other orders of iterating this integral

$$\begin{aligned} 0 &\le z \le 1-y \\ \sqrt{x} &\le y \le 1 \\ 0 &\le x \le 1 \end{aligned}$$

OR

$$\begin{aligned} 0 &\le z \le 1-y \\ 0 &\le x \le y^2 \\ 0 &\le y \le 1 \end{aligned}$$

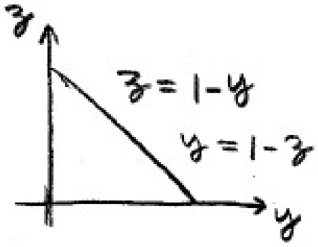


OR

$$\begin{aligned} 0 &\le x \le y^2 \\ 0 &\le y \le 1-z \\ 0 &\le z \le 1 \end{aligned}$$

OR

$$\begin{aligned} 0 &\le x \le y^2 \\ 0 &\le z \le 1-y \\ 0 &\le y \le 1 \end{aligned}$$

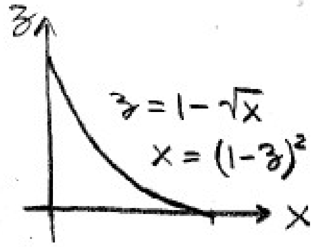


OR

$$\begin{aligned} \sqrt{x} &\le y \le 1-z \\ 0 &\le z \le 1-\sqrt{x} \\ 0 &\le x \le 1 \end{aligned}$$

OR

$$\begin{aligned} \sqrt{x} &\le y \le 1-z \\ 0 &\le x \le (1-z)^2 \\ 0 &\le z \le 1 \end{aligned}$$



these are equivalent views of the volume being integrated over. Thus

volume E's projections onto coordinates planes

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f\,dz\,dy\,dx = \int_0^1\int_0^{y^2}\int_0^{1-y} f\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{1-z}\int_0^{y^2} f\,dx\,dy\,dz = \int_0^1\int_0^{1-y}\int_0^{y^2} f\,dx\,dz\,dy$$

$$= \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f\,dy\,dz\,dx = \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f\,dy\,dx\,dz$$

$\boxed{\S 10.5\ \#13}$ Consider $r(u,v) = \langle u\cos v,\ u\sin v,\ v\rangle$ try to match with graph on p. 733 of 3^{rd} ed. of Stewart.

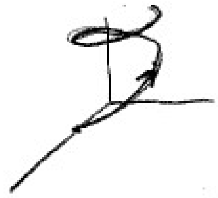
(1.) fix v and let u vary,

$$r(u, v_0) = \langle u\cos v_0,\ u\sin v_0,\ v_0\rangle \equiv \alpha(u)$$

this is a line in the $z = v_0$ plane in the $\langle \cos v_0, \sin v_0, 0\rangle$ direction

(2.) fix $u = u_0$ let v vary,

$$r(u_0, v) = \langle u_0\cos v,\ u_0 \sin v,\ v\rangle \equiv \beta(v)$$

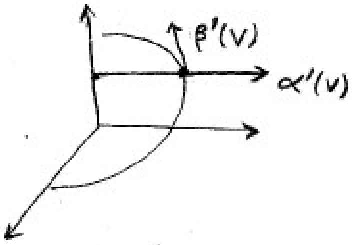


this is a helix of radius u_0 and slope one

Notice then that these u, v coordinate curves are perpendicular. This can be seen by checking their tangents,

$$\alpha'(u) = \langle \cos v_0,\ \sin v_0,\ 0\rangle$$

$$\beta'(v) = \langle -u_0\sin v,\ u_0\cos v,\ 1\rangle$$

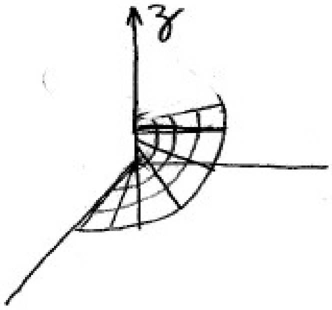


we want a point of intersection

$$\alpha(\tilde{u}) = \beta(\tilde{v})$$

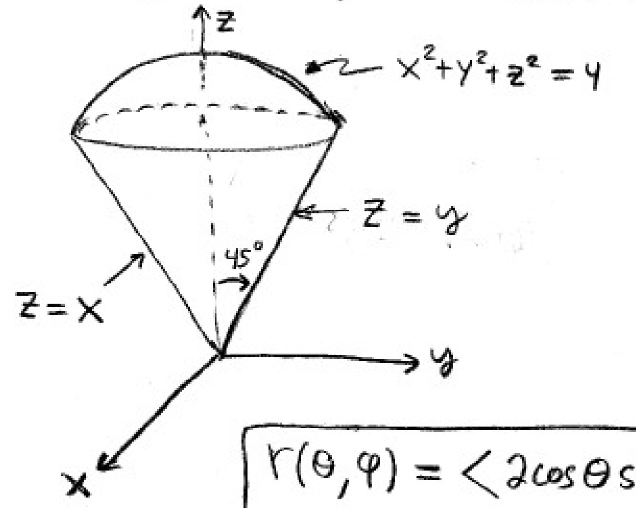
$$\langle \tilde{u}\cos v_0,\ \tilde{u}\sin v_0,\ v_0\rangle = \langle u_0\cos\tilde{v},\ u_0\sin\tilde{v},\ \tilde{v}\rangle$$

obviously we should choose $\hat{u} = u_0$ and $\hat{v} = v_0$ so $\alpha(u_0) = \beta(v_0) = r(u_0, v_0)$,
Note then $\alpha'(u_0)\cdot\beta'(v_0) = 0$ $\therefore$ the coordinate curves are $\perp$.



- the text's picture I.) is a better rendition.

§10.5 #21 Find a parametrization of $x^2+y^2+z^2=4$ above $z=\sqrt{x^2+y^2}$



Use spherical coordinates.

$$x = 2\cos\theta \sin\varphi$$
$$y = 2\sin\theta \sin\varphi$$
$$z = 2\cos\varphi$$

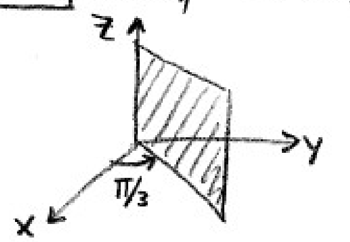
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \pi/4$$

$$\boxed{r(\theta,\varphi) = \langle 2\cos\theta\sin\varphi,\ 2\sin\theta\sin\varphi,\ 2\cos\varphi\rangle \quad \begin{array}{l} 0\le\theta\le 2\pi \\ 0\le\varphi\le\pi/4\end{array}}$$

• there are many other equally correct answers, parametrizations aren't unique.

§9.7 #12 The eq^n $\rho = 3$ is all $(x,y,z)\in\mathbb{R}^3$ such that
$\rho = \sqrt{x^2+y^2+z^2} = 3 \Rightarrow x^2+y^2+z^2=9$, this is a sphere of radius 3 centered at the origin.

§9.7 #14 The eq^n $\theta = \pi/3$ is a half-plane. It has Cartesian eq^{n}'s



$$\tan\theta = y/x \quad \therefore \tan(\pi/3) = y/x$$
$$\therefore \boxed{\sqrt{3}x - y = 0} \qquad x, y > 0$$

plane with normal $\langle\sqrt{3}, -1, 0\rangle$.

§9.7 #16 $\rho\sin\phi = 2$

$$\rho^2\sin^2\phi = \rho^2(1-\cos^2\phi) = 4 = \rho^2 - (\rho\cos\phi)^2$$
$$\Rightarrow 4 = x^2+y^2+z^2 - z^2 \Rightarrow \boxed{4 = x^2+y^2}$$

this is a cylinder with radius 2 and axis the z-axis.

Alternatively,

$$\begin{array}{l} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \end{array} \Rightarrow \begin{array}{l} x = 2\cos\theta \\ y = 2\sin\theta \end{array} \Rightarrow x^2+y^2 = 4(\cos^2\theta+\sin^2\theta) = 4.$$

• you may have found a completely different argument altogether.

$\boxed{\S 9.7 \# 19}$ $r^2+z^2 = x^2+y^2+z^2 = \rho^2 = 25$. This is a sphere centered at the origin with radius 5.

$\boxed{\S 9.7 \# 21}$ Write $z = x^2+y^2$ in cylindricals and sphericals.

$$z = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi\,(\cos^2\theta + \sin^2\theta) = (\rho \sin\phi)^2$$

and $z = \rho \cos\phi$ $\therefore$ $\boxed{\cos\phi = \rho \sin^2\phi}$. In cylindricals,

$$x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = \boxed{r^2 = z}$$

Remark: beware the other conventions. I usually use (r, ϕ, θ) in place of (ρ, θ, ϕ) and I like to use $s = \sqrt{x^2+y^2}$ and ϕ for polar coordinates. These conventions are common in physics for example. One excellent presentation of those conventions and vector calculus as it applies to Electrodynamics is Griffith's Intro to E&M text. I will behave in this course and use the inferior math conventions. (mostly)

$\boxed{\S 12.9 \# 1}$ Find the Jacobean of the transformation

$x = u + 4v$ and $y = 3u - 2v$

By definition,

$$\frac{\partial(x,y)}{\partial(u,v)} \equiv \det\begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix} = \det\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = -2-12 = \boxed{-14}$$

$\boxed{\S 12.9 \# 4}$ Let $x = \alpha \sin\beta$ and $y = \alpha\cos\beta$ find the Jacobean of $(x,y) \longmapsto (\alpha,\beta)$.

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} \equiv \det\begin{bmatrix} \partial x/\partial\alpha & \partial x/\partial\beta \\ \partial y/\partial\alpha & \partial y/\partial\beta \end{bmatrix} = \det\begin{bmatrix} \sin\beta & \alpha\cos\beta \\ \cos\beta & -\alpha\sin\beta \end{bmatrix} = -\alpha\sin^2\beta - \alpha\cos^2\beta$$

$$\therefore \boxed{\frac{\partial(x,y)}{\partial(\alpha,\beta)} = -\alpha}$$

$\boxed{\S 12.9 \# 6}$ Let $x = e^{u-v}$, $y = e^{u+v}$, $z = e^{u+v+w}$ find Jacobean, (H77)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv \det\begin{bmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{bmatrix}$$

$$= \det\begin{bmatrix} e^{u-v} & -e^{u-v} & 0 \\ e^{u+v} & e^{u+v} & 0 \\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w} \end{bmatrix}$$

$$= e^{u+v+w}\left(e^{u-v}e^{u+v} + e^{u+v}e^{u-v}\right)$$

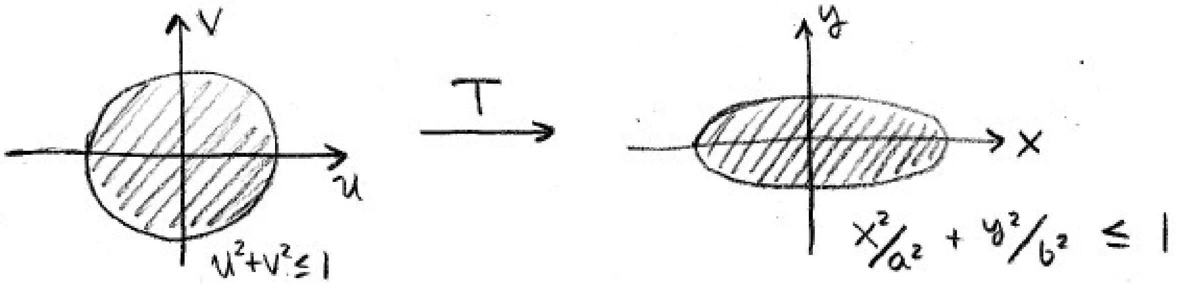
$$= e^{u+v+w}\left(e^{2u} + e^{2u}\right)$$

$$= \boxed{2e^{3u+v+w}}$$

$\boxed{\S 12.9 \# 10}$ $S = \{(u,v) \mid u^2+v^2 \leq 1\}$ and $x = au$, $y = bv$

notice $x^2/a^2 + y^2/b^2 = a^2u^2/a^2 + b^2v^2/b^2 = u^2+v^2 \leq 1$

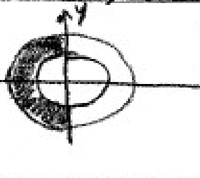
thus if we define $T(u,v) = (au, bv)$ we find



the transformation T deforms a disk to an oval.

Remark: I have changed the general ordering of the sections in Stewart. My hope was to make the treatment more logically ordered. For example, since we have already completed the Jacobeans we know how to change coordinates in a double, triple, etc... integration. In these sol$^{\underline{n}}$'s I will assume a few results from our lecture. I may expect you to prove those results on the test / final. (Don't worry I'll tell you what if any "proofs" are on our tests.)

$\boxed{\S 12.4 \# 10}$ Evaluate the following integral in polar coordinates,



$1 \leq r \leq 2$

$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

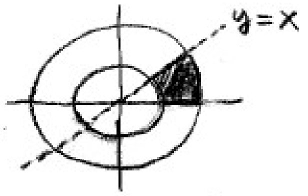
$$\iint_R (x+y)\,dA = \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)\,d\theta \int_1^2 r^2\,dr$$

$$= \Big[\sin\theta - \overset{0}{\cancel{\cos\theta}}\Big|_{\pi/2}^{3\pi/2} \frac{1}{3}r^3\Big|_1^2$$

$$= (-1-1)\left(\tfrac{1}{3}(8-1)\right) = \boxed{-14/3}$$

$\boxed{\S 12.4 \# 13}$ Let $R = \{(x,y) \mid 1 \leq x^2+y^2 \leq 4\ ,\ 0 \leq y \leq x\}$



$$R = \{(r,\theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \tfrac{\pi}{4}\}$$

$$\tan^{-1}(y/x) = \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta.$$

$$\iint_R \tan^{-1}(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \theta\, r\,dr\,d\theta$$

$$= \frac{1}{2}\theta^2\Big|_0^{\pi/4}\ \frac{1}{2}r^2\Big|_1^2$$

$$= \frac{1}{4}\left(\frac{\pi^2}{16}\right)(4-1)$$

$$= \boxed{\frac{3\pi^2}{64}}$$

notice that since the integrand can be factored into a purely θ and purely r dependendent portion **AND** the bounds are constants we can just multiply the r-part times the θ-part.

$\boxed{\S 12.4 \# 16}$ Find volume bounded by $z = 18 - 2x^2 - 2y^2$ and $z = 0$. The double integral will yield the volume, first convert to polars.

$$z = 18 - 2r^2\cos^2\theta - 2r^2\sin^2\theta = 18 - 2r^2$$

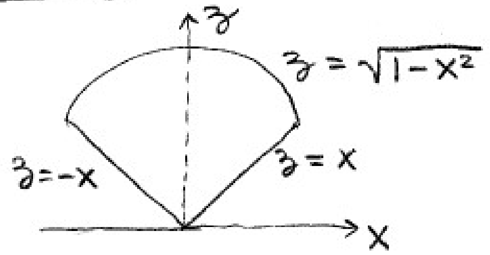
the surface intersects $z = 0$ to give bounds on x, y or better yet r, θ.

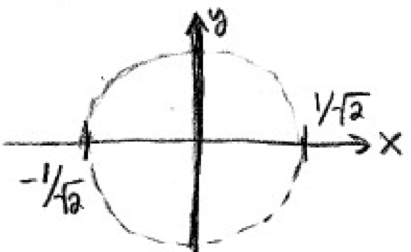
$$z = 0 = 18 - 2r^2 \Rightarrow 9 = r^2 \Rightarrow \boxed{r = 3}$$

the surface intersects $z = 0$ in a circle $r = 3$, $R = \left\{ (r,\theta) \,\middle|\, \begin{array}{l} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{array} \right\}$

$$
\begin{aligned}
V = \iint_R z\, dA &= \int_0^{2\pi}\int_0^3 (18 - 2r^2)\, r\, dr\, d\theta \\
&= (2\pi)\left(9r^2 - \tfrac{2}{4} r^4 \right|_0^3 \\
&= (2\pi)\left[81 - \tfrac{1}{2}(81)\right] \\
&= \boxed{81\pi}
\end{aligned}
$$

$\boxed{\S 12.4 \# 19}$ Find volume bounded by $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 1$





points of intersection have (on $y = 0$ slice)

$$\sqrt{1 - x^2} = x$$
$$1 - x^2 = x^2$$
$$1 = 2x^2 \therefore x = \pm 1/\sqrt{2}$$

the volume is obtained by rotating our picture about the z-axis. I give the top-view of the surface notice $0 \le r \le 1/\sqrt{2}$ & $0 \le \theta \le 2\pi$.

Since our shape goes between the cone and sphere we cannot just integrate z. Instead think a little.

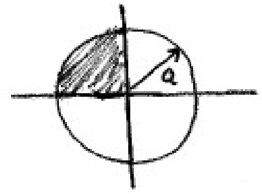
$$
\begin{aligned}
dV &= (z_{Top} - z_{Bottom})\, dA \quad \leftarrow \text{(typical infinitesimal volume.)} \\
&= \left(\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dx\, dy \\
&= \left(\sqrt{1 - r^2} - r\right) r\, dr\, d\theta
\end{aligned}
$$

$$V = \int dV = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(r\sqrt{1 - r^2} - r^2\right) dr\, d\theta = 2\pi\left(-\tfrac{1}{3}(1 - r^2)^{3/2} - \tfrac{1}{3} r^3 \right|_0^{\sqrt{2}/2}$$

$$= -\frac{2\pi}{3}\left(\left(\frac{1}{2}\right)^{3/2} + \left(\frac{1}{\sqrt{2}}\right)^3 - 1\right) = \boxed{\frac{\pi}{3}\left(2 - \sqrt{2}\right)}$$

§12.4 #26

$$\int_0^a \int_{-\sqrt{a^2-y^2}}^{0} x^2 y \, dx\, dy \longrightarrow \begin{matrix} 0 \le y \le a \\ \underbrace{-\sqrt{a^2-y^2} \le x \le 0}_{\substack{x^2+y^2=a^2 \\ \text{(the left half)}}} \end{matrix}$$



$0 \le r \le a$

$\frac{\pi}{2} \le \theta \le \pi$

$$\| $$

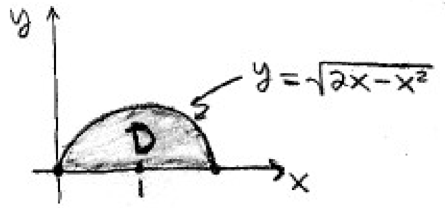
$$\int_{\frac{\pi}{2}}^{\pi} \int_0^a (r^2\cos^2\theta)(r\sin\theta)\, r\,dr\,d\theta = \int_{\pi/2}^{\pi} \cos^2\theta \sin\theta \, d\theta \int_0^a r^4\,dr$$

$$= \left(-\frac{1}{3}\cos^3\theta \Big|_{\pi/2}^{\pi}\right)\left(\frac{a^5}{5}\right)$$

$$= -\frac{1}{3}(-1)\left(a^5/5\right) = \boxed{a^5/15}$$

§12.4 #28

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx \qquad \begin{matrix} 0 \le y \le \sqrt{2x-x^2} = \sqrt{x(2-x)} \\ 0 \le x \le 2 \end{matrix}$$



$\Rightarrow y^2 = 2x - x^2$

$\Rightarrow x^2 - 2x + y^2 = 0$

$\Rightarrow (x-1)^2 + y^2 = 1$ (it's a circle of radius one at $(1,0)$)

It should be clear that D has $0 \le \theta \le \pi/2$.
It is also clear that the bound on r must depend on θ since we have differing radii for differing θ (for example $r=2$ $\theta=0$ while $r=0$ for $\theta = \pi/2$). We need to convert $y = \sqrt{2x-x^2}$ to a more useful form for us,

$$r^2 = x^2+y^2 = x^2 + 2x - x^2 = 2x = 2r\cos\theta \Rightarrow \underline{r = 2\cos\theta}$$

this checks with the limiting cases ($r = 0 = \cos(\pi/2)$ & $r = 2 = 2\cos(0)$)

$$\iint_D \sqrt{x^2+y^2}\, dA = \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}\cos^3\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}(1-\sin^2\theta)\cos\theta\, d\theta$$

$$= \frac{8}{3}\left(\sin\theta - \frac{1}{3}\sin^3\theta\Big|_0^{\pi/2}\right) = \frac{8}{3}\left(1-\frac{1}{3}\right) = \boxed{\frac{16}{9}}$$

H81

§12.4 #32 Let D_a be disk of radius a centered at origin, define

a.) $$I = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA = \lim_{a\to\infty} \iint_{D_a} e^{-x^2-y^2}\,dA$$

$$= \lim_{a\to\infty} \int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta$$

$$= \lim_{a\to\infty} \left[(2\pi)\frac{-1}{2}e^{-r^2}\Big|_0^a\right]$$

$$= \lim_{a\to\infty} \pi(-e^{-a^2}+1)$$

$$= \pi$$

b.) Let $S_a = [-a,a]\times[-a,a]$

$$I = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA = \lim_{a\to\infty}\iint_{S_a} e^{-x^2-y^2}\,dA$$

$$= \lim_{a\to\infty} \int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy$$

$$= \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}dx\right)\left(\int_{-a}^{a} e^{-y^2}dy\right)$$

$$= \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}dx\right)\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-y^2}dy\right)$$

$$= \left[\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}dx\right)\right]^2 = \pi$$

- I'll let you finish it. (there's not much left.)

§12.4 #33 Notice $\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$ follows from #32.

Assume that $\frac{d}{da}\int_{-\infty}^{\infty} e^{-ax^2}dx = \int_{-\infty}^{\infty}\frac{d}{da}(e^{-ax^2})\,dx$.

This will give you many integrals of the form

$$\int_{-\infty}^{\infty} z^n e^{-az^2}\,dz = \text{nice-formula}$$

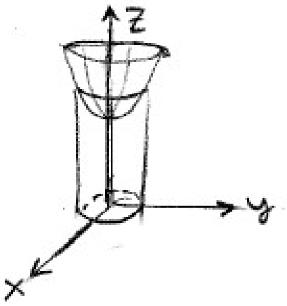
Otw we cannot always do this the details are technical

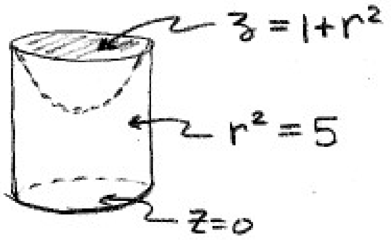
H82

§12.8 #6 Clearly $1 \le \rho \le 2$ and $0 \le \phi \le \pi/2$ and $\pi/2 \le \Theta \le 2\pi$ thus an integration over the pictured region would be,

$$\int_1^2 \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{2\pi} f(\rho, \phi, \Theta)\, d\Theta\, d\varphi\, d\rho$$

§12.8#9 $E \subseteq \mathbb{R}^3$ enclosed by $z = 1 + x^2 + y^2$, $x^2 + y^2 = 5$ and $z = 0$





$0 \le z \le 1 + r^2$
$0 \le r \le \sqrt{5}$
$0 \le \Theta \le 2\pi$

$$\iiint_E e^z\, dV = \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{1+r^2} e^z\, r\, dz\, dr\, d\Theta$$

$$= (2\pi) \int_0^{\sqrt{5}} r(e^{1+r^2} - e^0)\, dr$$

$$= 2\pi \left(\frac{1}{2} e^{1+r^2} \Big|_0^{\sqrt{5}} - \frac{1}{2} r^2 \Big|_0^{\sqrt{5}} \right)$$

$$= \boxed{\pi(e^6 - e - 5)}$$

§12.8#18 E is region with $0 \le \rho \le 1$; $0 \le \phi \le \pi/2$, $0 \le \Theta \le 2\pi$,

$$\iiint_E (x^2+y^2)\, dV = \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} (\rho^2 \sin^2\varphi \underline{\cos^2\Theta} + \rho^2 \sin^2\varphi \underline{\sin^2\Theta})\, \rho^2 \sin\varphi\, d\Theta\, d\varphi\, d\rho$$

$$= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \rho^4 \sin^3\varphi\, d\Theta\, d\varphi\, d\rho$$

$$= \int_0^1 \rho^4 d\rho \int_0^{2\pi} d\Theta \int_0^{\pi/2} (1 - \cos^2\varphi) \sin\varphi\, d\varphi$$

$$= \frac{1}{5} \cdot 2\pi \cdot \left(\frac{1}{3} \cos^3\varphi - \cos\varphi \Big|_0^{\pi/2} \right)$$

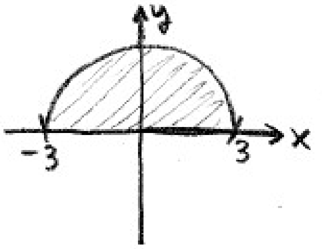
$$= \frac{2\pi}{5} \left(\frac{1}{3} \cancel{\cos^3\frac{\pi}{2}}^{\,0} - \cancel{\cos\frac{\pi}{2}}^{\,0} \underbrace{- \frac{1}{3} \cos^3(0) + \cos(0)}_{2/3} \right)$$

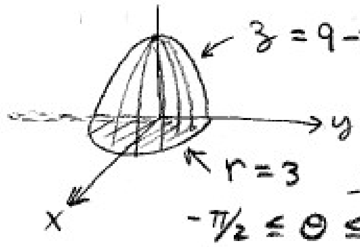
$$= \boxed{\frac{4\pi}{15}}$$

§12.8 #32

$$I = \int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\,dy\,dx \quad \Rightarrow \quad \begin{array}{l} -3 \le x \le 3 \\ 0 \le y \le \sqrt{9-x^2} \\ 0 \le z \le 9-x^2-y^2 \end{array}$$

$$z = 9 - x^2 - y^2 = 9 - r^2, \quad y^2 = 9 - x^2 \Rightarrow r^2 = 9$$





so $0 \le r \le 3$

$-\pi/2 \le \theta \le \pi/2$

$$I = \int_{-\pi/2}^{\pi/2}\int_{0}^{3}\int_{0}^{9-r^2} r^2\, dz\, dr\, d\theta \quad : \text{ using } dV = r\,dr\,d\theta\,dz \text{ and } \sqrt{x^2+y^2} = \sqrt{r^2} = r$$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{3} (9r^2 - r^4)\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(3r^3 - \frac{1}{5}r^5 \right|_0^3 d\theta$$

$$= \pi\left(81 - \frac{1}{5}(243)\right) = \left(\frac{405-243}{5}\right)\pi = \boxed{\frac{162\pi}{5}}$$

§12.8 #36 Suppressing the limit notation,

$$\iiint_{\mathbb{R}^3} \rho e^{-\rho^2}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho^3 e^{-\rho^2} \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= (2\pi)(-\cos\pi + \cos(0)) \int_0^{\infty} \rho^3 e^{-\rho^2}\, d\rho$$

Use integration by parts twice.

$$\int \underbrace{z^2}_{u}\, \underbrace{z e^{-z^2} dz}_{dv} = z^2\left(-\frac{1}{2}e^{-z^2}\right) + \int \frac{1}{2} e^{-z^2}\, 2z\,dz = -\frac{z^2}{2}e^{-z^2} + \int z e^{-z^2}\, dz$$

$$= -\frac{z^2}{2}e^{-z^2} - \frac{1}{2}e^{-z^2} + C$$

$$\therefore \int_0^{\infty} \rho^3 e^{-\rho^2} d\rho = -\frac{1}{2}\left(z^2 e^{-z^2} + e^{-z^2}\right|_0^{\infty} = \frac{1}{2} \quad \left(\text{using L'Hopital's on } z^2 e^{-z^2}\right)$$

$$\therefore \iiint_{\mathbb{R}^3} \rho e^{-\rho^2}\, dV = 4\pi/2 = \boxed{2\pi}$$