Multivariable Calculus

James S. Cook
Liberty University
Department of Mathematics

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preface

how to succeed in calculus

I do use the textbook, however, I follow these notes. You should use both. From past experience I can tell you that the students who excelled in my course were those students who both studied my notes and read the text. They also came to every class and paid attention. I recommend the following course of study:

1. submit yourself to learn, keep a positive attitude. This course is a lot of work. Yes, probably more than 3 others for most people. Most people have a lot of work to do in getting up to speed on real mathematical thinking. There is no substitute for time and effort. If you're complaining in your mind about the workload etc... then you're wasting your time.

2. come to class, take notes, **think**.

3. read these notes.

4. attempt Problem Sets, you will likely find forming a study group is essential for success here. I ask some very hard questions. The majority of the hwk grade comes from Problem Sets.

format of my notes

These notes were prepared with LaTeX. You'll notice a number of standard conventions in my notes:

1. definitions are usually in green.

2. remarks are in red.

3. theorems, propositions, lemmas and corollaries are in blue.

4. proofs start with a **Proof:** and are concluded with a $\square$.

5. often figures in these notes were prepared with **Graph**, a simple and free math graphing program, or **Maple**, or **Mathematica**. Or some online math tool, of which there are dozens.

By now the abbreviations below should be old news, but to be safe I replicate them here once more:

Notation	Meaning of Notation
§	section
$\exists$	there exists
$\nexists$	there does not exist
w.r.t.	with respect to
l.h.s.	left hand side
r.h.s.	right hand side
$x \in B$	the element x is inside the set B
$A \implies B$	A implies B
$A \iff B$	A and B are equivalent statements
$\therefore$	therefore
$\forall$	for all
$\equiv$	definition
$\approx$	approximately
eq$^{\underline{n}}$	equation
sol$^{\underline{n}}$	solution
$\mathbb{N}$	natural numbers; 1,2,3,...
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{Z}$	integers
$\mathbb{R}^2$	the Cartesian plane

Finally, please be warned these notes are a work in progress. I look forward to your input on how they can be improved, corrected and supplemented. This is my second set of notes for calculus III. I follow approximately the same LaTeXformat as I used for the Calculus I and II notes of 2010-2011. The old handwritten calculus III notes are not bad, but I want to improve the theoretical aspects of these notes. I have included most of the examples from the old notes and also a myriad of homework solutions in this version. There are far too many examples for lecture. My goal was to gather the examples in a convenient format for the student.

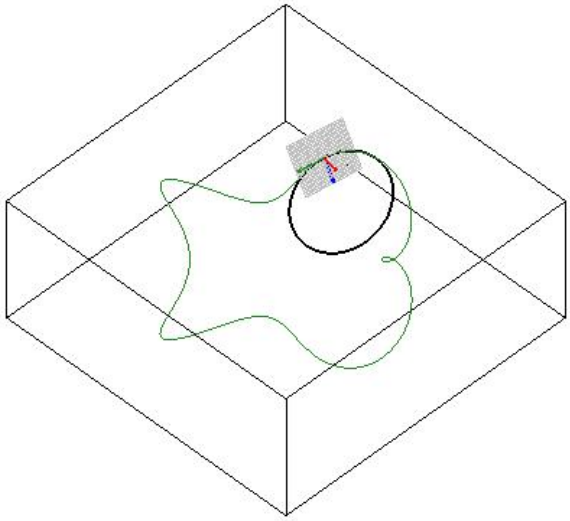
Please understand that the definitions given in these notes are primary. I recommend Stewart for additional examples, but not for definitions in this course. These notes take a different (and I hope clearer) path through the subject matter of multivariable calculus, I make no effort to be consistent with the text's development of concepts and I intend to add much detail on how calculus III works in curvelinear coordinates (not in Stewart). It is my intent these notes are a self-contained treatment of differential multivariate calculus. However, you should understand that I do assume you have a complete and working knowledge of Calculus I and II, basically Chapters

1-12 of Stewart, or take a look at my website for a breakdown of Math 131-132 as offered at Liberty University.

At this juncture there are about 285 pages. I would like to add another 50 pictures, but, I'll save them for class. These notes only cover the material for the first two tests. I will probably transition to the old notes after that point. However, I have more work to do on the last half of the course so I would kindly ask you refrain from printing the old notes on integration and vector calculus. It is highly likely that I am going to edit them and add discussion, proofs and totally new calculations. I will let you know when I have notes beyond those found here[1]

The old notes and much more can be found at my calculus III webpage. Every so often I mention the existence of an animation on my webpage. I am opening a zoo of gif-files where you can go see all sorts of mathematical creatures. If you create interesting creatures I will (with your permission) happily add them to my collection.

James Cook, August 21, 2011.
version 2.0



[1]of course, if you've got money to burn and/or you just want to offend Al Gore by wasteful printing then go for it, I always respect those who wish to make Al Gore cry.

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Chapter 1

analytic geometry

Euclidean space is named for the ancient mathematician Euclid. In Euclid's view geometry was a formal system with axioms and constructions. For example, the fact two parallel lines never interect is called the *parallel postulate*. If you take the course in modern geometry then you'll study more of the history of Euclid. Fortunately for us the axiomatic/constructive approach was replaced by a far easier view about 400 years ago. Rene Descartes made popular the idea of using numbers to label points in space. In this new Cartesian geometry the fact two parallel lines in a plane never intersect could be checked by some simple algebraic calculation. More than this, all sorts of curves, surfaces and even more fantastic objects became constructible in terms of a few simple algebraic equations. The idea of using numbers to label points and the resulting geometry governed by the analysis of those numbers is called **analytic geometry**. We study the basics of analytic geometry in this chapter.

In your previous mathematical studies you have already dealt with analytic geometry in the plane. Trigonometry provided a natural framework to decipher all sorts of interesting facts about triangles. Moreover, the study of trigonometric functions in turn has allowed us solutions to otherwise intractable integrals in calculus II. Trigonometric substitution is an example of where the geometry of triangles has allowed deeper analysis. It goes both ways. Geometry inspires analysis and analysis unravels geometry. These are two sides of something deeper.

In this course we need to tackel three dimensional problems. The proper notation which groups together concepts in the most efficient and clean manner is the vector notation. Historically, it was predated by the quaternionic analysis of Hamilton, but for about 120 year the vector notation has been the dominant framework for two and three dimensional analytic geometry[1]. In particular, the dot and cross products allow us to test for how parallel two lines are, or to project a line onto a plane, or even to calculate the direction which is perpendicular to a pair of given directions. Engineering and basic everyday physics all written in this vector langauge.

[1] general geometries are more naturally understood in the language of differential forms and manifolds, but this is where we all must begin

We also continue our study of functions in this chapter. We have studied functions $f : U \subseteq \mathbb{R} \to \mathbb{R}$ in the first two semesters of calculus. One goal in this course is to extend the analysis to **functions of many variables**. For example, $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x, y) = x^2 + y^2$. What can we say about this function? What calculus is known to analyze the properties of f? Before we begin to answer such questions in future chapters, we need to spend some time on the basic geometry of $\mathbb{R}^n$ and then especially $\mathbb{R}^3$. Naturally, we use $\mathbb{R}^3$ to model the three spatial dimensions which frame our everyday existence[2].

In this chapter we are concerned with understanding how to analytically describe points, curves and surfaces. We will examine what the solution set of $z = f(x, y)$ looks like for various f. Or, what is the solution set of $F(x, y) = k$, or $F(x, y, z) = k$? We learn how think about mappings $t \mapsto \langle x(t), y(t), z(t)\rangle$ or $(u, v) \mapsto \langle x(u, v), y(u, v), z(u, v)\rangle$. What is the geometry of such mappings? These are questions we hope to answer, at least in part, in this chapter.

1.1 vectors in euclidean space

We denote the real numbers as $\mathbb{R} = \mathbb{R}^1$. Naturally $\mathbb{R}$ is identified with a line as we are taught in our previous study of the number line. The *Cartesian products* of $\mathbb{R}$ with itself give us natural models for the plane, 3 dimensional space and more abstractly n-dimensional space:

Definition 1.1.1. .

1. **two-dimensional space:** is the set of all **ordered pairs** of real numbers:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

2. **three-dimensional space:** is the set of all **ordered triples** of real numbers:

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

3. **n-dimensional space:** is the set of all n**-tuples** of real numbers:

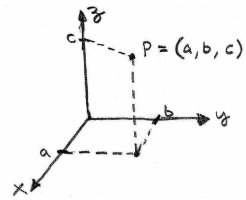
$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n\ copies} = \{(x_1, x_2, \ldots, x_n) \mid x_j \in \mathbb{R}\ for\ each\ j \in \mathbb{N}_n\}$$

The fact that the n-tuples above are ordered means that two n-tuples are equal iff each and every entry in the n-tuple matches.

[2] I would not say we live in $\mathbb{R}^3$, it's just a model, it's not reality. Respectable philosophers as recently as 200 years ago labored under the delusion that euclidean space must be reality since that was all they could imagine as reasonable

Definition 1.1.2. *vector equality, components.*

In particular, $(v_1, v_2, \ldots, v_n) = (w_1, w_2, \ldots, w_n)$ iff $v_1 = w_1,\ v_2 = w_2,\ \ldots, v_n = w_n$. In the context of $\mathbb{R}^2$ we say a is the **x-component** of (a, b) whereas b is the **y-component** of (a, b). In the context of $\mathbb{R}^3$ we say a is the **x-component** of (a, b, c) whereas b is the **y-component** of (a, b, c) and c is the **z-component** of (a, b, c). Generally, we say v_j the **j-th component** of $(v_1, v_2, \ldots, v_n)$.

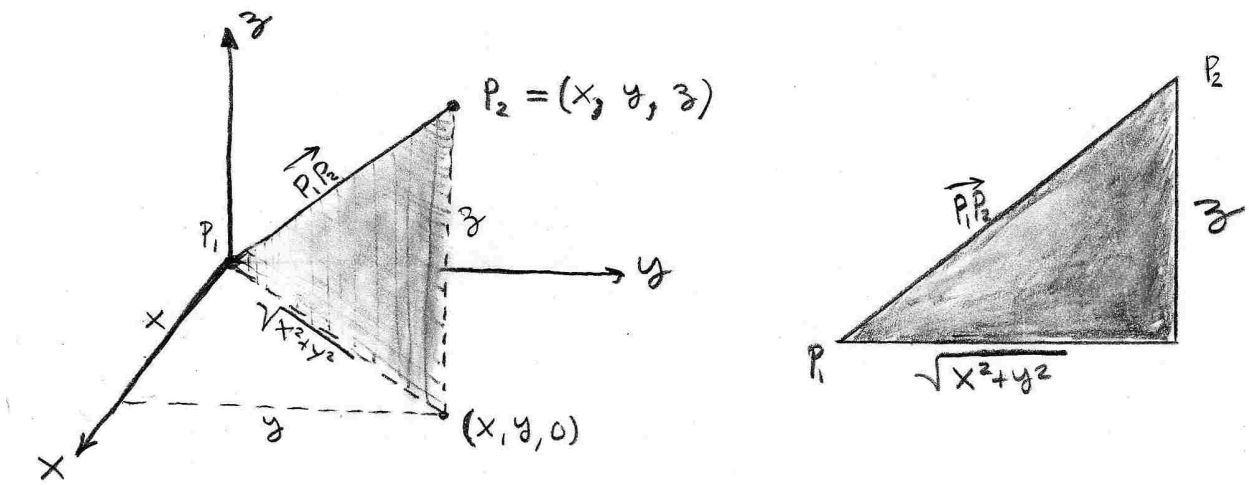


The sometimes the term **euclidean** is added to emphasize that we suppose distance between points is measured in the usual manner. Recall that in the one-dimensional case the distance between $x, y \in \mathbb{R}$ is given by the absolute value function; $d(x, y) = |y - x| = \sqrt{(y - x)^2}$. We define distance in n-dimensions by similar formulas:

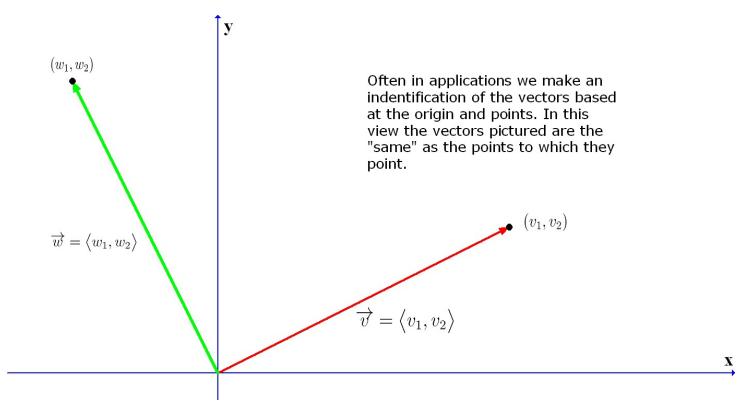
Definition 1.1.3. *euclidean distance.*

1. **distance in two-dimensional euclidean space:** if $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbb{R}^2$ then the distance between points p_1 and p_2 is
$$d(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
2. **distance in three-dimensional euclidean space:** if $p_1 = (x_1, y_1, z_1), p_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ then the distance between points p_1 and p_2 is
$$d(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
3. **distance in n-dimensional euclidean space:** if $a, b \in \mathbb{R}^n$ where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ then the distance between points a and point b is
$$d(a, b) = \sqrt{\sum_{j=1}^{n} (b_j - a_j)^2} = \sqrt{(b_1 - a_1)^2 + (b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}.$$

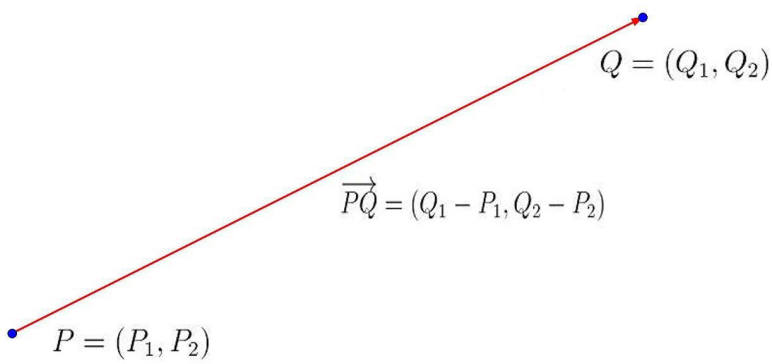
It is simple to verify that the definition above squares with our traditional ideas about distance from previous math courses. In particular, notice these follow from the Pythagorean theorem applied to appropriate triangles. The picture below shows the three dimensional distance formula is consistent with the two dimensional formula.



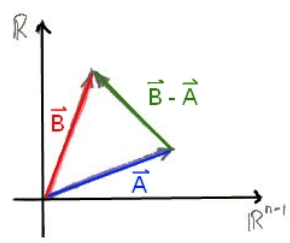
Notice that there is a natural correspondance between points and directed line-segments from the origin. We can view an n-tuple p as either representing the point $(p_1, p_2, \ldots, p_n)$ or the directed line-segment from the origin $(0, 0, \ldots, 0)$ to the point $(p_1, p_2, \ldots, p_n)$.



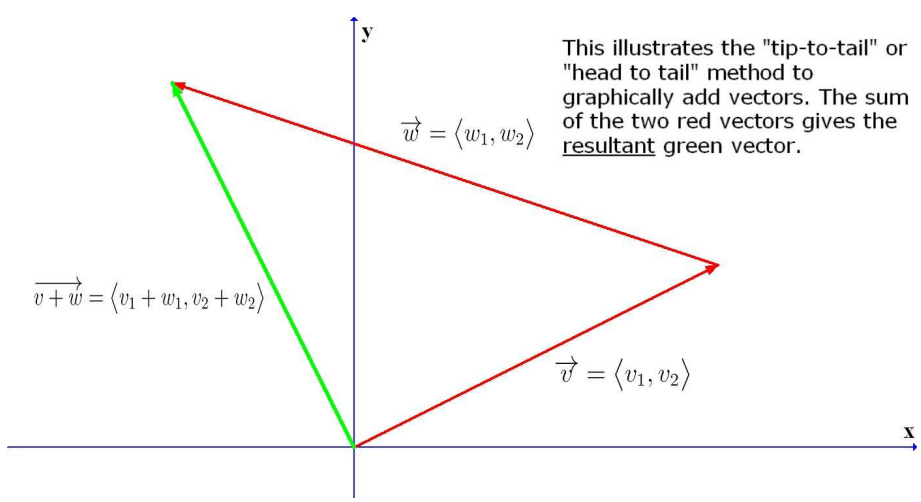
We will use the notation $\vec{p}$ for n-tuples throughout the remainder of these notes to emphasize the fact that $\vec{p}$ is a vector. Some texts use **bold** to denote vectors, but I prefer the over-arrow notation which is easily duplicated in hand-written work. The directed line-segment from point P to point Q is naturally identified with the vector $\vec{P} - \vec{Q}$ as illustrated below:



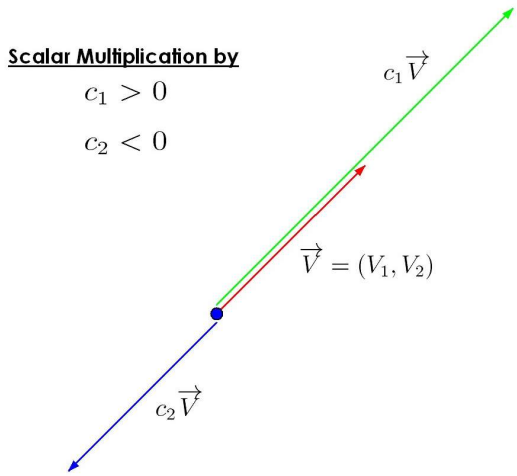
This is consistent with the identification of points and vectors based at the origin. See how the vector $\vec{a}$ and $\vec{b}$ are connected by the vector $\vec{b} - \vec{a}$



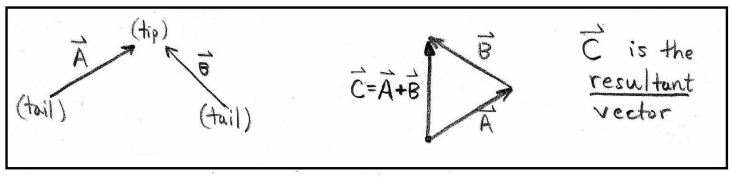
We add vectors geometrically by the tip-to-tail method as illustrated below.

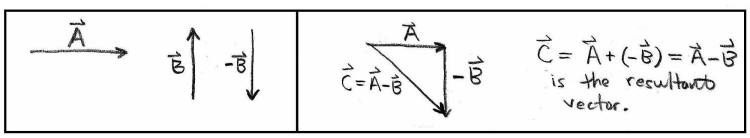


Also, we rescale them by shrinking or stretching their length by a scalar multiple:



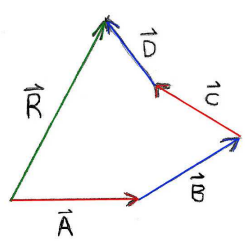
More pictures usually help:





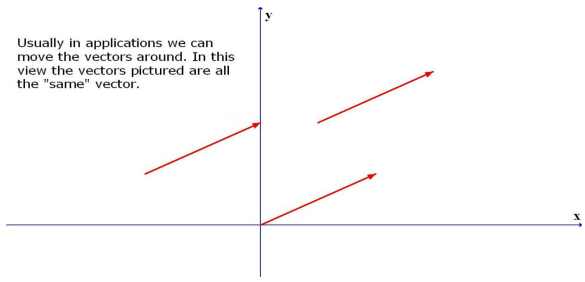
In the diagram below we illustrate the geometry behind the vector equation

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}.$$



Continuiing in this way we can add any finite number of vectors in the same *tip-2-tail* fashion.

One important point to notice here is that we can naturally move vectors from the origin to other points. Moreover, some people define vectors from the outset as directed-line-segments between points. In particular, the vector from A to B in $\mathbb{R}^n$ is denoted $\overrightarrow{AB}$ and is defined by $\overrightarrow{AB} = B - A$. It is natural to suppose the vector $\overrightarrow{AB}$ is based at A, however, we can equally well picture the vector $\overrightarrow{AB}$ based at any point in $\mathbb{R}^n$.



If we wish to keep track of the base point of vectors then additional notation is required. We could say that $(p, \vec{V})$ denotes a vector $\vec{V}$ which has basepoint p. Then addition, scalar multiplication, dot-products and vector lengths are all naturally defined for such objects. You just do those operations to the vector $\vec{V}$ and the point rides along. We will not use this notation in this course. Instead, we will use words or pictures to indicate where a given vector is based. Sometimes vectors are based at the origin, sometimes not. Sorry if this is confusing, but this is the custom of almost all authors and if I invent notation and am more careful on this point then I'm afraid I may put you at a disadvantage in other courses.

Algebraically, vector addition and scalar multiplication are easier to summarize concisely:

Definition 1.1.4. *vector addition and scalar multiplication.*

in $\mathbb{R}^2$,

$$\langle x_1,\ y_1\rangle + \langle x_2,\ y_2\rangle \ = \ \langle x_1 + x_2,\ y_1 + y_2\rangle$$

$$c\langle x_1,\ y_1\rangle \ = \ \langle cx_1,\ cy_1\rangle$$

Or for $\mathbb{R}^3$,

$$\langle x_1,\ y_1,\ z_1\rangle + \langle x_2,\ y_2,\ z_2\rangle \ = \ \langle x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2\rangle$$

$$c\langle x_1,\ y_1,\ z_1\rangle \ = \ \langle cx_1,\ cy_1,\ cz_1\rangle.$$

Generally, for $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we define $\vec{x} + \vec{y}$ and $c\vec{x}$ component-wise as follows:

$$(\ \vec{x} + \vec{y}\ )_j = \vec{x}_j + \vec{y}_j$$

$$(\ c\vec{x}\ )_j = c\vec{x}_j$$

for $j = 1, 2, \ldots, n$.

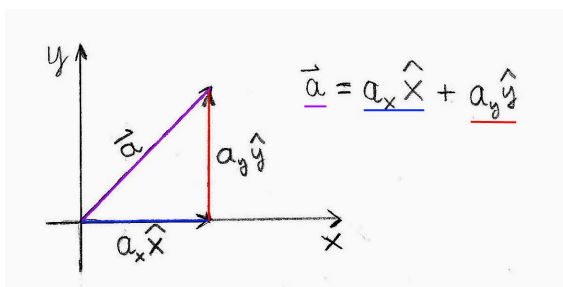
Given these definitions it is often convenient to break a vector down into its **vector components**. In particular, for $\mathbb{R}^2$, **define** $\widehat{x} = \langle 1, 0 \rangle$ **and** $\widehat{y} = \langle 0, 1 \rangle$ hence:

$$\begin{aligned} \langle a, b \rangle &= \langle a, 0 \rangle + \langle 0, b \rangle \\ &= a\langle 1, 0 \rangle + b\langle 0, 1 \rangle \\ &= a\,\widehat{x} + b\,\widehat{y} \end{aligned} \tag{1.1}$$

Definition 1.1.5. *vector and scalar components of two-vectors.*

The **vector component** of $\langle a, b \rangle$ in the x-direction[3] is simply $a\,\widehat{x}$ whereas the **vector component** of $\langle a, b \rangle$ in the y-direction is simply $b\,\widehat{y}$. In contrast, a, b are the **scalar components** of $\langle a, b \rangle$ in the x, y-directions respective.

Scalar components are scalars whereas vector components are vectors. These are entirely different objects if $n \neq 1$, please keep clear this distinction in your mind. Notice that the vector components are what we use to reproduce a given vector by the tip-to-tail sum:



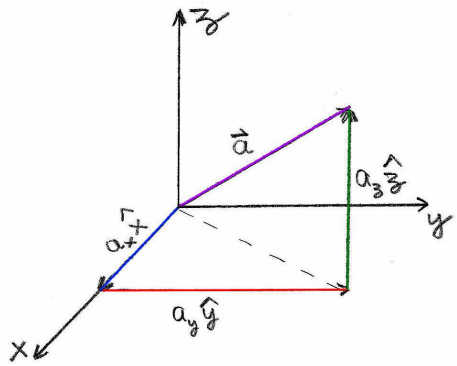
For $\mathbb{R}^3$ we define the following notation: $\widehat{x} = \langle 1, 0, 0 \rangle$, $\widehat{y} = \langle 0, 1, 0 \rangle$, **and** $\widehat{z} = \langle 0, 0, 1 \rangle$ hence:

$$\begin{aligned} \langle a, b, c \rangle &= \langle a, 0, 0 \rangle + \langle 0, b, 0 \rangle + \langle 0, 0, c \rangle \\ &= a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle + c\langle 0, 0, 1 \rangle \\ &= a\,\widehat{x} + b\,\widehat{y} + c\,\widehat{z} \end{aligned} \tag{1.2}$$

Definition 1.1.6. *vector and scalar components of three-vectors.*

The **vector components** of $\langle a, b, c \rangle$ are: $a\,\widehat{x}$ in the x-direction[4], $b\,\widehat{y}$ in the y-direction and $c\,\widehat{z}$ in the z-direction. In contrast, a, b, c are the **scalar components** of $\langle a, b, c \rangle$ in the x, y, z-directions respective.

Again, I emphasize that vector components are vectors whereas components or scalar components are by default scalars.



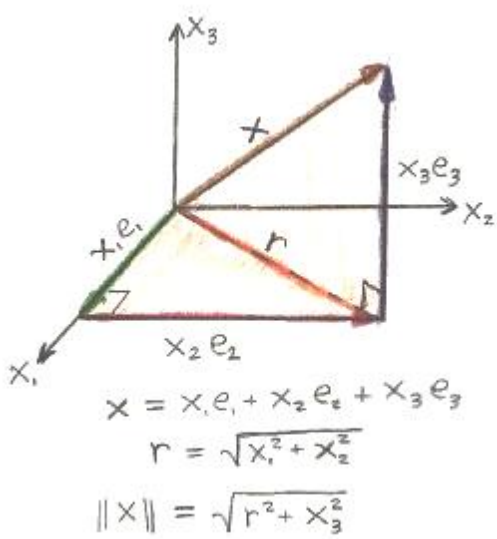
The story for $\mathbb{R}^n$ is not much different. **Define for $j = 1, 2, \ldots, n$ the vector $\widehat{x}_j = \langle 0, 0, \ldots, 1, \ldots, 0\rangle$ where the 1 appears in the j-th entry**, hence:

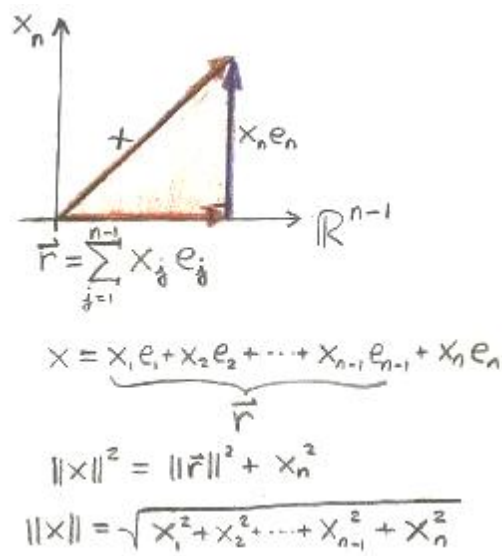
$$\begin{aligned}\langle a_1, a_2, \ldots, a_n\rangle &= \langle a_1, 0, \ldots, 0\rangle + \langle 0, a_2, \ldots, 0\rangle + \cdots + \langle 0, 0, \ldots, a_n\rangle \\ &= a_1\langle 1, 0, \ldots, 0\rangle + a_2\langle 0, 1, \ldots, 0\rangle + \cdots + a_n\langle 0, 0, \ldots, 1\rangle \\ &= a_1\,\widehat{x}_1 + \ a_2\,\widehat{x}_2 + \cdots + a_n\,\widehat{x}_n. \qquad (1.3)\end{aligned}$$

Definition 1.1.7. *vector and scalar components of n-vectors.*

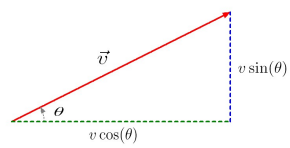
If $\vec{v} = \langle a_1, a_2, \ldots, a_n\rangle$ then $a_j\,\widehat{x}_j$ is the **vector component** in the x_j-direction of $\vec{v}$ whereas a_j is the **scalar component** in the x_j-direction of $\vec{v}$.

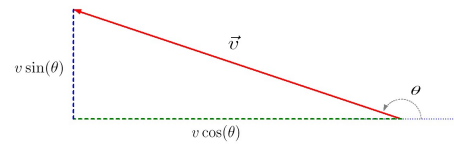
Here's an attempt at the picture for $n > 3$ (I use the linear algebra notation of $e_1 = \widehat{x}$ etc...):





Trigonometry is often useful in applied problems. It is not uncommon to be faced with vectors which are described by a length and a direction in the plane. In such a case we need to rely on trigonometry to *break-down* the vector into it's Cartesian components.





Definition 1.1.8. *dot product.*

The **dot-product** is a useful operation on vectors. In $\mathbb{R}^2$ we define,

$$\langle V_1, V_2\rangle \bullet \langle W_1, W_2\rangle = V_1W_1 + V_2W_2.$$

In $\mathbb{R}^3$ we define,

$$\langle V_1, V_2, V_3\rangle \bullet \langle W_1, W_2, W_3\rangle = V_1W_1 + V_2W_2 + V_3W_3.$$

In $\mathbb{R}^n$ we define, for $\vec{V} = \sum_{j=1}^n V_j\,\widehat{x}_j$ and $\vec{W} = \sum_{j=1}^n W_j\,\widehat{x}_j$

$$\vec{V} \bullet \vec{W} = V_1W_1 + V_2W_2 + \cdots + V_nW_n.$$

It is important to notice that the dot-product takes in two *vectors* and outputs a *scalar*. It has a number of interesting properties which we will often use:

Example 1.1.9. .

$$\vec{a} = \langle s, 2s, 3s\rangle \quad \text{and} \quad \vec{b} = \langle t, -t, 5t\rangle$$

$$\begin{aligned}\vec{a}\bullet\vec{b} &= \langle s, 2s, 3s\rangle \bullet \langle t, -t, 5t\rangle \\ &= st - 2st + 15st \\ &= \boxed{14st = \vec{a}\bullet\vec{b}}\end{aligned}$$

Proposition 1.1.10. *properties of the dot-product.*

let $\vec{A}, \vec{B}, \vec{C} \in \mathbb{R}^n$ be vectors and $c \in \mathbb{R}$

1. **commutative:** $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$,
2. **distributive:** $\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$,
3. **distributive:** $(\vec{A} + \vec{B}) \bullet \vec{C} = \vec{A} \bullet \vec{C} + \vec{B} \bullet \vec{C}$,
4. **scalars factor out:** $\vec{A} \bullet (c\vec{B}) = (c\vec{A}) \bullet \vec{B} = c\vec{A} \bullet \vec{B}$,
5. **non-negative:** $\vec{A} \bullet \vec{A} \geq 0$,
6. **no null-vectors:** $\vec{A} \bullet \vec{A} = 0$ iff $\vec{A} = 0$.

Proof: The proof of these properties is simple if we use the right notation. Observe

$$\vec{A} \bullet \vec{B} = \sum_{j=1}^{n} A_j B_j = \sum_{j=1}^{n} B_j A_j = \vec{B} \bullet \vec{A}.$$

Thus the dot-product is commutative. Next, note that $(\vec{B} + \vec{C})_j = B_j + C_j$ hence,

$$\vec{A} \bullet (\vec{B} + \vec{C}) = \sum_{j=1}^{n} A_j (B_j + C_j) = \sum_{j=1}^{n} A_j B_j + \sum_{j=1}^{n} A_j C_j = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}.$$

The proof of item (3.) actually follows from the commutative property and the right-distributive property we just proved since

$$(\vec{A} + \vec{B}) \bullet \vec{C} = \vec{C} \bullet (\vec{A} + \vec{B}) = \vec{C} \bullet \vec{A} + \vec{C} \bullet \vec{B} = \vec{A} \bullet \vec{C} + \vec{B} \bullet \vec{C}.$$

The proof of (4.) is left to the reader. Continue to (5.), note that

$$\vec{A} \bullet \vec{A} = \sum_{j=1}^{n} A_j A_j = A_1^2 + A_2^2 + \cdots + A_n^2$$

hence it is clear that $\vec{A} \bullet \vec{A}$ is the sum of squares of real numbers and consequently $\vec{A} \bullet \vec{A} \geq 0$. Moreover, if $\vec{A} \bullet \vec{A} = 0$ and $\vec{A} \neq 0$ then there must exist at least one component, say $A_j \neq 0$ hence $\vec{A} \bullet \vec{A} \geq A_j^2 > 0$ which is a contradiction. Therefore, (6.) follows. $\square$

The length of a vector $\vec{A}$ is simply the distance from the origin to the point which the vector points. In particular, we denote the length of the vector $\vec{A}$ by $||\vec{A}||$ and it's clear from the formula in the proof for $\vec{A} \bullet \vec{A}$ that

$$||\vec{A}|| = \sqrt{\vec{A} \bullet \vec{A}}$$

this formula holds for $\mathbb{R}^n$. Sometimes the length of the vector $\vec{A}$ is also called the *norm* of $\vec{A}$. The norm also has interesting properties which are quite similar to those which are known for the absolute value function on $\mathbb{R}$ (in fact, $||x|| = |x|$ for $x \in \mathbb{R}$).

Proposition 1.1.11. *properties of the norm (also known as length of vector).*

Suppose $\vec{A}, \vec{B} \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

1. **absolute value of scalar factors out:** $||c\vec{A}|| = |c|||\vec{A}||$,
2. **triangle inequality:** $||\vec{A} + \vec{B}|| \leq ||\vec{A}|| + ||\vec{B}||$,
3. **Cauchy-Schwarz inequality:** $|\vec{A} \bullet \vec{B}| \leq ||\vec{A}||\,||\vec{B}||$.
4. **non-negative:** $||\vec{A}|| \geq 0$,
5. **only zero vector has zero length:** $||\vec{A}|| = 0$ iff $\vec{A} = 0$.

Proof: The proof of (1.) is simple,

$$||c\vec{A}|| = \sqrt{(c\vec{A}) \bullet (c\vec{A})} = \sqrt{c^2 \vec{A} \bullet \vec{A}} = \sqrt{c^2}\sqrt{\vec{A} \bullet \vec{A}} = |c|\ ||\vec{A}||.$$

To prove the triangle Let $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\begin{aligned}
||\vec{x} + \vec{y}||^2 &= |(\vec{x} + \vec{y}) \bullet (\vec{x} + \vec{y})| && \text{defn. of norm} \\
&= |\vec{x} \bullet (\vec{x} + \vec{y}) + \vec{y} \bullet (\vec{x} + \vec{y})| && \text{prop. of dot-product} \\
&= |\vec{x} \bullet \vec{x} + \vec{x} \bullet \vec{y} + \vec{y} \bullet \vec{x} + \vec{y} \bullet \vec{y}| && \text{prop. of dot-product} \\
&= |\ ||\vec{x}||^2 + 2\vec{x} \bullet \vec{y} + ||\vec{y}||^2\ | && \text{prop. of dot-product} \\
&\leq ||\vec{x}||^2 + 2|\vec{x} \bullet \vec{y}| + ||\vec{y}||^2 && \text{triangle ineq. for } \mathbb{R} \\
&\leq ||\vec{x}||^2 + 2||\vec{x}||\,||\vec{y}|| + ||\vec{y}||^2 && \text{Cauchy-Schwarz ineq.} \\
&\leq (||\vec{x}|| + ||\vec{y}||)^2 && \text{algebra}
\end{aligned}$$

Notice that both $||\vec{x}+\vec{y}||$ and $||\vec{x}||+||\vec{y}||$ are nonnegative hence the inequality above yields $||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$. Continue to item (3.). Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. If either $\vec{x} = 0$ or $\vec{y} = 0$ then the inequality is clearly true. Suppose then that both $\vec{x}$ and $\vec{y}$ are nonzero vectors. It follows that $||\vec{x}||, ||\vec{y}|| \neq 0$ and we can define vectors of unit-length; $\hat{x} = \frac{\vec{x}}{||\vec{x}||}$ and $\hat{y} = \frac{\vec{y}}{||\vec{y}||}$. Notice that $\hat{x} \bullet \hat{x} = \frac{\vec{x}}{||\vec{x}||} \bullet \frac{\vec{x}}{||\vec{x}||} = \frac{1}{||\vec{x}||^2}\vec{x} \bullet \vec{x} = \frac{\vec{x} \bullet \vec{x}}{\vec{x} \bullet \vec{x}} = 1$ and likewise $\hat{y} \bullet \hat{y} = 1$. Consider,

$$\begin{aligned}
0 \leq ||\hat{x} \pm \hat{y}||^2 &= (\hat{x} \pm \hat{y}) \bullet (\hat{x} \pm \hat{y}) \\
&= \hat{x} \bullet \hat{x} \pm 2(\hat{x} \bullet \hat{y}) + \hat{y} \bullet \hat{y} \\
&= 2 \pm 2(\hat{x} \bullet \hat{y}) \\
&\Rightarrow \quad -2 \leq \pm 2(\hat{x} \bullet \hat{y}) \\
&\Rightarrow \quad \pm \hat{x} \bullet \hat{y} \leq 1 \\
&\Rightarrow \quad |\hat{x} \bullet \hat{y}| \leq 1
\end{aligned}$$

Therefore, noting that $\vec{x} = ||\vec{x}||\hat{x}$ and $\vec{y} = ||\vec{y}||\hat{y}$,

$$|\vec{x} \bullet \vec{y}| = |\ ||\vec{x}||\hat{x} \bullet ||\vec{y}||\hat{y}\ | = ||\vec{x}||\,||\vec{y}||\,|\hat{x} \bullet \hat{y}| \leq ||\vec{x}||\,||\vec{y}||\ \square.$$

As I introduced in the proof above[5], if a vector has a length of one then it is called a **unit-vector**.

Definition 1.1.12. *unit vectors.*

Any nonzero vector $\vec{A}$ defines a unit-vector in the same direction which is denoted $\widehat{A}$ and is defined by:

$$\widehat{A} = \frac{1}{||\vec{A}||}\vec{A}.$$

I invite the reader to check that $||\widehat{A}|| = 1$. Moreover, we should observe that any nonzero vector can be written as the product of its unit-vector $\widehat{A}$ and its length $||\vec{A}||$:

$$\vec{A} = ||\vec{A}||\widehat{A}$$

When it is convenient and unambiguous we use the notation $||\vec{A}|| = A$ and it follows

$$\boxed{\vec{A} = A\widehat{A}.}$$

We already used unit-vectors in the vector component discussion. Notice that

$$\widehat{x} \bullet \widehat{x} = 1, \qquad \widehat{y} \bullet \widehat{y} = 1, \qquad \widehat{z} \bullet \widehat{z} = 1.$$

Example 1.1.13. .

$$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k} \quad \text{and} \quad \vec{b} = 5\hat{i} + 9\hat{k}$$

$$\begin{aligned}
\vec{a}\cdot\vec{b} &= (\hat{i} - 2\hat{j} + 3\hat{k})\bullet(5\hat{i} + 9\hat{k}) \\
&= 5\underbrace{\hat{i}\cdot\hat{i}}_{1} + 9\underbrace{\hat{i}\cdot\hat{k}}_{0} - 10\underbrace{\hat{j}\cdot\hat{i}}_{0} - 18\underbrace{\hat{j}\cdot\hat{k}}_{0} + 15\underbrace{\hat{k}\cdot\hat{i}}_{0} + 27\underbrace{\hat{k}\cdot\hat{k}}_{1} \\
&= 5 + 27 \\
&= \boxed{32 = \vec{a}\cdot\vec{b}}
\end{aligned}$$

In summary, we have that $\widehat{x}_i \bullet \widehat{x}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. This is a very interesting formula. It shows that set of vectors $\{\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\}$ are all of unit-length and distinct vectors are have dot-products which are zero.

[5] I happened to find this argument in Insel, Spence and Friedberg's undergraduate linear algebra text.

Definition 1.1.14. *orthogonal vectors.*

We say $\vec{A}$ is **orthogonal** to $\vec{B}$ iff $\vec{A} \bullet \vec{B} = 0$. A set of vectors which is both orthogonal and all of unit length is said to be an **orthonormal set** of vectors.

The interesting formula $\widehat{x}_i \bullet \widehat{x}_j = \delta_{ij}$ compactly expresses the orthonormality of the standard basis[6] $\{\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\}$.

Orthogonality makes for interesting formulas. Let $\vec{V} = \langle V_1, V_2 \rangle \in \mathbb{R}^2$ and calculate,

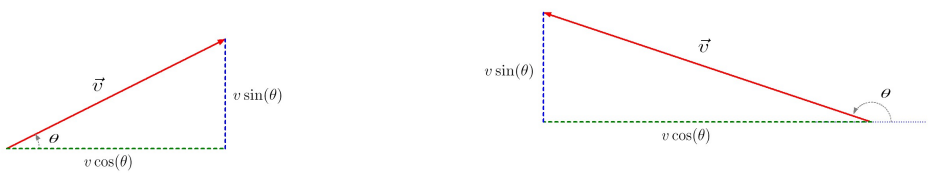
$$\vec{V} \bullet \widehat{x} = (V_1 \widehat{x}_1 + V_2 \widehat{x}_2) \bullet \widehat{x}_1 = V_1 \widehat{x}_1 \bullet \widehat{x}_1 + V_2 \widehat{x}_2 \bullet \widehat{x}_1 = \delta_{11} V_1 + \delta_{12} V_2 = V_1$$

$$\vec{V} \bullet \widehat{x}_2 = (V_1 \widehat{x}_1 + V_2 \widehat{x}_2) \bullet \widehat{x}_2 = V_1 \widehat{x}_1 \bullet \widehat{x}_2 + V_2 \widehat{x}_2 \bullet \widehat{x}_2 = \delta_{12} V_1 + \delta_{22} V_2 = V_2$$

This means we can use the dot-product to select the scalar components of a given vector.

$$\vec{V} = \langle \vec{V} \bullet \widehat{x}_1,\ \vec{V} \bullet \widehat{x}_2 \rangle = \left(\vec{V} \bullet \widehat{x}_1 \right) \widehat{x}_1 + \left(\vec{V} \bullet \widehat{x}_2 \right) \widehat{x}_2.$$

Let's pause to make a connection to the standard angle θ and the cartesian components.



Note that $\vec{V} = \cos(\theta)\,\widehat{x} + \sin(\theta)\,\widehat{y}$ and $\vec{V} = \left(\vec{V} \bullet \widehat{x} \right) \widehat{x} + \left(\vec{V} \bullet \widehat{y} \right) \widehat{y}$. It follows that:

$$\cos(\theta) = \vec{V} \bullet \widehat{x} \qquad \text{and} \qquad \sin(\theta) = \vec{V} \bullet \widehat{y}.$$

You could use these equations to define the standard angle in retrospect. Naturally, the decomposition above equally well applies to $\mathbb{R}^n$:

$$\vec{V} = \langle \vec{V} \bullet \widehat{x}_1,\ \vec{V} \bullet \widehat{x}_2,\ \ldots\ ,\ \vec{V} \bullet \widehat{x}_n \rangle = \sum_{j=1}^{n} \left(\vec{V} \bullet \widehat{x}_j \right) \widehat{x}_j.$$

This is called an **orthogonal decomposition** of $\vec{V}$ because it gives $\vec{V}$ as a sum of vectors which are pairwise orthogonal. Intuitively, I think of this as breaking the vector into it's basic parts. So far, all of this is with respect to Cartesian coordinates. Perhaps we will also see how similar decompositions are possible for curvelinear coodinate systems or moving coordinate systems.

[6] a term which is studied in linear algebra

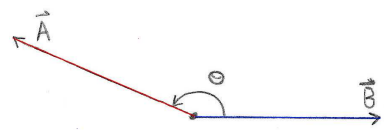
The study of geometry involves lengths and angles of shapes. We have all the tools we need to define the angle θ between nonzero vectors

Definition 1.1.15. *angle between a pair of vectors.*

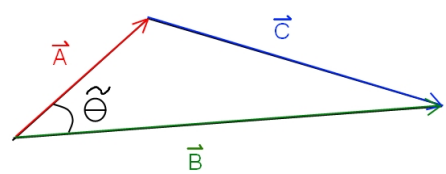
Let $\vec{A}, \vec{B}$ be nonzero vectors in $\mathbb{R}^n$. We define the angle between $\vec{A}$ and $\vec{B}$ by

$$\theta = \cos^{-1}\left[\frac{\vec{A} \bullet \vec{B}}{||\vec{A}||\,||\vec{B}||}\right].$$

Note nonzero vectors $\vec{A}, \vec{B}$ have $||\vec{A}|| \neq 0$ and $||\vec{B}|| \neq 0$ thus the Cauchy-Schwarz inequality $|\vec{A} \bullet \vec{B}| \leq ||\vec{A}||\,||\vec{B}||$ implies $\frac{\vec{A} \bullet \vec{B}}{||\vec{A}||\,||\vec{B}||} \leq 1$. It follows that the argument of the inverse cosine is within its domain. Moreover, since the standard inverse cosine has range $[0, \pi]$ it follows the angle which is given by the formula above is the smallest angle *between* the vectors. Of course, if θ is the angle between $\vec{A}, \vec{B}$ then geometry clearly indicates $2\pi - \theta$ is the angle on the other side of the θ vertex. I think a picture helps:



The careful reader will question how I know the formula really recovers the idea of angle that we have previously used in our studies of trigonometry. All I have really argued thus far is that the formula for θ is reasonable. Examine the triangle formed by $\vec{A}, \vec{B}$ and $\vec{C} = \vec{B} - \vec{A}$. Notice that $\vec{A} + \vec{C} = \vec{B}$. Picture $\vec{A}$ and $\vec{B}$ as adjacent sides to an angle $\tilde{\theta}$ which has opposite side $\vec{C}$. Let the lengths of $\vec{A}, \vec{B}, \vec{C}$ be A, B, C respective.



Applying[7] the **Law of Cosines** to the triangle above yields

$$C^2 = A^2 + B^2 - 2AB\cos(\tilde{\theta}).$$

Solve for $\tilde{\theta}$,

$$\tilde{\theta} = \cos^{-1}\left[\frac{A^2 + B^2 - C^2}{2AB}\right]$$

[7] if you had Math 131 with me then you proved the Law of Cosines in one of your first Problem Sets.

Is this consistent, does $\theta = \tilde{\theta}$? Choose coordinates[8] which place the vectors $\vec{A}, \vec{B}, \vec{C}$ are in the xy-plane and let $\vec{A} = \langle A_1, A_2 \rangle, \vec{B} = \langle B_1, B_2 \rangle$ hence $\vec{C} = \langle B_1 - A_1, B_2 - A_2 \rangle$ we calculate

$$C^2 = (B_1 - A_1)^2 + (B_2 - A_2)^2 = B_1^2 - 2A_1B_1 + A_1^2 + B_2^2 - 2A_2B_2 + A_2^2$$

Thus, $C^2 = A^2 + B^2 - 2\vec{A} \bullet \vec{B}$ and we find:

$$\tilde{\theta} = \cos^{-1}\left[\frac{2\vec{A} \bullet \vec{B}}{2AB}\right] = \cos^{-1}\left[\frac{\vec{A} \bullet \vec{B}}{||\vec{A}||\,||\vec{B}||}\right] = \theta.$$

Thus, we find the generalization of angle for $\mathbb{R}^n$ agrees with the two-dimensional concept we've explored in previous courses. Moreover, we discover a geometrically lucid formula for the dot-product:

$$\boxed{\vec{A} \bullet \vec{B} = ||\vec{A}||\,||\vec{B}|| \cos(\theta)}$$

or if we denote $\vec{A} = A\widehat{A}$ and $\vec{B} = B\widehat{B}$ then

$$\boxed{\vec{A} \bullet \vec{B} = AB\cos(\theta).}$$

The connection between this formula and the definition is nontrivial and is essentially equivalent to the Law of Cosines. This means that this is a powerful formula which allows deep calculation of geometrically non-obvious angles through the machinery of vectors. Notice:

$$\boxed{\text{If } \vec{A}, \vec{B} \text{ are nonzero orthogonal vectors then the angle between them is } \pi/2.}$$

this observation is an immediate consequence of the the definition of orthogonal vectors and the fact $\cos(\pi/2) = 0$. We find that orthogonal vectors are in fact perpendicular (which is a known term from geometry). In addition,

$$\boxed{\text{If } \vec{A}, \vec{B} \text{ are parallel vectors then } \vec{A} \bullet \vec{B} = AB \text{ and } \theta = 0.}$$

likewise,

$$\boxed{\text{If } \vec{A}, \vec{B} \text{ are antiparallel vectors then } \vec{A} \bullet \vec{B} = -AB \text{ and } \theta = \pi.}$$

The dot-product gives us a concrete method to test for whether two vectors point in the same direction, opposite directions or are purely perpendiular.

[8] even in the context of $\mathbb{R}^n$ we can place $\vec{A}, \vec{B}$ and $\vec{B} - \vec{A}$ in a particular plane, this argument actually extends to n-dimensions provided you accept the Law of Cosines is known in any plane

Example 1.1.16. .

Determine if the vectors are orthogonal, parallel, or neither.

$\vec{u}\cdot\vec{v} = \langle -5, 3, 7\rangle \cdot \langle 6, -8, 2\rangle = -30-24+14 = -40 \neq 0$

these are not orthogonal. Note $|\vec{u}| = \sqrt{25+9+49} = \sqrt{83}$

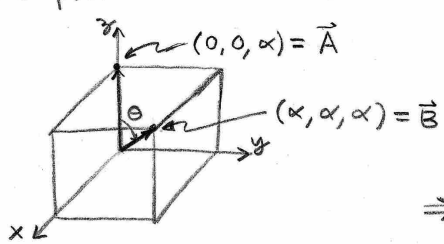
and $|\vec{v}| = \sqrt{36+64+4} = \sqrt{104}$ then $|\vec{u}||\vec{v}| = \sqrt{(104)(83)} = \sqrt{8932}$

if $\vec{u}$ was parallel to $\vec{v}$ then $\vec{u}\cdot\vec{v} = \pm|\vec{u}||\vec{v}|$ yet we find $\vec{u}\cdot\vec{v} = -40 \neq \pm\sqrt{8932}$. So $\vec{u}$ & $\vec{v}$ are neither orthogonal or parallel.

Example 1.1.17. .

$\boxed{E4}$ find angle between diagonal of cube and a line on the corner.

We place the cube at the origin and let its sides have length α.



$|\vec{A}| = \sqrt{\alpha^2} = \alpha$

$|\vec{B}| = \sqrt{3\alpha^2} = \sqrt{3}\,\alpha$

$\vec{A}\cdot\vec{B} = \langle 0, 0, \alpha\rangle \cdot \langle \alpha, \alpha, \alpha\rangle = \alpha^2$

$\Rightarrow \alpha^2 = \sqrt{3}\,\alpha^2 \cos\theta$

$\therefore \cos\theta = \frac{1}{\sqrt{3}}$

$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \cong \boxed{54.74^\circ}$

We can do more than just measure angles. We can also use the dot-product to project vectors to lines or even planes. In particular:

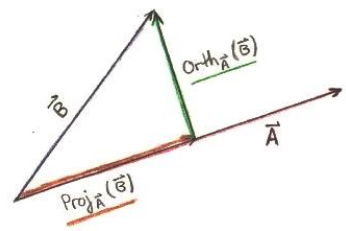
Definition 1.1.18. *projection onto vector.*

$\vec{A} \neq 0$ then $Proj_{\vec{A}}(\vec{B}) = \big(\vec{B}\bullet\widehat{A}\big)\widehat{A}$ defines the **vector projection of $\vec{B}$ onto $\vec{A}$**. We also define the orthogonal complement of $\vec{B}$ with respect to $\vec{A}$ by $Orth_{\vec{A}}(\vec{B}) = \vec{B} - Proj_{\vec{A}}(\vec{B})$.

We've already seen the projection formula implicitly in the formula $\vec{V} = (\vec{V}\bullet\widehat{x})\,\widehat{x} + (\vec{V}\bullet\widehat{y})\,\widehat{y}$ note that $Proj_{\widehat{x}}(\vec{V}) = (\vec{V}\bullet\widehat{x})\,\widehat{x}$ since the unit vector of the unit vector $\widehat{x}$ is just $\widehat{x}$. Likewise, $Proj_{\widehat{y}}(\vec{V}) = (\vec{V}\bullet\widehat{y})\,\widehat{y}$. Very well, you might not find this terribly interesting. However, if was to ask where a perpendicular bisector of $\langle 2,2,1\rangle$ intersects $\langle 3,6,9\rangle$ then I doubt you could do it with geometry alone. It's simple with the projection formula:

$$\begin{aligned} Proj_{\langle 2,2,1\rangle}\big(\langle 1,2,3\rangle\big) &= \big[\langle 1,2,3\rangle\bullet\langle \tfrac{2}{3},\tfrac{2}{3},\tfrac{1}{3}\rangle\big]\langle \tfrac{2}{3},\tfrac{2}{3},\tfrac{1}{3}\rangle \\ &= \big[\tfrac{2}{3}+\tfrac{4}{3}+\tfrac{3}{3}\big]\langle \tfrac{2}{3},\tfrac{2}{3},\tfrac{1}{3}\rangle \\ &= \langle 2,2,1\rangle. \end{aligned}$$

That was actually an accident. I'll try to do an uglier example in lecture. We'll use the projection from time to time, it is a really nice tool to calculate things that are at times hard to picture directly. Generically the projection can be pictured by:



You should think of $Orth_{\vec{A}}(\vec{B})$ as the way to obtain the piece of $\vec{B}$ which is perpendicular to $\vec{A}$.

Example 1.1.19. .

$\boxed{E3}$ Given $\vec{A} = \langle 3, 4\rangle$ and $\vec{B} = \langle 5, 12\rangle$ find the angle between them and the unit vectors $\hat{A}$ & $\hat{B}$ and the vector/scalar projections,

$$|\vec{A}| = \sqrt{9+16} = \sqrt{25} = 5 \quad \therefore \boxed{\hat{A} = \tfrac{1}{5}\langle 3, 4\rangle}.$$

$$|\vec{B}| = \sqrt{25+144} = \sqrt{169} = 13 \quad \therefore \boxed{\hat{B} = \tfrac{1}{13}\langle 5, 12\rangle}$$

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = 65\cos\theta \quad \& \quad \vec{A}\cdot\vec{B} = 3(5)+4(12) = 15+48 = 63.$$

$$\Rightarrow \cos\theta = 63/65 \quad \therefore \boxed{\theta = \cos^{-1}(63/65) = 0.2487 = 14.25^\circ}$$

Then we find

$$comp_{\vec{A}}(\vec{B}) = \vec{B}\cdot\hat{A} = \tfrac{1}{5}\langle 5, 12\rangle\cdot\langle 3, 4\rangle = \tfrac{1}{5}(15+48) = 63/5$$

$$proj_{\vec{A}}(\vec{B}) = comp_{\vec{A}}(\vec{B})\cdot\hat{A} = \tfrac{63}{5}\tfrac{1}{5}\langle 3, 4\rangle = \tfrac{63}{25}\langle 3, 4\rangle$$

Example 1.1.20. .

Let $\vec{a} = \hat{i}+\hat{j}+\hat{k} = \langle 1,1,1\rangle$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k} = \langle 1,-1,1\rangle$ then find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$

$$comp_a(\vec{b}) = \vec{b}\cdot\hat{a} = \langle 1,-1,1\rangle\cdot\left[\tfrac{1}{\sqrt{3}}\langle 1,1,1\rangle\right] = \tfrac{1}{\sqrt{3}}(1-1+1) = \boxed{\tfrac{1}{\sqrt{3}}}$$

we find the "scalar projection of $\vec{b}$ onto $\vec{a}$" is $\boxed{comp_a(b) = 1/\sqrt{3}}$

In other words the component of $\vec{b}$ in the $\vec{a}$-direction is $1/\sqrt{3}$.

$$proj_a(\vec{b}) = (comp_a(\vec{b}))\hat{a} = \tfrac{1}{\sqrt{3}}\left(\tfrac{1}{\sqrt{3}}\langle 1,1,1\rangle\right) = 1/3\langle 1,1,1\rangle.$$

The projection of $\vec{b}$ onto the $\vec{a}$-axis is the vector projection of $\vec{b}$ onto $\vec{a}$, in particular $\boxed{proj_a(\vec{b}) = 1/3\langle 1,1,1\rangle}$

Definition 1.1.21. *projection and orthogonal complement with respect to a plane.*

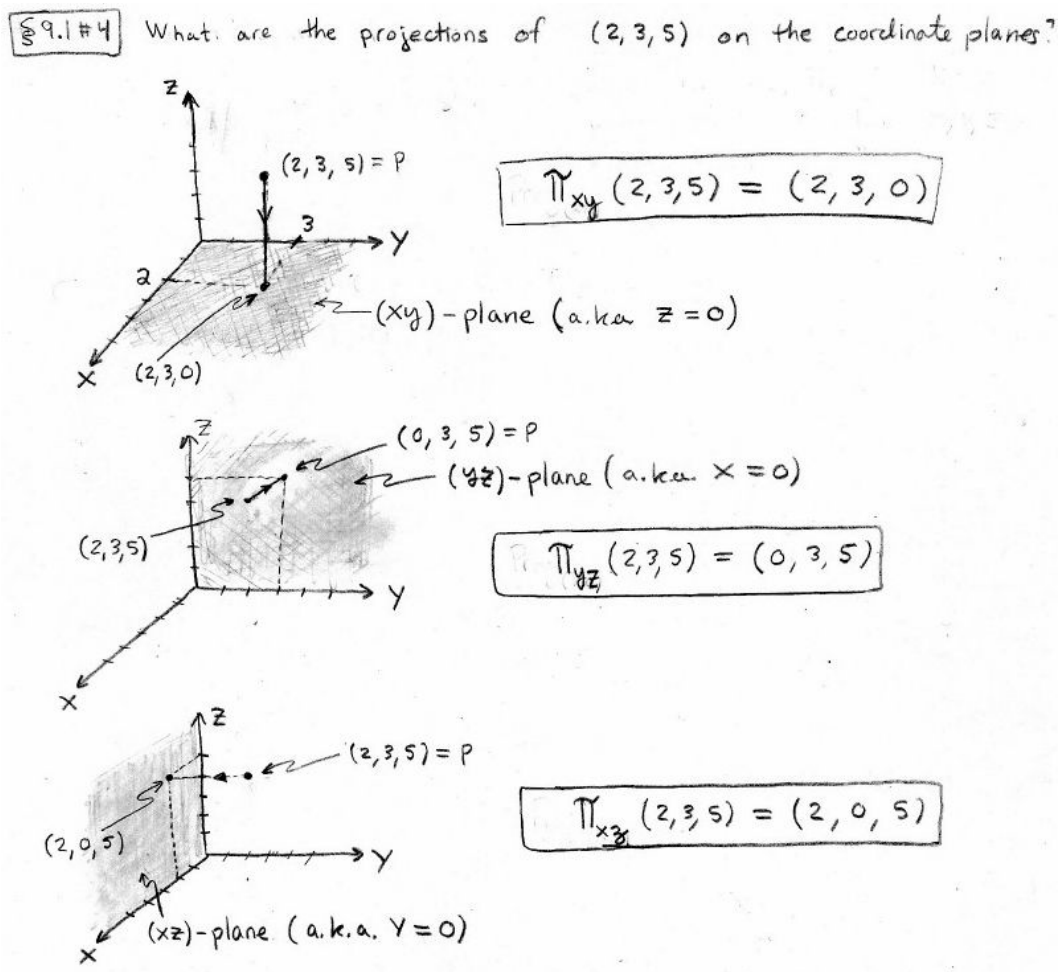
If $\vec{A}, \vec{B}$ are non-parallel vectors in some plane S then $Proj_S(\vec{R}) = (\vec{R} \bullet \widehat{A})\widehat{A} + (\vec{R} \bullet \widehat{B})\widehat{B}$ defines the **projection of $\vec{R}$ onto the plane** S. The orthogonal projection of $\vec{R}$ off the plane S is given by $Orth_S(\vec{R}) = \vec{R} - Proj_S(\vec{R})$.

The projections onto the coordinate planes are sometimes interesting. Clearly $\widehat{x}$, $\widehat{y}$ fit in the xy-plane hence

$$Proj_{xy-plane}(\langle v_1, v_2, v_3\rangle) = (\widehat{x} \bullet \langle v_1, v_2, v_3\rangle)\,\widehat{x} + (\widehat{y} \bullet \langle v_1, v_2, v_3\rangle)\,\widehat{y} = \langle v_1, v_2, 0\rangle.$$

Likewise, $Proj_{zx-plane}(\langle v_1, v_2, v_3\rangle) = \langle v_1, 0, v_3\rangle$ and $Proj_{yz-plane}(\langle v_1, v_2, v_3\rangle) = \langle 0, v_2, v_3\rangle$.

Example 1.1.22. .



Perhaps this material belongs with the larger discussion of planes. I included it here simply to illustrate the utility of the dot-product.

Example 1.1.23. *Judging the colinearity of two vectors is important to physics. The work done by a force is maximized when the force is applied over a displacement which is precisely parallel to the force. On the other hand, the work done by a perpendicular force is zero. The dot-product captures all these concepts in a nice neat formula: the work W done by a constant force $\vec{F}$ applied to an object undergoing a displacement $\triangle\vec{x}$ is given by* $\boxed{W = \vec{F} \bullet \triangle\vec{x}}$.

$\boxed{\text{E5}}$ If a <u>constant</u> force of $\vec{F} = \langle 1, 1, 1 \rangle$ is applied to a particle that is displaced by $\overrightarrow{\Delta x} = \langle 1, 0, 0 \rangle$ find the work done.

$$W \equiv \vec{F} \bullet \overrightarrow{\Delta x} = \langle 1,1,1 \rangle \bullet \langle 1,0,0 \rangle = \boxed{1 = W}$$

Much later in this course we turn to the question of calculating work done by nonconstant forces over arbitrary curves.

Example 1.1.24. .

Let $\vec{F} = \langle 10, 18, -6 \rangle$ be a constant force field which moves an object from $(2, 3, 0)$ to $(4, 9, 15)$.

$$\begin{aligned} W = \vec{F} \bullet (\overrightarrow{\Delta x}) &= \langle 10, 18, -6 \rangle \bullet \langle 4-2, 9-3, 15-0 \rangle \\ &= \langle 10, 18, -6 \rangle \bullet \langle 2, 6, 15 \rangle \\ &= 20 + 108 - 90 \\ &= \boxed{38 = W} \quad \text{with units } \boxed{W = 38\,J} \end{aligned}$$

• assuming $[F]$ = Newtons, $[\Delta x]$ = meters, we'll avoid units in this course, I'm happy to tell you how to add them correctly if you're interested, its just more writing and I think it distracts from <u>the</u> math here

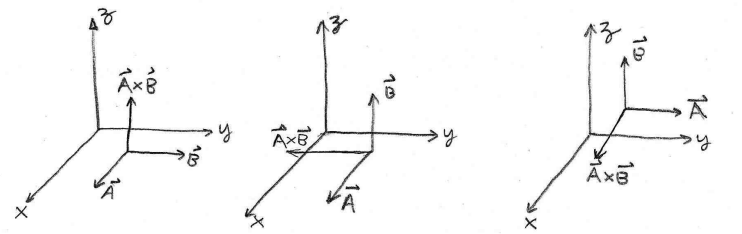
There are many dot-products in basic physics.

Example 1.1.25. *If $\vec{v}$ is the velocity of a mass m then the kinetic energy is given by $K = \frac{1}{2} m\vec{v} \bullet \vec{v}$.*

Example 1.1.26. *Or, if $\vec{v}$ is the velocity of a mass m and $\vec{F}$ is the net-force on m then the power developed by $\vec{F}$ is given by $P = \vec{v} \bullet \vec{F}$.*

1.2 the cross product

We saw that the dot-product gives us a natural way to check if a pair of vectors is orthogonal. You should remember: $\vec{A}, \vec{B}$ are orthogonal iff $\vec{A} \bullet \vec{B} = 0$. We turn to a slightly different goal in this section: given a pair of nonzero, nonparallel vectors $\vec{A}, \vec{B}$ how can we find another vector $\vec{A} \times \vec{B}$ which is perpendicular to both $\vec{A}$ and $\vec{B}$? Geometrically, in $\mathbb{R}^3$ it's not too hard to picture it:



My intent in this section is to motivate the standard formula for this product and to prove some of the standard properties of this cross product. These calculations are special to $\mathbb{R}^3$.

Remark 1.2.1.

Forbidden jitzu ahead, In $\mathbb{R}^n$ the story is a bit more involved, we can calculate the orthogonal complement to $span\{\vec{A}, \vec{B}\}$ and this produces an $(n-2)$-dimensional space of orthogonal vectors to $\vec{A}, \vec{B}$. If $n = 4$ this means there is a whole plane of vectors which we could choose. Only in the case $n = 3$ is the orthogonal complement simply a one-dimensional space, a line.

Therefore, suppose $\vec{A}, \vec{B}$ are nonzero, nonparallel vectors in $\mathbb{R}^3$. I'll calculate conditions on $\vec{A} \times \vec{B}$ which insure it is perpendicular to both $\vec{A}$ and $\vec{B}$. Let's denote $\vec{A} \times \vec{B} = \vec{C}$. We should expect $\vec{C}$ is some function of the components of $\vec{A}$ and $\vec{B}$. I'll use $\vec{A} = \langle A_1, A_2, A_3 \rangle$ and $\vec{B} = \langle B_1, B_2, B_3 \rangle$ whereas $\vec{C} = \langle C_1, C_2, C_3 \rangle$

$$0 = \vec{C} \bullet \vec{A} = C_1 A_1 + C_2 A_2 + C_3 A_3$$
$$0 = \vec{C} \bullet \vec{B} = C_1 B_1 + C_2 B_2 + C_3 B_3$$

Suppose $A_1 \neq 0$, then we may solve $0 = \vec{C} \bullet \vec{A}$ as follows,

$$C_1 = -\frac{A_2}{A_1} C_2 - \frac{A_3}{A_1} C_3$$

Suppose $B_1 \neq 0$, then we may solve $0 = \vec{C} \bullet \vec{B}$ as follows,

$$C_1 = -\frac{B_2}{B_1} C_2 - \frac{B_3}{B_1} C_3$$

It follows, given the assumptions $A_1 \neq 0$ and $B_1 \neq 0$,

$$\frac{A_2}{A_1} C_2 + \frac{A_3}{A_1} C_3 = \frac{B_2}{B_1} C_2 + \frac{B_3}{B_1} C_3$$

Multiply by A_1B_1 to obtain:

$$B_1A_2C_2 + B_1A_3C_3 = A_1B_2C_2 + A_1B_3C_3$$

Thus,

$$(A_1B_2 - B_1A_2)C_2 + (A_1B_3 - B_1A_3)C_3 = 0$$

One solution is simply $C_2 = A_1B_3 - B_1A_3$ and $C_3 = A_1B_2 - B_1A_2$ and it follows that $C_1 = A_2B_3 - B_2A_3$. Of course, generally we could have vectors which are nonzero and yet have $A_1 = 0$ or $B_1 = 0$. The point of the calculation is not to provide a general derivation. Instead, my intent is simply to show you how you might be led to make the following definition:

Definition 1.2.2. *cross product.*

Let $\vec{A}, \vec{B}$ be vectors in $\mathbb{R}^3$. The vector $\vec{A} \times \vec{B}$ is called the **cross product** of $\vec{A}$ with $\vec{B}$ and is defined by

$$\vec{A} \times \vec{B} = \langle\ A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1\ \rangle.$$

We say $\vec{A}$ cross $\vec{B}$ is $\vec{A} \times \vec{B}$.

It is a simple exercise to verify that

$$\vec{A} \bullet (\vec{A} \times \vec{B}) = 0 \qquad \text{and} \qquad \vec{B} \bullet (\vec{A} \times \vec{B}) = 0.$$

Both of these identities should be utilized to check your calculation of a given cross product. Let's think about the formula for the cross product a bit more. We have

$$\vec{A} \times \vec{B} = (A_2B_3 - A_3B_2)\,\widehat{x}_1 + (A_3B_1 - A_1B_3)\,\widehat{x}_2 + (A_1B_2 - A_2B_1)\,\widehat{x}_3$$

distributing,

$$\vec{A} \times \vec{B} = A_2B_3\,\widehat{x}_1 - A_3B_2\,\widehat{x}_1 + A_3B_1\,\widehat{x}_2 - A_1B_3\,\widehat{x}_2 + A_1B_2\,\widehat{x}_3 - A_2B_1\,\widehat{x}_3$$

The pattern is clear. Each term has indices $1, 2, 3$ without repeat and we can generate the signs via the antisymmetric symbol ϵ_{ijk} which is defined be zero if any indices are repeated and

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \qquad \text{whereas} \qquad \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1.$$

With this convenient shorthand we find the nice formula for the cross product that follows:

$$\boxed{\vec{A} \times \vec{B} = \sum_{i,j,k=1}^{3} A_iB_j\epsilon_{ijk}\,\widehat{x}_k}$$

Interestingly the Cartesian unit-vectors $\widehat{x}_1$, $\widehat{x}_2$, $\widehat{x}_3$ satisfy the simple relation:

$$\boxed{\widehat{x}_i \times \widehat{x}_j = \sum_{k=1}^{3} \epsilon_{ijk}\,\widehat{x}_k,}$$

which is just a fancy way of saying that

$$\boxed{\widehat{x} \times \widehat{y} = \widehat{z}, \qquad \widehat{y} \times \widehat{z} = \widehat{x}, \qquad \widehat{z} \times \widehat{x} = \widehat{y}}$$

There are many popular mneumonics to remember these. The basic properties of the cross product together with these formula allow us to quickly calculate some cross products (see Example 1.2.7)

Proposition 1.2.3. *basic properties of the cross product.*

Let $\vec{A}, \vec{B}, \vec{C}$ be vectors in $\mathbb{R}^3$ and $c \in \mathbb{R}$

1. **anticommutative:** $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$,
2. **distributive:** $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$,
3. **distributive:** $(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}$,
4. **scalars factor out:** $\vec{A} \times (c\vec{B}) = (c\vec{A}) \times \vec{B} = c\vec{A} \times \vec{B}$,

Proof: once more, the proof is easy with the right notation. Begin with (1.),

$$\vec{A} \times \vec{B} = \sum_{i,j,k=1}^{3} A_i B_j \epsilon_{ijk} \, \widehat{x}_k = -\sum_{i,j,k=1}^{3} A_i B_j \epsilon_{jik} \, \widehat{x}_k = -\sum_{i,j,k=1}^{3} B_j A_i \epsilon_{jik} \, \widehat{x}_k = -\vec{B} \times \vec{A}.$$

The key observation was that $\epsilon_{ijk} = -\epsilon_{jik}$ for all i, j, k. If you don't care for this argument then you could also give the brute-force argument below:

$$\begin{aligned} \vec{A} \times \vec{B} &= \langle\ A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1\ \rangle \\ &= -\langle\ A_3B_2 - A_2B_3,\ A_1B_3 - A_3B_1,\ A_2B_1 - A_1B_2\ \rangle \\ &= -\langle\ B_2A_3 - B_3A_2,\ B_3A_1 - B_1A_3,\ B_1A_2 - B_2A_1\ \rangle \\ &= -\vec{B} \times \vec{A}. \end{aligned} \tag{1.4}$$

Next, to prove (2.) we once more use the compact notation,

$$\begin{aligned} \vec{A} \times (\vec{B} + \vec{C}) &= \sum_{i,j,k=1}^{3} A_i (B_j + C_j) \epsilon_{ijk} \, \widehat{x}_k \\ &= \sum_{i,j,k=1}^{3} (A_i B_j \epsilon_{ijk} \, \widehat{x}_k + A_i C_j \epsilon_{ijk} \, \widehat{x}_k) \\ &= \sum_{i,j,k=1}^{3} A_i B_j \epsilon_{ijk} \, \widehat{x}_k + \sum_{i,j,k=1}^{3} A_i C_j \epsilon_{ijk} \, \widehat{x}_k \\ &= \vec{A} \times \vec{B} + \vec{A} \times \vec{C}. \end{aligned}$$

The proof of (3.) follows naturally from (1.) and (2.), note:

$$(\vec{A}+\vec{B})\times\vec{C} = -\vec{C}\times(\vec{A}+\vec{B}) = -\vec{C}\times\vec{A}-\vec{C}\times\vec{B} = \vec{A}\times\vec{C}+\vec{B}\times\vec{C}.$$

I leave the proof of (4.) to the reader. □

The properties above basically say that the cross product behaves the same as the usual addition and multiplication of numbers with the caveat that the order of factors matters. If we switch the order then we must include a minus due to the anticommutivity of the cross product.

Example 1.2.4. .

$$\begin{aligned}(\vec{A}-\vec{B})\times(\vec{A}+\vec{B}) &= \vec{A}\times(\vec{A}+\vec{B}) - \vec{B}\times(\vec{A}+\vec{B}) \\ &= \vec{A}\times\vec{A} + \vec{A}\times\vec{B} - \vec{B}\times\vec{A} + \vec{B}\times\vec{B} \\ &= 0 + \vec{A}\times\vec{B} - (-\vec{A}\times\vec{B}) + 0 \\ &= \underline{2\vec{A}\times\vec{B}} \;//.\end{aligned}$$

Notice $\vec{A}\times\vec{A} = -\vec{A}\times\vec{A} \therefore 2\vec{A}\times\vec{A} = 0 \therefore \vec{A}\times\vec{A} = 0.$
We should always keep this in mind.

There are a number of popular tricks to remember this rule. Let's look at a particular example a couple different ways:

Example 1.2.5. .

Let $\vec{A} = \langle 1, 2, 3\rangle$ and $\vec{B} = \langle 4, 5, 6\rangle$.
Calculate $\vec{A}\times\vec{B}$. Following the definition,

$$\begin{aligned}\vec{A}\times\vec{B} &= \langle 1, 2, 3\rangle \times \langle 4, 5, 6\rangle \\ &= \langle 2(6)-3(5),\ 3(4)-1(6),\ 1(5)-2(4)\rangle \\ &= \underline{\langle -3, 6, -3\rangle}.\end{aligned}$$

Example 1.2.6. .

The determinant of a matrix can be used to capture the pattern of the cross-product,

$$\boxed{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc}$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$\boxed{\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)}$$

Following the pattern above,

$$\det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix} = \hat{x}(A_2B_3 - A_3B_2) + \hat{y}(A_3B_1 - A_1B_3) + \hat{z}(A_1B_2 - A_2B_1)$$

Thus, another way to remember the formula is given by the determinant.

Therefore,

$$\vec{A} \times \vec{B} = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \hat{x}(12 - 15) + \hat{y}(12 - 6) + \hat{z}(5 - 8)$$
$$= \underline{-3\hat{x} + 6\hat{y} - 3\hat{z}}.$$

Technically, this formula is not really a determinant since genuine determinants are formed from matrices filled with objects of the same type. In the hybrid expression above we actually have one row of vectors and two rows of scalars. That said, I include it hear since many people use it and I also have found it useful in past calculations. If nothing else at least it helps you learn what a determinant is. That is a calculation which is worthwhile since determinants have application far beyond mere cross products.

Example 1.2.7. .

Another method to calculate cross-products is to remember the fundamental products $\hat{x}\times\hat{y}=\hat{z}$, $\hat{y}\times\hat{z}=\hat{x}$ and $\hat{z}\times\hat{x}=\hat{y}$ and then use distributivity and anticommutivity,

$$\begin{aligned}
\vec{A}\times\vec{B} &= (\hat{x}+2\hat{y}+3\hat{z})\times(4\hat{x}+5\hat{y}+6\hat{z}) \\
&= 5\hat{x}\times\hat{y}+6\hat{x}\times\hat{z}+8\hat{y}\times\hat{x}+12\hat{y}\times\hat{z}+ \\
&\quad +12\hat{z}\times\hat{x}+15\hat{z}\times\hat{y} \\
&= 5\hat{z}-6\hat{y}-8\hat{z}+12\hat{x}+12\hat{y}-15\hat{x} \\
&= \underline{-3\hat{z}+6\hat{y}-3\hat{x}}.
\end{aligned}$$

The calculation above is probably not the quickest for the example at hand here, but it is faster for other computations. For example, suppose $\vec{A}=\langle 1,2,3\rangle$ and $\vec{B}=\widehat{x}$ then

$$\vec{A}\times\vec{B}=(\widehat{x}+2\,\widehat{y}+3\,\widehat{z})\times\widehat{x}=2\,\widehat{y}\times\widehat{x}+3\,\widehat{z}\times\widehat{x}=-2\,\widehat{z}+3\,\widehat{y}.$$

For me, this is quicker than the determinant formula.

Example 1.2.8. *once more I contrast the calculational strategies:*

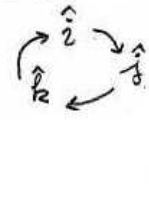
Let $\vec{a}=\langle 3,2,4\rangle$ and $\vec{b}=\langle 1,-2,-3\rangle$

$$\begin{aligned}
\vec{a}\times\vec{b} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & 4\\ 1 & -2 & -3\end{vmatrix} = \hat{i}\begin{vmatrix}2 & 4\\ -2 & -3\end{vmatrix}-\hat{j}\begin{vmatrix}3 & 4\\ 1 & -3\end{vmatrix}+\hat{k}\begin{vmatrix}3 & 2\\ 1 & -2\end{vmatrix} \\
&= \hat{i}(-6+8)-\hat{j}(-9-4)+\hat{k}(-6-2) \\
&= \boxed{2\hat{i}+13\hat{j}-8\hat{k}=\langle 2,13,-8\rangle=\vec{a}\times\vec{b}}
\end{aligned}$$

Notice $\vec{a}\cdot(\vec{a}\times\vec{b})=\langle 3,2,4\rangle\cdot\langle 2,13,-8\rangle=6+26-32=0$
and $\vec{b}\cdot(\vec{a}\times\vec{b})=\langle 1,-2,-3\rangle\cdot\langle 2,13,-8\rangle=2-26+24=0.$ $\Big\}$ $\vec{a}$ & $\vec{b}$ orthog. to $\vec{a}\times\vec{b}$.

Alternatively we could use the algebra of cross products

$$\begin{aligned}
\vec{a}\times\vec{b} &= (3\hat{i}+2\hat{j}+4\hat{k})\times(\hat{i}-2\hat{j}-3\hat{k}) \\
&= 3\hat{i}\times\hat{i} - 6\hat{i}\times\hat{j} - 9\hat{i}\times\hat{k} + 2\hat{j}\times\hat{i} - 4\hat{j}\times\hat{j} - 6\hat{j}\times\hat{k} \\
&\qquad + 4\hat{k}\times\hat{i} - 8\hat{k}\times\hat{j} - 12\hat{k}\times\hat{k} \\
&= -6\hat{k} + 9\hat{j} - 2\hat{k} - 6\hat{i} + 4\hat{j} + 8\hat{i} \\
&= \langle 8-6,\ 9+4,\ -6-2\rangle \\
&= \langle 2,\ 13,\ -8\rangle
\end{aligned}$$



There are a number of identities which connect the dot and cross products. These formulas require considerable effort if you choose to use brute-force proof methods.

Proposition 1.2.9. *nontrivial properties of the cross product.*

Let $\vec{A}, \vec{B}, \vec{C}$ be vectors in $\mathbb{R}^3$

1. $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\bullet\vec{C})\vec{B} - (\vec{A}\bullet\vec{B})\vec{C}$
2. **Jacobi Identity:** $\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0,$
3. **cyclicity of triple product:** $\vec{A}\bullet(\vec{B}\times\vec{C}) = \vec{B}\bullet(\vec{C}\times\vec{A}) = \vec{C}\bullet(\vec{A}\times\vec{B})$
4. **Lagrange's identity:** $||\vec{A}\times\vec{B}||^2 = ||\vec{A}||^2\,||\vec{B}||^2 - \left[\ \vec{A}\bullet\vec{B}\ \right]^2$

Proof: I leave proof of (1.) and (2.) to the reader. Let's see how (3.) is shown in the compact notation. Note $(\vec{B}\times\vec{C})_k = \sum_{ij} B_i C_j \epsilon_{ijk}$ hence

$$\begin{aligned}
\vec{A}\bullet(\vec{B}\times\vec{C}) &= \sum_{k=1}^{3} A_k \sum_{i,j=1}^{3} B_i C_j \epsilon_{ijk} \\
&= \sum_{i,j,k=1}^{3} B_i C_j A_k \epsilon_{ijk} \\
&= \sum_{i,j,k=1}^{3} C_j A_k B_i \epsilon_{jki} \\
&= \sum_{i,j,k=1}^{3} A_k B_i C_j \epsilon_{kij}
\end{aligned}$$

where we have used the cyclicity of the antisymmetric symbol $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$. The cyclicity of the triple product follows. Try to prove this without summations, it takes considerable patience.

Now we turn our attention to Lagrange's identity. I begin by quoting a useful identity connecting the antisymmetric symbol and the Kroenecker delta,

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Consider that,

$$\begin{aligned}
||\vec{A} \times \vec{B}||^2 &= \sum_{k=1}^{3} (\vec{A} \times \vec{B})_k^2 \\
&= \sum_{k=1}^{3} \left(\sum_{i,j=1}^{3} A_i B_j \epsilon_{ijk} \right) \left(\sum_{l,m=1}^{3} A_l B_m \epsilon_{lmk} \right) \\
&= \sum_{i,j,k,l,m=1}^{3} \sum_{k=1}^{3} A_i A_l B_j B_m \epsilon_{ijk}\epsilon_{lmk} \\
&= \sum_{i,j,l,m=1}^{3} A_i A_l B_j B_m (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \\
&= \sum_{i,j,l,m=1}^{3} A_i A_l \delta_{il} B_j B_m \delta_{jm} \; - \sum_{i,j,l,m=1}^{3} A_i B_m \delta_{im} B_j A_l \delta_{jl} \\
&= \sum_{i,j=1}^{3} A_i^2 B_j^2 - \sum_{i,j=1}^{3} A_i B_i B_j A_j \\
&= \sum_{i=1}^{3} A_i^2 \sum_{i=1}^{3} B_j^2 - \sum_{i=1}^{3} A_i B_i \sum_{j=1}^{3} B_j A_j \\
&= ||\vec{A}||^2 ||\vec{B}||^2 - \left[\; \vec{A} \bullet \vec{B} \; \right]^2.
\end{aligned}$$

I leave derivation of the crucial identity to the reader. $\square$.

Use Lagrange's identity together with $\vec{A} \bullet \vec{B} = AB\cos(\theta)$,

$$||\vec{A} \times \vec{B}||^2 = A^2B^2 - [AB\cos(\theta)]^2 = A^2B^2(1 - \cos^2(\theta)) = A^2B^2\sin^2(\theta)$$

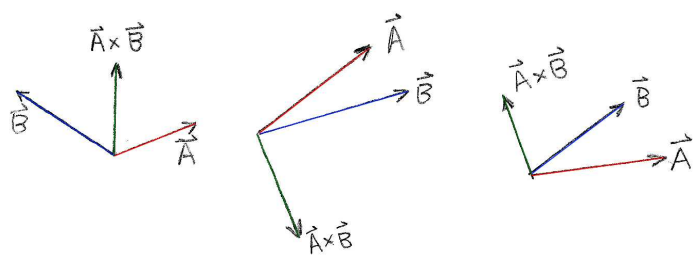
It follows there exists some unit-vector $\hat{\mathbf{n}}$ such that

$$\boxed{\vec{A} \times \vec{B} = AB\sin(\theta)\hat{\mathbf{n}}}$$

The direction of the unit-vector $\hat{\mathbf{n}}$ is conveniently indicated by the **right-hand-rule**. I typically perform the rule as follows:

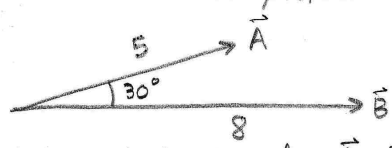
1. point fingers of **right hand** in direction $\vec{A}$
2. cross the fingers into thr direction of $\vec{B}$
3. the direction your thumb points is the approximate direction of $\hat{\mathbf{n}}$

I say *approximate* because $\vec{A} \times \vec{B}$ is strictly perpendicular to both $\vec{A}$ and $\vec{B}$ whereas your thumb's direction is a little ambiguous. But, it does pick one side of the plane in which the vectors $\vec{A}$ and $\vec{B}$ reside.



Example 1.2.10. .

$\boxed{E7}$ Consider the vectors pictured below. Find $\vec{A}\times\vec{B}$

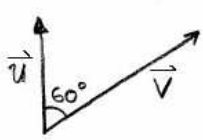


$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta = 40\sin 30° = 20.$

By the right hand rule $\vec{A}\times\vec{B} = 20\hat{n}$ where $\hat{n}$ points into page.

Example 1.2.11. .

Find $|\vec{u}\times\vec{v}|$ and $\widehat{\vec{u}\times\vec{v}}$ (the unit vector in $u\times v$ direction) given that



$|\vec{u}| = 5$ and $|\vec{v}| = 10$ and $\theta = 60°$

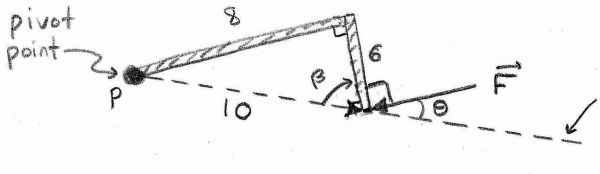
$|u\times v| = |u||v|\sin\theta = (5)(10)\sin 60° = 50\frac{\sqrt{3}}{2} = \boxed{25\sqrt{3}}$

point fingers in direction of $\vec{u}$ then curl them into the direction of $\vec{v}$ (use right hand). your thumb will point into the page.

$\boxed{\vec{u}\times\vec{v} \text{ goes into the page}}$

Example 1.2.12. .

$\boxed{E8}$ The torque $\vec{\tau}$ about a radial arm $\vec{r}$ created by a force $\vec{F}$ is defined to be $\vec{\tau} = \vec{r} \times \vec{F}$. Calculate the torque generated by $\vec{F}$ in the picture below,



Notice that $\theta + 90^\circ + \beta = 180^\circ$. Then notice $\tan\beta = 8/6$ thus $\beta = \tan^{-1}(8/6) = 53.13^\circ$. Calculate then

$$\theta = 90^\circ - \beta = 36.87^\circ$$

we see that the acute angle at P is in fact θ so $\sin\theta = \frac{6}{10}$.

$$|\vec{\tau}| = |\vec{r}||\vec{F}| \sin\theta = 10F\frac{6}{10} = \boxed{6F = \tau}$$

and $\vec{\tau}$ is into the page by right hand rule applied to.

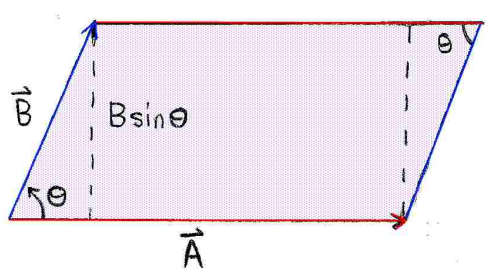
this means this force will produce a motion like ↻.

Example 1.2.13. *Another important application of the cross product to physics is the Lorentz force law. If a charge q has velocity $\vec{v}$ and travels through a magnetic field $\vec{B}$ then the force due to the electromagnetic interaction between q and the field is $\vec{F} = q\vec{v} \times \vec{B}$.*

Finally, we should investigate how the dot and cross product give nice formulas for the area of a parallelogram or the volume of a parallel piped. Suppose $\vec{A}, \vec{B}$ give the sides of a parallelogram.

$$Area = || \vec{A} \times \vec{B} ||$$

The picture below shows why the formula above is true,



$$\text{AREA} = (\text{BASE})(\text{HEIGHT}) = AB\sin\theta$$

$$\therefore \quad \text{AREA} = \|\vec{A} \times \vec{B}\|$$

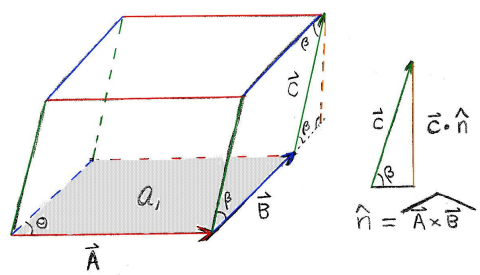
On the other hand, if $\vec{A}, \vec{B}, \vec{C}$ give the corner-edges of a parallelogram then

$$Volume = \mid \vec{A} \bullet (\vec{B} \times \vec{C}) \mid$$

These formulas are connected by the following thought: the volume subtended by $\vec{A}, \vec{B}$ and the unit-vector $\hat{\mathbf{n}}$ from $\vec{A} \times \vec{B} = AB\sin(\theta)\hat{\mathbf{n}}$ is equal to the area of the parallelogram with sides $\vec{A}, \vec{B}$. Algebraically:

$$\mid \hat{\mathbf{n}} \bullet (\vec{A} \times \vec{B}) \mid = \mid \hat{\mathbf{n}} \bullet (AB\sin(\theta)\hat{\mathbf{n}}) \mid = |AB\sin(\theta)| = ||\vec{A} \times \vec{B}||.$$

The picture below shows why the triple product formula is valid.



$a_1 = \|\vec{A} \times \vec{B}\| =$ area of base.

$h_1 = |\vec{C} \cdot \hat{n}| = \left| \vec{C} \cdot \left[\frac{\vec{A} \times \vec{B}}{\|\vec{A} \times \vec{B}\|} \right] \right| =$ height

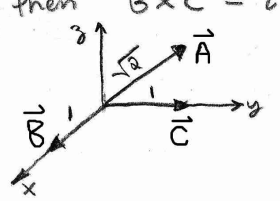
$$\text{Volume} = h_1 a_1 = \left| (\vec{A} \times \vec{B}) \bullet \vec{C} \right|$$

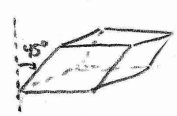
Example 1.2.14. .

$\boxed{E9}$ The volume of a parallelepiped with edges $\vec{A}, \vec{B}$ and $\vec{C}$ out of a common vertex is given by

$$V = |\vec{A} \cdot (\vec{B} \times \vec{C})|$$

For example let $\vec{A} = \langle 0, 1, 1 \rangle$, $\vec{B} = \langle 1, 0, 0 \rangle$, $\vec{C} = \langle 0, 1, 0 \rangle$ then $\vec{B} \times \vec{C} = \hat{i} \times \hat{j} = \hat{k}$ and $\vec{A} \cdot (\vec{B} \times \vec{C}) = 1$. Thus





has volume $\boxed{V = 1}$

Moreover, given this geometric interpretation we find a new proof (up to a sign) for the cyclic property. By the symmetry of the edges it follows that $\mid \vec{A} \bullet (\vec{B} \times \vec{C}) \mid = \mid \vec{B} \bullet (\vec{C} \times \vec{A}) \mid = \mid \vec{C} \bullet (\vec{A} \times \vec{B}) \mid$. We should find the same volume no matter how we label width, depth and height.

1.3 lines and planes in $\mathbb{R}^3$

There are two main viewpoints to describe lines and planes. The parametric viewpoint introduces parameters which label points on the object of interest. For a line we need one parameter, for a plane we need two parameters. On the other hand, we can view lines and planes just in terms of the solution sets of the cartesian coordinates x, y, z. In constrast, we need one equation to describe a plane whereas we need two equations to fix a line. In between these two viewpoints is the concept of a graph. A graph takes one or more of the Cartesian coordinates as parameter(s) and as such it can easily be thought of as a parametrization. On the other hand, a graph is given by an equation involving only cartesian coordinates so it is easy to think of it as a solution set[9]. Connecting these viewpoints and gaining a geometric appreciation for both is one of the main themes of this course. Finally, I return to the projection to the plane and we examine the connection between the dot-product based projection and the normal to the plane.

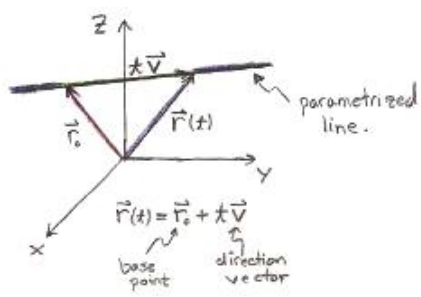
1.3.1 parametrized lines and planes

The parametric equations[10] for lines and planes are very natural if you have a proper understanding of vector addition.

Definition 1.3.1. *parametrized line.*

The line L which points in the $\vec{v}$-direction and passes through some point $\vec{r}_o$ has the natural parametrization given by

$$\boxed{\vec{r}(t) = \vec{r}_o + t\vec{v}}$$



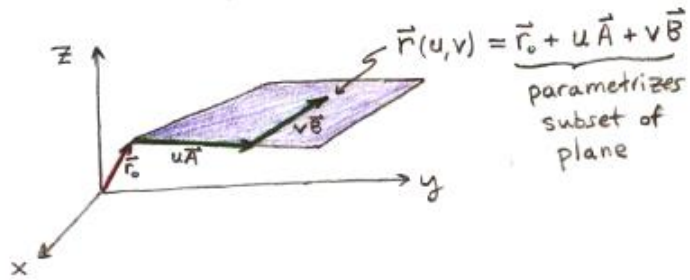
Definition 1.3.2. *parametrized plane.*

The plane S which contains the point $\vec{r}_o$ and the vectors $\vec{A}, \vec{B}$ has a natural parametrization is given by

$$\boxed{\vec{r}(u, v) = \vec{r}_o + u\vec{A} + v\vec{B}.}$$

[9] in later sections we will refer to the solution set formulation as a **level set** formulation

[10] I assume the reader has familarity with these terms from calculus II, although, I do provide general definitions later in this chapter. These are the two most basic examples



If you wish to select a subset of the line or plane above you can appropriately restrict the domain of the parameters. For example, one is often asked to find the parametrization of a line-segment from a point P to a point Q. I recommend the following approach: for $0 \leq t \leq 1$ let

$$\vec{r}(t) = P(1-t) + tQ.$$

It's easy to calculate $\vec{r}(0) = P$ and $\vec{r}(1) = Q$. This formula can also be written as

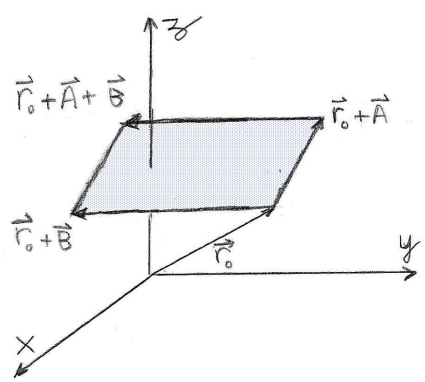
$$\vec{r}(t) = P + t(Q - P) = P + t\ [\overrightarrow{PQ}].$$

If we let t go beyond the unit-interval then we trace out the line which contains the line-segment PQ.

Example 1.3.3. Find the parametrization of a line segment which goes from $(1,3)$ **to** $(5,2)$. *We use the comment preceding this example and construct:*

$$\vec{r}(t) = (1,3) + t[(5,2) - (1,3)] = \langle 1 + 4t, 3 - t \rangle$$

On the other hand, if we wish to parametrize just the parallelogram in the plane with corners $\vec{r}_o$, $\vec{r}_o + \vec{A}$, $\vec{r}_o + \vec{B}$ and $\vec{r}_o + \vec{A} + \vec{B}$ we may limit the values of the parameters u, v to the unit square $[0,1] \times [0,1]$; that is, we demand $0 \leq u \leq 1$ and $0 \leq v \leq 1$.



Example 1.3.4. Find parametrization of plane containing the vectors $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$ and the point $(1,2,0)$. *We use the natural parametrization:*

$$\vec{r}(u,v) = (1,2,0) + u\langle 1,0,0\rangle + v\langle 0,1,0\rangle = (1+u, 2+v, 0).$$

If we allow (u,v) to trace out all of $\mathbb{R}^2$ then we will find the parametrization above covers the xy-plane. Scalar equations which capture the same are $x = 1+u, y = 2+v, z = 0$. If we restrict the parameters to $0 \leq u, v \leq v$ then the mapping $\vec{r}$ just covers $[1,2]\times[2,3]\times\{0\}$.

Example 1.3.5. Suppose $\vec{r}(u,v) = (1+u+v, 2-u, 3+v)$ with $(u,v) \in \mathbb{R}^2$ parametrizes a plane. Find the two vectors which lie in the plane and a point on its surface. *The solution is to work backwards in comparison to the last example. We wish to rip apart the formula so that we can identify $\vec{r}_o$ and $\vec{A}, \vec{B}$ for the given $\vec{r}$.*

$$\vec{r}(u,v) = (1+u+v, 2-u, 3+v) = (1,2,3) + u(1,-1,0) + v(1,0,1)$$

Identify the point $(1,2,3)$ is on the plane, in fact it $\vec{r}(0,0) = \vec{r}_o$. Moreover, the vectors $\langle 1,-1,0\rangle$ and $\langle 1,0,1\rangle$ lie on the plane. You can verify that $\langle 1,-1,0\rangle$ connects $\vec{r}(0,0) = (1,2,3)$ and $\vec{r}(1,0) = (2,1,3)$ whereas $\langle 1,1,0\rangle$ connects $\vec{r}(0,0) = (1,2,3)$ and $\vec{r}(0,1) = (2,2,4)$.

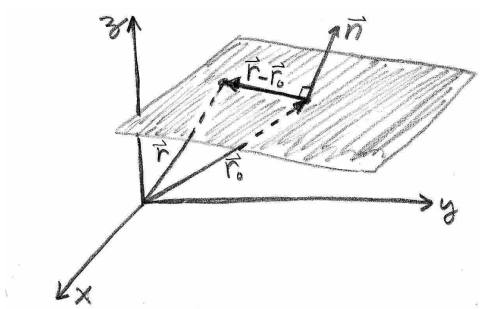
1.3.2 lines and planes as solution sets

Definition 1.3.6. *vector equation of plane.*

We say that $S \subset \mathbb{R}^3$ is a **plane** with **base point** $\vec{r}_o$ and **normal vector** $\vec{n}$ iff each $\vec{r} \in S$ satisfies

$$\boxed{(\vec{r} - \vec{r}_o)\bullet\vec{n} = 0.} \qquad \textbf{(vector equation of plane)}$$

The geometric motivation for this definition is simple enough: the normal vector is a vector which is perpendicular to all vectors in the plane. If we take the difference $\vec{r}-\vec{r}_o$ then this will be a vector which lies on the plane and consequently we must insist they are orthogonal. Here's a picture of why the definition is reasonable:



Note that the same set of points S can be given many different base points and many different normals. This reflects the fact that we can choose the base point anywhere on the plane and the

normal either above or below the plane and can be given many different lengths.

Let $\vec{r} = \langle x, y, z\rangle$ be an arbitrary point on the plane S with base point $\vec{r}_o = \langle x_o, y_o, z_o\rangle$ and normal $\vec{n} = \langle a, b, c\rangle$ then we can write our plane equation explicitly:

$$\begin{aligned}(\vec{r} - \vec{r}_o) \bullet \vec{n} = 0 \quad &\Leftrightarrow \quad \langle x - x_o, y - y_o, z - z_o\rangle \bullet \langle a, b, c\rangle = 0 \\ &\Leftrightarrow \quad \boxed{a(x - x_o) + b(y - y_o) + c(z - z_o) = 0.} \quad \textbf{(scalar equation of plane)}\end{aligned}$$

I expect you to know the vector and scalar equations for a plane. We will use both in concept and calculation.

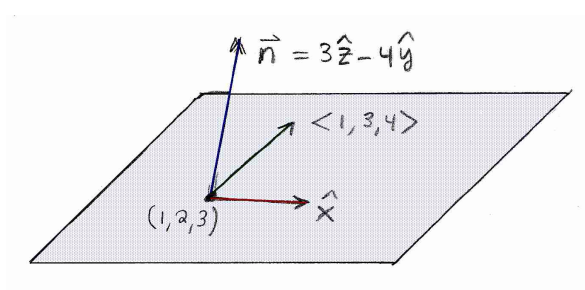
Example 1.3.7. *Suppose the plane S contains the point $(1, 2, 3)$ and the vectors $\vec{A} = \widehat{x}$ and $\vec{B} = \langle 1, 3, 4\rangle$.* **Find the scalar equation of S.**

Solution: *we are given the base point $(1, 2, 3)$ so we need only find a normal for S. Recall that the cross product of $\vec{A}$ with $\vec{B}$ gives a vector which is perpendicular to both vectors and by the geometry of $\mathbb{R}^3$ it must be colinear with the normal vector we seek. After all, one we use up two of the dimensions then there is only one left to use in $\mathbb{R}^3$. Calculate:*

$$\vec{A} \times \vec{B} = \widehat{x} \times \left[\widehat{x} + 3\,\widehat{y} + 4\,\widehat{z}\right] = \widehat{x} \times \widehat{x} + 3\,\widehat{x} \times \widehat{y} + 4\,\widehat{x} \times \widehat{z} = 3\,\widehat{z} - 4\,\widehat{y}.$$

Thus, we choose $\vec{n} = \langle 0, -4, 3\rangle$ meaning $a = 0, b = -4, c = 3$ hence

$$\boxed{-4(y - 2) + 3(z - 3) = 0.}$$



equations of intersecting planes

Given two planes they may intersect in a line. Suppose S_1, S_2 are planes which intersect along some line L then we have that L is the simulataneous solution to the equations of both planes. That is to say $(x, y, z) \in L$ iff $(x, y, z) \in S_1 \cap S_2$. In particular, if $(x_o, y_o, z_o) \in S_1 \cap S_2$ then we can write the equations of S_1, S_2 as:

$$a_1(x - x_o) + b_1(y - y_o) + c_1(z - z_o) = 0 \qquad \text{and} \qquad a_2(x - x_o) + b_2(y - y_o) + c_2(z - z_o) = 0$$

We must solve both at once to find an equation for L. Generally there is no simple formula, however if $a_1, b_1, c_1, a_2, b_2, c_2 \neq 0$ then we are free to divide by those constants. First to get the equations to match divide both by the coefficient of their $(z - z_o)$ factor,

$$\frac{a_1}{c_1}(x - x_o) + \frac{b_1}{c_1}(y - y_o) + z - z_o = 0 \qquad \text{and} \qquad \frac{a_2}{c_2}(x - x_o) + \frac{b_2}{c_2}(y - y_o) + z - z_o = 0$$

Thus, solving both equations for $z - z_o$ we find,

$$\frac{a_1}{c_1}(x - x_o) + \frac{b_1}{c_1}(y - y_o) = \frac{a_2}{c_2}(x - x_o) + \frac{b_2}{c_2}(y - y_o)$$

Multiply by c_1c_2 and rearrange to find

$$\big[a_1c_2 - a_2c_1\big](x - x_o) + \big[b_1c_2 - b_2c_1\big](y - y_o) = 0$$

Consequently,

$$\frac{x - x_o}{b_1c_2 - b_2c_1} = \frac{y - y_o}{a_2c_1 - a_1c_2} \quad \star\,.$$

Following the same algebra we can equally well solve for $x - x_o$,

$$\frac{b_1}{a_1}(y - y_o) + \frac{c_1}{a_1}(z - z_o) = \frac{b_2}{a_2}(y - y_o) + \frac{c_2}{a_2}(z - z_o)$$

and multiply by a_1a_2 to find

$$\big[b_1a_2 - b_2a_1\big](y - y_o) + \big[c_1a_2 - c_2a_1\big](z - z_o) = 0$$

Hence,

$$\frac{y - y_o}{c_1a_2 - c_2a_1} = \frac{z - z_o}{b_2a_1 - b_1a_2} \quad \star\star.$$

We combine $\star$ and $\star\star$ to obtain the **symmetric equations for the line** L

$$\boxed{\frac{x - x_o}{b_1c_2 - b_2c_1} = \frac{y - y_o}{c_2a_1 - c_1a_2} = \frac{z - z_o}{c_2a_1 - c_1a_2}}$$

If we denote $\vec{n}_1 = \langle a_1, b_1, c_1\rangle$ and $\vec{n}_2 = \langle a_2, b_2, c_2\rangle$ then recognize that

$$\vec{w} = \vec{n}_1 \times \vec{n}_2 = \langle b_1c_2 - b_2c_1, c_2a_1 - c_1a_2, c_2a_1 - c_1a_2\rangle$$

Therefore, with the notation $\vec{w} = \langle a, b, c\rangle$, the symmetric equation is simply:

$$\boxed{\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c}.}$$

We can use these equations to parametrize the line L. Let $t = \frac{x - x_o}{a}$ hence $x = x_o + at$ is the parametric equation for x, likewise, $y = y_o + bt$ and $z = z_o + ct$. We identify that $\langle a, b, c\rangle$ is precisely the direction-vector for the line L since we can group the scalar parametric equations above to obtatin the vector parametric equation below:

$$\vec{r}(t) = \ \langle x_o + at,\ y_o + bt,\ z_o + ct\rangle \ = \ \langle x_o, y_o, z_o\rangle + \ t\langle a, b, c\rangle.$$

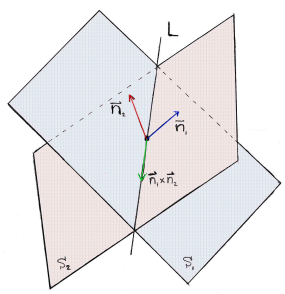
We find the following interesting geometric result:

The line of intersection for two planes has a direction vector which is colinear with the cross product of the normals of the intersected planes.

If we take a step back and analyze this by pure geometric visualization this is rediculously obvious. The line of intersection lies in both planes. Therefore, if $\vec{v}$ is the direction vector of L and $\vec{n}_1$ is the normal of plane S_1 and $\vec{n}_2$ is the normal of plane S_2 then

1. $\vec{v} \bullet \vec{n}_1 = 0$ because $\vec{v}$ lies on S_1
2. $\vec{v} \bullet \vec{n}_2 = 0$ because $\vec{v}$ lies on S_2
3. if $\vec{v}$ is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$ then it must be colinear with $\vec{n}_1 \times \vec{n}_2$.

This is an example of how geometry is sometimes easier than algebra. In fact, that is often the case, however, you must get used to both lines of logic in this course. This is the beauty of analytic geometry.



Let's examine how we can get from the parametric viewpoint to the symmetric equations. Suppose we are given the vector parametric equations for a line with base point $\vec{r}_o = (x_o, y_o, z_o)$ and direction vector $\vec{v} = \langle a, b, c \rangle$ with $a, b, c \neq 0$:

$$\vec{r}(t) = \vec{r}_o + t\vec{v} = (x_o + ta, y_o + tb, z_o + tc).$$

Further suppose we denote $\vec{r} = \langle x, y, z \rangle$ then the scalar parametric equations for the line are:

$$x = x_o + ta, \quad y = y_o + tb, \quad z = z_o + tc.$$

These can be solved for t,

$$\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c}.$$

and once more we find that the symmetric equations for a line reveal the direction of the line. Well, to be careful, we can multiply this equation by $1/k \neq 0$ and we'd still have the same solution set but $a \to ka$, $b \to kb$ and $c \to kc$ hence the direction naturally identified would be $\langle ka, kb, kc \rangle$. This is an ambiguity we always face with lines. The direction vector is not unique, unless we add further criteria. For example,

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$$

suggest the line has direction $\langle 2, 3, 4\rangle$ whereas

$$\frac{x}{-2} = \frac{y}{-3} = \frac{z}{-4}$$

suggests the line has direction $\langle -2, -3, -4\rangle$.

Finally, recall that I insisted that the intersection of the planes was a line from the outset of this discussion. There is another possibility. It could be that two planes are either parallel and have no point of intersection, or they could simply be the same plane. In both of those cases the cross product of the normals is trivial since the normals of parallel planes are colinear.

1.3.3 lines and planes as graphs

Suppose S is the plane $a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$ then if $c \neq 0$ we can solve for z to find

$$z = z_o - \frac{a}{c}(x - x_o) - \frac{b}{c}(y - y_o).$$

If we define $f(x, y) = z_o - \frac{a}{c}(x - x_o) - \frac{b}{c}(y - y_o)$ then we can write S as the graph $z = f(x, y)$.

Definition 1.3.8. *graphs of $f(x, y), g(x, z)$ or $h(y, z)$.*

$$\begin{aligned} graph(f) &= \{(x, y, f(x, y)) \mid (x, y) \in dom(f)\} \\ graph(g) &= \{(x, g(x, z), z) \mid (x, z) \in dom(g)\} \\ graph(h) &= \{(h(y, z), y, z) \mid (y, z) \in dom(h)\} \end{aligned}$$

clearly $S = graph(f)$ in this context. You could also write S as the graph $y = g(x, z)$ or $x = h(y, z)$ provided $b \neq 0$ and $a \neq 0$ respective. I hope you can find the formulas for g or h. These graphs provide parametrizations as follows, once more consider the case $c \neq 0$, let

$$x = u, \quad y = v, \quad z = f(u, v)$$

Equivalently,

$$\boxed{\vec{r}(u, v) = \langle u, v, f(u, v)\rangle.}$$

In this way we find a natural parametrization of a graph. Likewise, if $a \neq 0$ or $b \neq 0$ then

$$\vec{r}(u, v) = \langle u, g(u, v), v)\rangle \qquad \text{or} \qquad \vec{r}(u, v) = \langle h(u, v), u, v\rangle$$

provide natural parametrizations. Parametrizations created in these ways are said to be **induced** from the graphs g, h respective.

Writing the line as a graph requires us to solve the symmetric equations for two of the cartesian coordinates in terms of the remaining coordinate. For example, solve for y, z in terms of x:

$$\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c} \quad \Rightarrow \quad a(y - y_o) = b(x - x_o) \;\; \& \;\; a(z - z_o) = c(x - x_o).$$

hence,

$$y = \underbrace{y_o + \frac{b}{a}(x - x_o)}_{\textit{let this be } h(x)} \quad \& \quad z = \underbrace{z_o + \frac{c}{a}(x - x_o)}_{\textit{let this be } g(x)}$$

Define $f(x) = (h(x), g(x))$ then $\boxed{graph(f) = \{(x, f(x)) \mid x \in \mathbb{R}\}}$ and it is clear that $graph(f) = L$. We say $f : \mathbb{R} \to \mathbb{R}^2$ is a $\mathbb{R}^2$-valued function of a real variable. Some texts call such functions mappings whereas they inist that *functions* are real-valued. I make no such restriction in these notes. In any event, there is a natural parametrization which is induced from the graph for L, use $x = t$ hence[11]

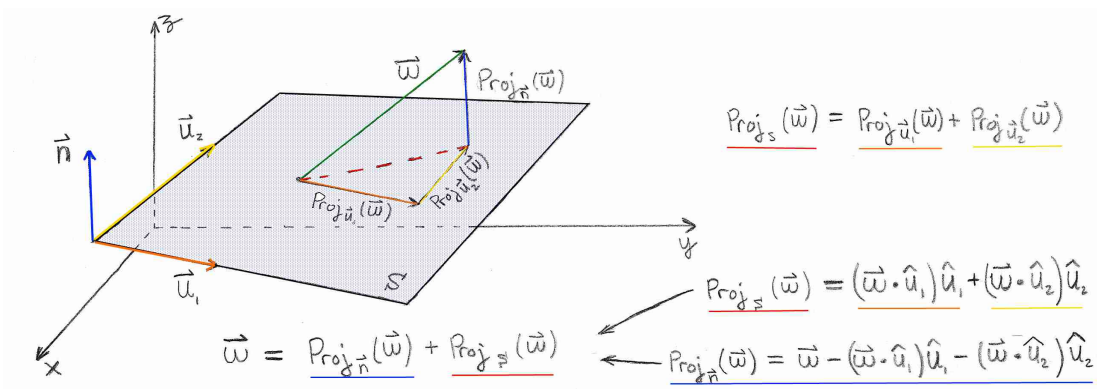
$$\boxed{\vec{r}(t) \;=\; \langle t, f(t)\rangle \;=\; \langle t, g(t), h(t)\rangle}$$

parametrizes L. We could also solve for y or z provided $b \neq 0$ or $c \neq 0$. I leave those to the reader.

1.3.4 on projections onto a plane

There are two ways to look at projections onto a plane. Let S be the plane of interest. If we have a pair of orthogonal unit-vectors $\widehat{u}_1, \widehat{u}_2$ on S then we can project a general vector $\vec{v}$ to the plane by

$$Proj_S(\vec{v}) = (\vec{v} \bullet \widehat{u}_1)\widehat{u}_1 + (\vec{v} \bullet \widehat{u}_2)\widehat{u}_2.$$

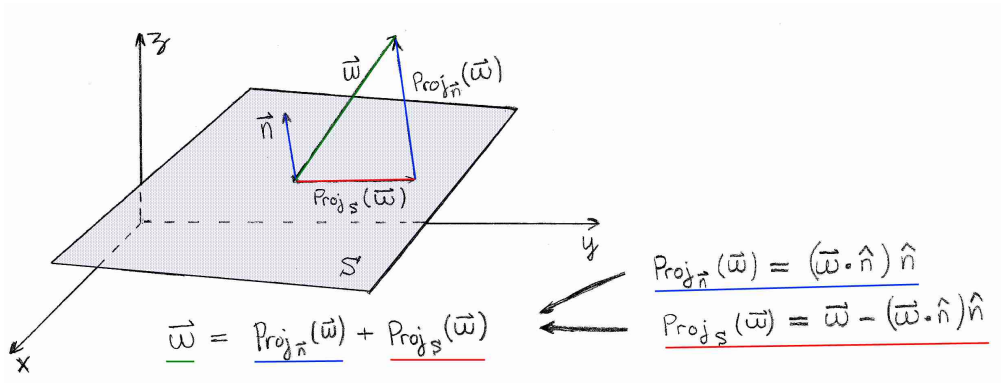


In three dimensions we can calculate this projection via subtracting off the piece of the vector which is in the normal direction. If $\vec{n}$ is the normal to S then I claim

$$Proj_S(\vec{v}) = \vec{v} - Proj_{\vec{n}}(\vec{v})$$

Here is a picture for this formula:

[11] we use the notation $((a, b), c) = (a, (b, c)) = (a, b, c)$ which implicits a bijective correspondance between these technically distinct triples, this is a common notation.



Let's see why in $n = 3$ these formulas are equivalent. Construct the normal corresponding to the unit vectors in the usual manner: $\vec{n} = \widehat{u}_1 \times \widehat{u}_2$. Orthogonality of the unit vectors implies $\theta = \frac{\pi}{2}$ hence $||\widehat{u}_1 \times \widehat{u}_2|| = 1$ and it follows $\vec{n} = \widehat{n}$. Let us define

$$\overline{Proj}_S(\vec{v}) = \vec{v} - (\vec{v} \bullet \widehat{n})\widehat{n} \quad \star$$

Observe this formula produces a vector on the plane,

$$\overline{Proj}_S(\vec{v}) \bullet \widehat{n} = \vec{v} \bullet \widehat{n} - (\vec{v} \bullet \widehat{n})\widehat{n} \bullet \widehat{n} = \vec{v} \bullet \widehat{n} - \vec{v} \bullet \widehat{n} = 0.$$

Notice that $\widehat{n} \bullet \widehat{u}_1 = 0$ and $\widehat{n} \bullet \widehat{u}_2 = 0$. Take the dot-product of $\star$ with $\widehat{u}_1$ and $\widehat{u}_2$ to obtain,

$$\overline{Proj}_S(\vec{v}) \bullet \widehat{u}_1 = \vec{v} \bullet \widehat{u}_1 \qquad \text{and} \qquad \overline{Proj}_S(\vec{v}) \bullet \widehat{u}_2 = \vec{v} \bullet \widehat{u}_2$$

It follows that $\overline{Proj}_S(\vec{v}) = \big(\vec{v} \bullet \widehat{u}_1\big)\widehat{u}_1 + \big(\vec{v} \bullet \widehat{u}_2\big)\widehat{u}_2$. The formulas agree. They both produce the same projection onto the plane. If we attach $\vec{v}$ to the plane at some point $P \in S$ then $Proj_S(\vec{v})$ attached to P will point to the point on the plane S which is closest to the end of the vector $\vec{v}$.

Perhaps the example that follows will help you understand the discussion above.

Example 1.3.9. *Let S be the plane through $(0, 0, 1)$ with normal $\vec{n} = \langle 1, 0, 1\rangle$. Notice that $\vec{u}_1 = \langle 1, 0, -1\rangle$ and $\vec{u}_2 = \langle 0, 1, 0\rangle$ are orthogonal as $\vec{u}_1 \bullet \vec{u}_2 = 0$ and they are both on S as $\vec{n} \bullet \vec{u}_1 = 0$ and $\vec{n} \bullet \vec{u}_2 = 0$. Normalize the vectors,*

$$\widehat{u}_1 = \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle, \qquad \widehat{u}_2 = \langle 0, 1, 0\rangle \qquad \widehat{n} = \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle$$

Let $\vec{v} = \langle a, b, c\rangle$. Calculate,

$$\begin{aligned} Proj_S(\vec{v}) &= \big(\vec{v} \bullet \widehat{u}_1\big)\widehat{u}_1 + \big(\vec{v} \bullet \widehat{u}_2\big)\widehat{u}_2 \\ &= \frac{1}{2}\big(a - c\big)\langle 1, 0, -1\rangle + b\langle 0, 1, 0\rangle \\ &= \langle\ \frac{1}{2}(a - c),\ b,\ \frac{1}{2}(c - a)\ \rangle. \end{aligned}$$

On the other hand, the normal formula says,

$$\begin{aligned} Proj_S(\vec{v}) &= \vec{v} - (\vec{v} \bullet \widehat{n})\widehat{n} \\ &= \langle a, b, c\rangle - \frac{1}{2}(a+c)\langle 1, 0, 1\rangle \\ &= \langle a - \frac{1}{2}(a+c), b, c - \frac{1}{2}(a+c)\rangle \\ &= \langle\ \frac{1}{2}(a-c),\ b,\ \frac{1}{2}(c-a)\ \rangle. \end{aligned}$$

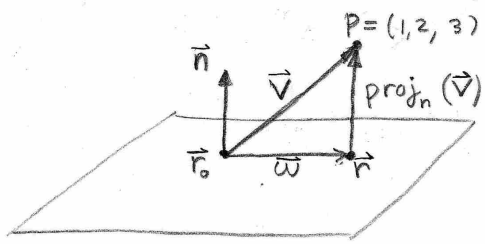
We typically use the normal formulation of the projection since it's easier to find a normal than it is to find a pair of orthonormal vectors on the plane. That said, the orthonormal projection formula naturally generalizes to higher dimensional studies. We discuss applications of such formulas in linear algebra. It is the math behind least squares data fitting and Fourier analysis.

Example 1.3.10. .

$\boxed{E17}$ If we have the plane $x - y + 10z = 10$ then what point on plane is closest to $P = (1,2,3)$? Lets pick a point on the plane. Choose,

$$x = 0,\ y = 0 \Rightarrow 10z = 10 \therefore z = 1 \Rightarrow \vec{r}_0 = (0,0,1)$$

the normal to the plane is $\vec{n} = \langle 1, -1, 10\rangle$ so we can draw a schematic picture of what we have and want



$\vec{v} = P - r_0 = \langle 1, 2, 2\rangle$

$\vec{r}$ = the desired point

$\vec{r} = \vec{r}_0 + \vec{w} = \vec{P} - proj_{\vec{n}}(\vec{v})$

notice $|\vec{n}| = \sqrt{102}$

$$\begin{aligned} \vec{r} &= \vec{P} - proj_{\vec{n}}(\vec{v}) \\ &= (1,2,3) - \frac{\vec{n}\bullet\vec{v}}{|\vec{n}|^2}\vec{n} \\ &= \langle 1,2,3\rangle - \frac{1}{102}(\langle 1,-1,10\rangle \bullet \langle 1,2,2\rangle)\langle 1,-1,10\rangle \\ &= \langle 1,2,3\rangle - (19/102)\langle 1,-1,10\rangle \\ &= \frac{1}{102}\langle 102-19,\ 204+19,\ 306-190\rangle \\ &= \frac{1}{102}\langle 83, 223, 116\rangle \ \therefore\ \boxed{(83/102,\ 223/102,\ 116/102)\ \text{closest point.}} \end{aligned}$$

As a check is $\vec{r}$ on the plane? $\frac{83}{102} - \frac{223}{102} + \frac{1160}{102} = \frac{1020}{102} = 10$, yep.

1.3.5 additional examples

Example 1.3.11. .

$\boxed{E10}$ find parametric eq^{n}'s of line with direction $\langle 1, 0, 1\rangle = \vec{v}$ and initial point $\vec{r}_0 = \langle \pi, \pi, \pi\rangle$. Well $r(t) = r_0 + tv$ so $\vec{r}(t) = \langle \pi, \pi, \pi\rangle + t\langle 1, 0, 1\rangle = \langle \pi + t, \pi, \pi + t\rangle$

$$\boxed{\begin{aligned} x &= \pi + t \\ y &= \pi \\ z &= \pi + t \end{aligned}}$$

Example 1.3.12. .

$\boxed{E11}$ Suppose $L: \vec{r}(t) = \langle 3 - t, t + 5, 2t + 8\rangle$ then

$$\left.\begin{aligned} x &= 3 - t \quad \Rightarrow t = 3 - x \\ y &= t + 5 \quad \Rightarrow t = 5 - y \\ z &= 2t + 8 \quad \Rightarrow t = \tfrac{1}{2}(z - 8) \end{aligned}\right\} \Rightarrow t = \boxed{3 - x = 5 - y = \tfrac{1}{2}(z - 8)}$$

Ok, to be more picky we should write $\frac{x-3}{-1} = \frac{y-5}{-1} = \frac{z-8}{2}$ then we can identify $\vec{v} = \langle -1, -1, 2\rangle$ and $\vec{r}_0 = \langle 3, 5, 8\rangle$ of course those facts are obvious from $\vec{r}(t)$ to begin.

Example 1.3.13. .

$\boxed{E13}$ Show $L_1: \vec{r}_1(t) = \langle 3 + t, 2 + t, 1 + t\rangle$ and $\vec{r}_2(t) = \langle 3t, 3t, 3t + 6\rangle$ are parallel. We'll notice

$$\vec{r}_1(t) = \langle 3, 2, 1\rangle + t\langle 1, 1, 1\rangle = \vec{r}_1(0) + t\vec{v}_1$$
$$\vec{r}_2(t) = \langle 0, 0, 6\rangle + t\langle 3, 3, 3\rangle = \vec{r}_2(0) + t\vec{v}_2$$

Clearly $\vec{v}_2 = 3\vec{v}_1$ thus L_1 & L_2 are parallell.

Example 1.3.14. .

Find eq^{n} of line through $(-2, 4, 10)$ and parallel to $\langle 3, 1, -8\rangle$,

$$\vec{r}(t) = \langle -2, 4, 10\rangle + t\langle 3, 1, -8\rangle = \boxed{\langle 3t - 2, t + 4, -8t + 10\rangle = \vec{r}(t)}$$

A.k.a. $\underline{x = 3t - 2,\ y = t + 4 \text{ and } z = -8t + 10}$.

Example 1.3.15. .

$\boxed{E14}$ Show $L_1 : \vec{r}_1(t) = \langle 1+t,\ 3t-2,\ 4-t \rangle$ and $L_2 : \vec{r}_2(t) = \langle 2t,\ 3+t,\ 4t-3 \rangle$ do not intersect. To be fair we should <u>not</u> check if $\exists t$ such that $\vec{r}_1(t) = \vec{r}_2(t)$, because they might intersect at different t. So introduce s for $\vec{r}_2$. Examine $\vec{r}_1(t) = \vec{r}_2(s)$. This gives

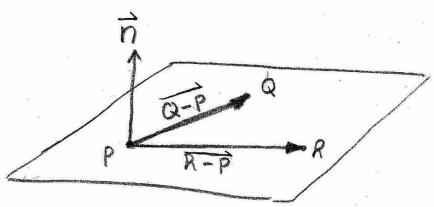
$$\begin{aligned} 1+t &= 2s \\ 3t-2 &= 3+s \\ 4-t &= 4s-3 \end{aligned} \quad \begin{aligned} &\to 3t = 6s-3 = 3+s+5 \Rightarrow 5s = 11 \therefore \underline{s = 11/5} \\ &\to 3t = 12-12s+9 = 3+s+5 \Rightarrow 13s = 13 \therefore \underline{s=1} \end{aligned}$$

Therefore since $1 \neq 11/5$ it is seen these eq^{n}'s have no sol^{n} hence $\nexists\, s,t$ such that $\vec{r}_1(t) = \vec{r}_2(s)$ $\therefore$ L_1 & L_2 do not intersect.

Moreover, these lines are skew since L_1 has direction vector $\langle 1, 3, -1 \rangle$ whereas L_2 has direction vector $\langle 2, 1, 4 \rangle$. It is simple to show there does not exist a $k \in \mathbb{R}$ such that $\langle 1, 3, -1 \rangle = k \langle 2, 1, 4 \rangle$. (Thus L_1 and L_2 point in different directions)

Example 1.3.16. .

$\boxed{E15}$ Find eq^{n}'s of plane possessing points $(0,0,0)$, $(0,1,0)$ and $(0,1,1)$. Let me draw a picture, define



$P = (0,0,0)$
$Q = (0,1,0) = \hat{j}$
$R = (1,1,1) = \hat{i} + \hat{j} + \hat{k}$

$Q - P = Q$.
$R - P = R$.

$$\vec{n} = (Q-P)\times(R-P) = \hat{j} \times (\hat{i} + \hat{j} + \hat{k}) = \hat{j}\times\hat{i} + \hat{j}\times\hat{k} = \langle 1, 0, -1 \rangle.$$

Thus the eq^{n}'s of the plane are (using $\vec{P} = \vec{r}_0$) $\boxed{x - z = 0}$

Example 1.3.17. .

$\boxed{E16}$ Find plane through $(1,2,3)$ with normal parallel to the intersection line of the planes

$$x + y + z = 10 \quad \text{and} \quad 2x + 3y + z = 20.$$

<u>General Principle</u>: intersection is where both eq^{n}'s are true, to quantify it pick something in common and equate, here z is a natural choice,

$$z = 10 - x - y = 20 - 2x - 3y$$

$$\Rightarrow 2y + x = 20 - 10 \Rightarrow x = 10 - 2y.$$

So we can parametrize the line by the y-coordinate,

$$\vec{r}(y) = \langle 10 - 2y,\ y,\ 10 - x - y \rangle \quad , x = 10 - 2y$$
$$= \langle 10 - 2y,\ y,\ y \rangle$$
$$= \langle 10, 0, 0 \rangle + y\langle -2, 1, 1 \rangle.$$

The direction of the line of intersection is $\langle -2, 1, 1 \rangle$ our plane is thus

$$\boxed{-2(x-1) + (y-2) + (z-3) = 0}$$

<u>Remark</u>: there is a shorter more efficient sol^{n}. Just pick two points on line, don't try to find $\vec{r}(y)$.

Example 1.3.18. .

Find line through $(2,1,0)$ and perpendicular to $\vec{A} = \hat{i} + \hat{j}$ and $\vec{B} = \hat{j} + \hat{k}$. Recall $\vec{A} \times \vec{B}$ is $\perp$ to both $\vec{A}$ & $\vec{B}$.

Choose direction vector $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k} = \langle 1, -1, 1 \rangle.$

As a check, notice $\vec{A} \cdot \langle 1, -1, 1 \rangle = 0$ and $\vec{B} \cdot \langle 1, -1, 1 \rangle = 0$.

We find

$$\boxed{\vec{r}(t) = \langle 2, 1, 0 \rangle + t\langle 1, -1, 1 \rangle}$$

No need to bother with symmetric equations.

Example 1.3.19. .

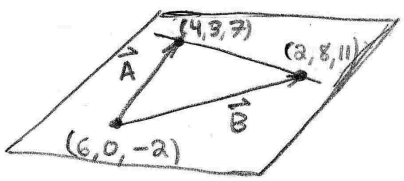
find line through $(1,0,6)$ and perpendicular to the plane $x+3y+z=5$. Notice the normal to this plane is $\vec{N}=\langle 1,3,1\rangle$. To say line is perpendicular to a plane is to say the line is parallel to the normal.

$$\vec{r}(t) = \langle 1,0,6\rangle + t\langle 1,3,1\rangle = \boxed{\langle t+1,\ 3t,\ t+6\rangle = \vec{r}(t)}$$

parametric eq$^{\underline{n}}$'s are $x=t+1$, $y=3t$ and $z=t+6$.

<u>WARNING</u>: in my notes & lecture $z=z$. Sorry, but not so sorry to change. ☺.

Example 1.3.20. .



Given a point $(6,0,-2)$ and a line $x=4-2t$, $y=3+5t$, $z=7+4t$ contained in the plane find the equation for this plane. Begin with picture. Pick two points on line, lets use the points on the line where $t=0$ and $t=1$, $\vec{r}(0)=(4,3,7)$, $\vec{r}(1)=(2,8,11)$

$$\vec{A} = \langle 4-6,\ 3-0,\ 7-(-2)\rangle = \langle -2,\ 3,\ 9\rangle$$
$$\vec{B} = \langle 2-6,\ 8-0,\ 11-(-2)\rangle = \langle -4,\ 8,\ 13\rangle$$

We want the normal vector $\perp$ to both $\vec{A}$ & $\vec{B}$ thus we

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 9 \\ -4 & 8 & 13 \end{vmatrix} = \langle 39-72,\ -(-26+36),\ -16+12\rangle$$

Thus $\vec{A}\times\vec{B} = \langle -33,\ -10,\ -4\rangle$ is a normal for the plane. Thus, the equation for plane is $\boxed{-33(x-6)-10y-4(z+2)=0}$

1.4 curves

A curve is a one-dimensional subset of some space. There are at least three common, but distinct, ways to frame the mathematics of a curve. These viewpoints were already explored in the previous section but I list them once more: we can describe a curve:

1. as a path, that is as a parametrized curve.
2. as a level curve, also known as a solution set.
3. as a graph.

I expect you master all three viewpoints in the two-dimensional context. However, for three or more dimensions we primarily use the parametric viewpoint in this course. Exceptions to this rule are fairly rare: the occasional homework problem where you are asked to find the curve of intersection for two surfaces, or the symmetric equations for a line. In contrast, the parametric decription of a curve in three dimensions is far more natural. Do you want to describe a curve as where two surfaces intersect or would you rather describe a curve as a set of points formed by pasting a copy of the real line through your space? I much prefer the parametric view.

Definition 1.4.1. *vector-valued functions, curves and paths.*

A vector valued function of a real variable is an assignment of a vector for each real number in some domain. It's a mapping $t \mapsto \vec{f}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t) \rangle$ for each $t \in J \subset \mathbb{R}$. We say $f_j : J \subseteq \mathbb{R} \to \mathbb{R}$ is the j-th component function of $\vec{f}$. Let $C = \vec{f}(J)$ then C is said to be a **curve which is parametrized by** $\vec{f}$. We can also say that $t \mapsto \vec{f}(t)$ is a **path** in $\mathbb{R}^n$. Equivalently, but not so usefully, we can write the scalar parametric equations for C above as

$$x_1 = f_1(t), \quad x_2 = f_2(t), \ \ldots, \ x_n = f_n(t)$$

for all $t \in J$.

When we define a parametrization of a curve it is important to give the <u>formula</u> for the path **and** the <u>domain</u> of the parameter. Note that when I say the word *curve* I mean for us to think about some set of points, whereas when I say the word *path* I mean to refer to the particular mapping whose image is a curve. We may cover a particular curve with infinitely many different paths.

1.4.1 curves in two-dimensional space

We have several viewpoints to consider. Graphs, parametrized curves and level sets.

graphs in the plane

Let's begin by reminding ourselves of the definition of a graph:

Definition 1.4.2. *Graph of a function.*

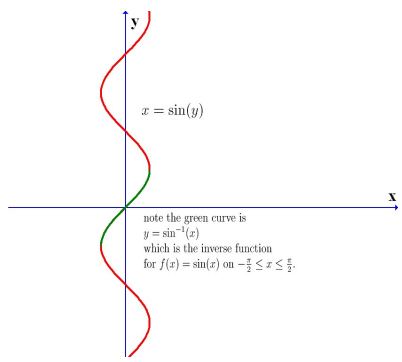
Let $f : dom(f) \to \mathbb{R}$ be a function then

$$graph(f) = \{(x, f(x)) \mid x \in dom(f)\}.$$

We know this is quite restrictive. We must satisfy the vertical line test if we say our curve is the graph of a function.

Example 1.4.3. *To form a circle centered at the origin of radius R we need to glue together two graphs. In particular we solve the equation $x^2 + y^2 = R^2$ for $y = \sqrt{R^2 - x^2}$ or $y = -\sqrt{R^2 - x^2}$. Let $f(x) = \sqrt{R^2 - x^2}$ and $g(x) = -\sqrt{R^2 - x^2}$ then we find $graph(f) \cup graph(g)$ gives us the whole circle.*

Example 1.4.4. *On the other hand, if we wish to describe the set of all points such that $\sin(y) = x$ we also face a similar difficulty if we insist on functions having independent variable x. Naturally, if we allow for functions with y as the independent variable then $f(y) = \sin(y)$ has graph $graph(f) = \{(f(y), y) \mid y \in dom(f)\}$. You might wonder, is this correct? I would say a better question is, "is this allowed?". Different books are more or less imaginative about what is permissible as a function. This much we can say, if a shape fails both the vertical and horizontal line tests then it is not the graph of a single function of x or y.*



Example 1.4.5. *Let $f(x) = mx + b$ for some constants m, b then $y = f(x)$ is the line with slope m and y-intercept b.*

level curves in two-dimensions

Level curves are amazing. The full calculus of level curves is only partially appreciated even in calculus III, but trust me, this viewpoint has many advantages as you learn more. For now it's simple enough:

Definition 1.4.6. *Level Curve.*

A level curve is given by a function of two variables $F : dom(F) \subseteq \mathbb{R}^2 \to \mathbb{R}$ and a constant k. In particular, the set of all $(x, y) \in \mathbb{R}^2$ such that $F(x, y) = k$ is called the level-set of F, but more commonly we just say $F(x, y) = k$ is a level curve.

In an algebra class you might have called this the "graph of an equation", but that terminology is dead to us now. For us, it is a level curve. Moreover, for a particular set of points $C \subseteq \mathbb{R}^2$ we can find more than one function F which produces C as a level set. Unlike functions, for a particular curve there is not just one function which returns that curve. This means that it might be important to give both the level-function F and the level k to specify a level curve $F(x, y) = k$.

Example 1.4.7. *A circle of radius R centered at the origin is a level curve $F(x, y) = R^2$ where $F(x, y) = x^2 + y^2$. We call F the level function (of two variables).*

Example 1.4.8. *To describe $\sin(y) = x$ as a level curve we simply write $\sin(y) - x = 0$ and identify the level function is $F(x, y) = \sin(y) - x$ and in this case $k = 0$. Notice, we could just as well say it is the level curve $G(x, y) = 1$ where $G(x, y) = x - \sin(y) + 1$.*

Note once more this type of ambiguity is one distinction of the level curve langauge, in constrast, the graph $graph(f)$ of a function $y = f(x)$ and the function f are interchangeable. Some mathematicians insist the rule $x \mapsto f(x)$ defines a function whereas others insist that a function is a set of pairs $(x, f(x))$. I prefer the mapping rule because it's how I think about functions in general whereas the idea of a graph is much less useful in general.

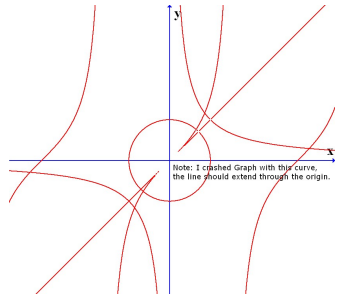
Example 1.4.9. *A line with slope m and y-intercept b can be described by $F(x, y) = mx + b - y = 0$. Alternatively, a line with x-intercept x_o and y-intercept y_o can be described as the level curve $G(x, y) = \frac{x}{x_o} + \frac{y}{y_o} = 1$.*

Example 1.4.10. *Level curves need not be simple things. They can be lots of simple things glued together in one grand equation:*

$$F(x, y) = (x - y)(x^2 + y^2 - 1)(xy - 1)(y - \tan(x)) = 0.$$

Solutions to the equation above include the line $y = x$, the unit circle $x^2 + y^2 = 1$, the tilted-hyperbola known more commonly as the reciprocal function $y = \frac{1}{x}$ and finally the graph of the tangent. Some of these intersect, others are disconnected from each other.

It is sometimes helpful to use software to plot equations. However, we must be careful since they are not as reliable as you might suppose. The example above is not too complicated but look what happens with Graph:



Theorem 1.4.11. *any graph of a function can be wriiten as a level curve.*

If $y = f(x)$ is the graph of a function then we can write $F(x, y) = f(x) - y = 0$ hence the graph $y = f(x)$ is also a level curve.

Not much of a theorem. But, it's true. The converse is not true without a lot of qualification. I'll state that theorem (it's called the implicit function theorem) in a future chapter after we've studied partial differentiation.

parametrized curves in two-dimensions

Example 1.4.12. *Suppose $a, b\rangle 0$ and $h, k \in \mathbb{R}$. The parametrization*

$$\vec{r}(t) = \langle h + a\cos(t), k + b\sin(t)\rangle$$

for $t \in [0, 2\pi]$ coveres and the ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Example 1.4.13. *Suppose $a, b\rangle 0$ and $h, k \in \mathbb{R}$. The parametrization*

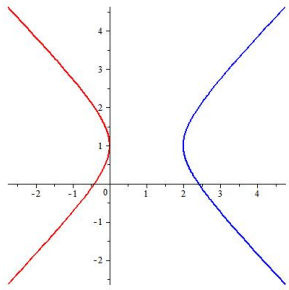
$$\vec{r}_1(t) = \langle h + a\cosh(t), k + b\sinh(t)\rangle$$

for $t \in \mathbb{R}$ coveres one branch of the hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

Note $x = h + a\cosh(t)$ implies $\frac{x-h}{a} = \cosh(t) \geq 1$ therefore it follows $x \geq h + a$. We've covered the right branch. If we wish to cover the left branch of this hyperbola then use:

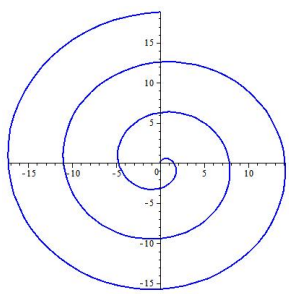
$$\vec{r}_2(t) = \langle h - a\cosh(t), k + b\sinh(t)\rangle.$$



Example 1.4.14. *A spiral can be thought of as a sort of circle with a variable radius. With that in mind I write: for $t \geq 0$,*

$$\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle$$

to give a spiral whose "radius" is proportional to the angle t subtended from $t = 0$.

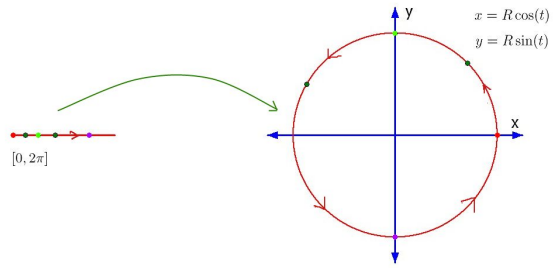


Finding the parametric equations for a curve does require a certain amount of creativity. However, it's almost always some slight twist on the examples I give in this section. The remaining examples I also give in calculus II, I add some detail to emphasize how the parametrization matches the already known identities of certain curves and I add pictures which emphasize the idea that the parametrization pastes a line into $\mathbb{R}^2$.

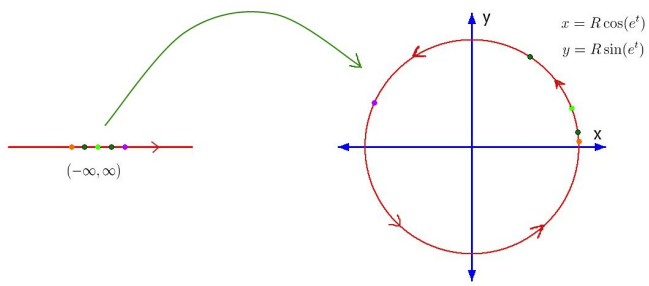
Example 1.4.15. *Let $x = R\cos(t)$ and $y = R\sin(t)$ for $t \in [0, 2\pi]$. This is a parametrization of the circle of radius R centered at the origin. We can check this by substituting the equations back into our standard Cartesian equation for the circle:*

$$x^2 + y^2 = (R\cos(t))^2 + (R\sin(t))^2 = R^2(\cos^2(t) + \sin^2(t))$$

Recall that $\cos^2(t) + \sin^2(t) = 1$ therefore, $x(t)^2 + y(t)^2 = R^2$ for each $t \in [0, 2\pi]$. This shows that the parametric equations do return the set of points which we call a circle of radius R. Moreover, we can identify the parameter in this case as the standard angle from standard polar coordinates.



Example 1.4.16. *Let $x = R\cos(e^t)$ and $y = R\sin(e^t)$ for $t \in \mathbb{R}$. We again cover the circle at t varies since it is still true that $(R\cos(e^t))^2 + (R\sin(e^t))^2 = R^2(\cos^2(e^t) + \sin(e^t)) = R^2$. However, since range$(e^t) = [1, \infty)$ it is clear that we will actually wrap around the circle infinitly many times. The parametrizations from this example and the last do cover the same set, but they are radically different parametrizations: the last example winds around the circle the circle ∞-ly many times.*

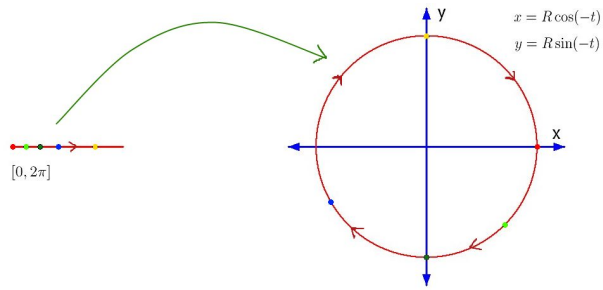


Example 1.4.17. *Let $x = R\cos(-t)$ and $y = R\sin(-t)$ for $t \in [0, 2\pi]$. This is a parametrization of the circle of radius R centered at the origin. We can check this by substituting the equations back into our standard Cartesian equation for the circle:*

$$x^2 + y^2 = (R\cos(-t))^2 + (R\sin(-t))^2 = R^2(\cos^2(-t) + \sin^2(-t))$$

Recall that $\cos^2(-t)+\sin^2(-t) = 1$ therefore, $x(t)^2+y(t)^2 = R^2$ for each $t \in [0, 2\pi]$. This shows that the parametric equations do return the set of points which we call a circle of radius R. Moreover, we can identify the parameter an angle measured CW[12] from the positive x-axis. In contrast, the standard polar coordinate angle is measured CCW from the postive x-axis. Note that in this example we cover the circle just once, but the direction of the curve is opposite that of Example 1.4.15.

[12]CW is an abbreviation for ClockWise,whereas CCW is an abbreviation for CounterClockWise.



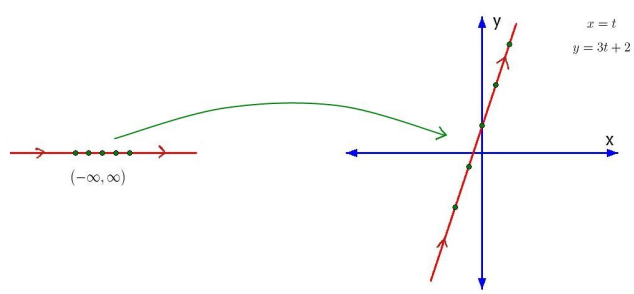
The idea of directionality is not at all evident from Cartesian equations for a curve. Given a graph $y = f(x)$ or a level-curve $F(x, y) = k$ there is no intrinsic concept of direction ascribed to the curve. For example, if I ask you whether $x^2 + y^2 = R^2$ goes CW or CCW then you ought not have an answer. I suppose you could ad-hoc pick a direction, but it wouldn't be natural. This means that if we care about giving a direction to a curve we need the concept of the parametrized curve. We can use the ordering of the real line to induce an ordering on the curve.

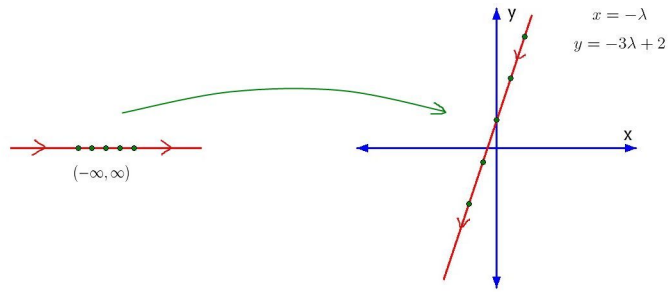
Definition 1.4.18. *oriented curve.*

Suppose $f, g : J \subseteq \mathbb{R} \to \mathbb{R}$ are $1-1$ functions. We say the set $\{(f(t), g(t)) \mid t \in J\}$ is an **oriented curve** and say $t \to (f(t), g(t))$ is a consistently oriented **path** which covers C. If $J = [a, b]$ and $(f(a), g(a)) = p$ and $(f(b), g(b)) = q$ then we can say that C is a curve from to p to q.

I often illustrate the orientation of a curve by drawing little arrows along the curve to indicate the direction. Furthermore, in my previous definition of parametrization I did not insist the parametric functions were $1-1$, this means that those parametrizations could reverse direction and go back and forth along a given curve. What is meant by the terms "path", "curve" and "parametric equations" may differ from text to text so you have to keep a bit of an open mind and try to let context be your guide when ambguity occurs. I will try to be uniform in my langauge within this course.

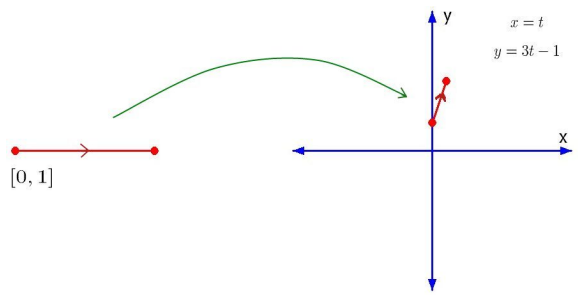
Example 1.4.19. *The line $y = 3x + 2$ can be parametrized by $x = t$ and $y = 3t + 2$ for $t \in \mathbb{R}$. This induces an orientation which goes from left to right for the line. On the other hand, if we use $x = -\lambda$ and $y = -3\lambda + 2$ then as λ increases we travel from right to left on the curve. So the λ-equations give the line the opposite orientation.*

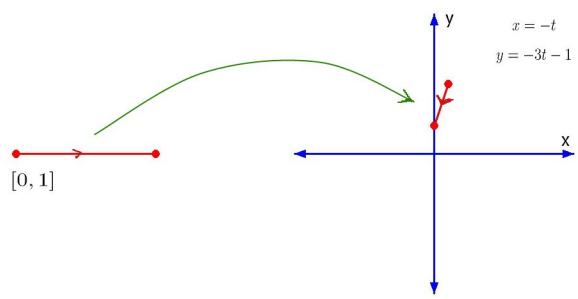




To reverse orientation for $x = f(t), y = g(t)$ for $t \in J = [a, b]$ one may simply replace t by $-t$ in the parametric equations, this gives new equations which will cover the same curve via $x = f(-t), y = g(-t)$ for $t \in [-a, -b]$.

Example 1.4.20. *The line-segment from* $(0, -1)$ *to* $(1, 2)$ *can be parametrized by* $x = t$ *and* $y = 3t - 1$ *for* $0 \leq t \leq 1$. *On the other hand, the line-segment from* $(1, 2)$ *to* $(0, -1)$ *is parametrized by* $x = -t, y = -3t - 1$ *for* $-1 \leq t \leq 0$.





The other method to graph parametric curves is simply to start plugging in values for the parameter and assemble a table of values to plot. I have illustrated that in part by plotting the green dots in the domain of the parameter together with their images on the curve. Those dots are the results of plugging in the parameter to find corresponding values for x, y. I don't find that is a very reliable approach in the same way I find plugging in values to $f(x)$ provides a very good plot of $y = f(x)$. That sort of brute-force approach is more appropriate for a CAS system. There are many excellent tools for plotting parametric curves, hopefully I will have some posted on the course website. In addition, the possibility of animation gives us an even more exciting method for visualization of the

time-evolution of a parametric curve. In the next chapter we connect the parametric viewpoint with physics and such an animation actually represents the physical motion of some object. My focus in the remainder of this chapter is almost purely algebraic, I could draw pictures to explain, but I wanted the notes to show you that the pictures are not necessary when you understand the algebraic process. That said, the best approach is to do some combination of algebraic manipulation/figuring and graphical reasoning.

converting to and from the parametric viewpoint in 2D

Let's change gears a bit, we've seen that parametric equations for curves give us a new method to describe particular geometric concepts such as orientability or multiple covering. Without the introduction of the parametric concept these geometric ideas are not so easy to describe. That said, I now turn to the question of how to connect parametric descriptions with Cartesian descriptions of a curve. We'd like to understand how to go both ways if possible:

1. how can we find the Cartesian form for a given parametric curve?

2. how can we find a parametrization of a given Cartesian curve?

In case (2.) we mean to include the ideas of level curves and graphs. It turns out that both questions can be quite challenging for certain examples. However, in other cases, not so much: for example any graph $y = f(x)$ is easily recast as the set of parametric equations $x = t$ and $y = f(t)$ for $t \in dom(f)$. For the standard graph of a function we use x as the parameter.

1.4.2 how can we find the Cartesian form for a given parametric curve?

Example 1.4.21. *What curve has parametric equations $x = t$ for $y = t^2$ for $t \in \mathbb{R}$? To find Cartesian equation we eliminate the parameter (when possible)*

$$t^2 = x^2 = y \quad \Rightarrow \quad y = x^2$$

Thus the Cartesian form of the given parametrized curve is simply $y = x^2$.

Example 1.4.22. *Example 15.2.2: Find parametric equations to describe the graph $y = \sqrt{x+3}$ for $0 \leq x < \infty$. We can use $x = t^2$ and $y = \sqrt{t^2+3}$ for $t \in \mathbb{R}$. Or, we could use $x = \lambda$ and $y = \sqrt{\lambda+3}$ for $\lambda \in [0, \infty)$.*

Example 1.4.23. *What curve has parametric equations $x = t$ for $y = t^2$ for $t \in [0, 1]$? To find Cartesian equation we eliminate the parameter (when possible)*

$$t^2 = x^2 = y \quad \Rightarrow \quad y = x^2$$

Thus the Cartesian form of the given parametrized curve is simply $y = x^2$, however, given that $0 \leq t \leq 1$ and $x = t$ it follows we do not have the whole parabola, instead just $y = x^2$ for $0 \leq x \leq 1$.

Example 1.4.24. *Identify what curve has parametric equations* $x = \tan^{-1}(t)$ *and* $y = \tan^{-1}(t)$ *for* $t \in \mathbb{R}$. *Recall that range*$(\tan^{-1}(t)) = (-\pi/2, \pi/2)$. *It follows that* $-\pi/2 < x < \pi/2$. *Naturally we just equate inverse tangent to obtain* $\tan^{-1}(t) = y = x$. *The curve is the open line-segment with equation* $y = x$ *for* $-\pi/2 < x < \pi/2$. *This is an interesting parameterization, notice that as* $t \to \infty$ *we approach the point* $(\pi/2, \pi/2)$, *but we never quite get there.*

Example 1.4.25. *Consider* $x = \ln(t)$ *and* $y = e^t - 1$ *for* $t \geq 1$. *We can solve both for* t *to obtain*

$$t = e^x = \ln(y+1) \quad \Rightarrow \quad y = -1 + exp(exp(x)).$$

The domain for the expression above is revealed by analyzing $x = \ln(t)$ *for* $t \geq 1$, *the image of* $[1, \infty)$ *under natural log is precisely* $[0, \infty)$; $\ln[1, \infty) = [0, \infty)$.

Example 1.4.26. *Suppose* $x = \cosh(t) - 1$ *and* $y = 2\sinh(t) + 3$ *for* $t \in \mathbb{R}$. *To eliminate* t *it helps to take an indirect approach. We recall the most important identity for the hyperbolic sine and cosine:* $\cosh^2(t) - \sinh^2(t) = 1$. *Solve for hyperbolic cosine;* $\cosh(t) = x + 1$. *Solve for hyperbolic sine;* $\sinh(t) = \frac{y-3}{2}$. *Now put these together via the identity:*

$$\cosh^2(t) - \sinh^2(t) = 1 \quad \Rightarrow \quad (x+1)^2 - \frac{(y-3)^2}{4} = 1.$$

Note that $\cosh(t) \geq 1$ *hence* $x + 1 \geq 1$ *thus* $x \geq 0$ *for the curve described above. On the other hand* y *is free to range over all of* $\mathbb{R}$ *since hyperbolic sine has range* $\mathbb{R}$. *You should*[13] *recognize the equation as a hyperbola centered at* $(-1, 3)$.

how can we find a parametrization of a given Cartesian curve?

I like this topic more, the preceding bunch of examples, while needed, are boring. The art of parameterizing level curves is much more fun.

Example 1.4.27. Find parametric equations for the circle centered at (h, k) **with radius** R.

To begin recall the equation for such a circle is $(x-h)^2 + (y-k)^2 = R^2$. *Our inspiration is the identity* $\cos^(t) + \sin^2(t) = 1$. *Let* $x - h = R\cos(t)$ *and* $y - k = R\sin(t)$ *thus*

$$\boxed{x = h + R\cos(t)} \qquad \text{and} \qquad \boxed{y = k + R\sin(t).}$$

I invite the reader to verify these do indeed parametrize the circle by explicitly plugging in the equations into the circle equation. Notice, if we want the whole circle then we simply choose any interval for t *of length* 2π *or longer. On the other hand, if you want to select just a part of the circle you need to think about where sine and cosine are positive and negative. For example, if I want to parametrize just the part of the circle for which* $x > h$ *then I would choose* $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

[13] many students need to review these at this point, we use circles, ellipses and hyperbolas as examples in this course. I'll give examples of each in this chapter.

The reason I choose that intuitively is that the parametrization given for the circle above is basically built from polar coordinates[14] centered at (h, k). That said, to be sure about my choice of parameter domain I like to actually plug in some of my proposed domain and make sure it matches the desired criteria. I think about the graphs of sine and cosine to double check my logic. I know that $\cos(-\frac{\pi}{2}, \frac{\pi}{2}) = (0, 1]$ whereas $\sin(-\frac{\pi}{2}, \frac{\pi}{2}) = (-1, 1)$, I see it in my mind. Then I think about the parametric equations in view of those facts,

$$x = h + R\cos(t) \qquad \text{and} \qquad y = k + R\sin(t).$$

I see that x will range over $(h, h + R]$ and y will range over $(k - R, k + R)$. This is exactly what I should expect geometrically for half of the circle. Visualize that $x = h$ is a vertical line which cuts our circle in half. These are the thoughts I think to make certain my creative leaps are correct. I would encourage you to think about these matters. Don't try to just memorize everything, it will not work for you, there are simply too many cases. It's actually way easier to just understand these as a consequence of trigonometry, algebra and analytic geometry.

[14] we will discuss further in a later section, but this should have been covered in at least your precalculus course.

Example 1.4.28. Find parametric equations for the level curve $x^2 + 2x + \frac{1}{4}y^2 = 0$ which give the ellipse a CW orientation.

To begin we complete the square to understand the equation:

$$x^2 + 2x + \frac{1}{4}y^2 = 0 \quad \Rightarrow \quad (x+1)^2 + \frac{1}{4}y^2 = 1.$$

We identify this is an ellipse centered at $(-1, 0)$. Again, I use the pythagorean trig. identity as my guide: I want $(x+1)^2 = \cos^2(t)$ and $\frac{1}{4}y^2 = \sin^2(t)$ because that will force the parametric equations to solve the ellipse equation. However, I would like for the equations to describe CW direction so I replace the t with $-t$ and propose:

$$\boxed{x = -1 + \cos(-t)} \qquad \text{and} \qquad \boxed{y = 2\sin(-t)}$$

If we choose $t \in [0, 2\pi)$ then the whole ellipse will be covered. I could simplify $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$ but I have left the minus to emphasize the idea about reversing the orientation. In the preceding example we gave the circle a CCW orientation.

Example 1.4.29. Find parametric equations for the part of the level curve $x^2 - y^2 = 1$ which is found in the first quadrant.

We recognize this is a hyperbola which opens horizontally since $x = 0$ gives us $-y^2 = 1$ which has no real solutions. Hyperbolic trig. functions are built for a problem just such as this: recall $\cosh^2(t) - \sinh^2(t) = 1$ thus we choose $x = \cosh(t)$ and $y = \sinh(t)$. Furthermore, the hyperbolic sine function $\sinh(t) = \frac{1}{2}\big(e^t - e^{-t}\big)$ is everywhere increasing since it has derivative $\cosh(t)$ which is everywhere positive. Moreover, since $\sinh(0) = 0$ we see that $\sinh(t) \geq 0$ for $t \geq 0$. Choose non-negative t for the domain of the parametrization:

$$\boxed{x = \cosh(t), \qquad y = \sinh(t), \qquad t \in [0, \infty).}$$

Example 1.4.30. Find parametric equations for the part of the level curve $x^2 - y^2 = 1$ which is found in the third quadrant.

Based on our thinking from the last example we just need to modify the solution a bit:

$$\boxed{x = -\cosh(t), \qquad y = \sinh(t), \qquad t \in (-\infty, 0].}$$

Note that if $t \in (-\infty, 0]$ then $-\cosh(t) \leq -1$ and $\sinh(t) \leq 0$, this puts us in the third quadrant. It is also clear that these parametric equations to solve the hyperbola equation since

$$(-\cosh(t))^2 - (\sinh(t))^2 = \cosh^2(t) - \sinh^2(t) = 1.$$

The examples thus far are rather specialized, and in general there is no method to find parametric equations. This is why I said it is an art.

Example 1.4.31. Find parametric equations for the level curve $x^2y^2 = x - 2$.

This example is actually pretty easy because we can solve for $y^2 = \frac{x-2}{x^2}$ hence $y = \pm\sqrt{\frac{x-2}{x^2}}$. We can choose x as parameter so the parametric equations are just

$$x = t \quad \text{and } y = \sqrt{\frac{t-2}{t^2}}$$

for $t \geq 2$. Or, we could give parametric equations

$$x = t \quad \text{and } y = -\sqrt{\frac{t-2}{t^2}}$$

for $t \geq 2$. These parametrizations simply cover different parts of the same level curve.

Remark 1.4.32. *but... what is t ?*

If you are at all like me when I first learned about parametric curves you're probably wondering what is t ? You probably, like me, suppose incorrectly that t should be just like x or y. There is a crucial difference between x and y and t. The notations x and y are actually shorthands for the Cartesian coordinate maps $x : \mathbb{R}^2 \to \mathbb{R}$ and $y : \mathbb{R}^2 \to \mathbb{R}$ where $x(a, b) = a$ and $y(a, b) = b$. When I use the notaion $x = 3$ then you know what I mean, you know that I'm focusing on the vertical line with first coordinate 3. On the other hand, if I say $t = 3$ and ask where is it? Then you should say, you question doesn't make sense. The concept of t is tied to the curve for which it is the parameter. There are infinitely many geometric meanings for t. In other words, if you try to find t in the xy-plane without regard to a curve then you'll never find an answer. It's a meaningless question.

On the other hand if we are given a curve and ask what the meaning of t is for that curve then we ask a meaningful question. There are two popular meanings.

1. the parameter s measures the arclength from some base point on the given curve.
2. the parameter t gives the time along the curve.

In case (1.) for an oriented curve this actually is uniquely specified if we have a starting point. Such a parameterization is called the **arclength parametrization** or **unit-speed** parametrization of a curve. These play a fundamental role in the study of the differential geometry of curves. In case (2.) we have in mind that the curve represents the physical trajectory of some object, as t increases, time goes on and the object moves. I tend to use (2.) as my conceptual backdrop. But, keep in mind that these are just applications of parametric curves. In general, the parameter need not be time or arclength. It might just be what is suggested by algebraic convenience.

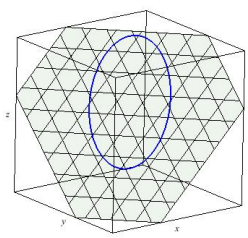
1.4.3 curves in three dimensional space

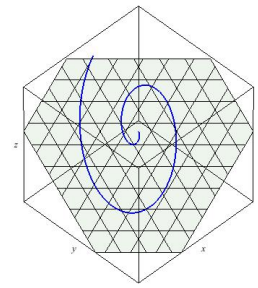
Other interesting curves can be obtained by feeding a simple curve like a circle into the parameterization of a plane.

Example 1.4.33. *Suppose $\vec{R}(u,v) = \vec{r}_o + u\vec{A} + v\vec{B}$ is the parametrization of a plane S then if we compose $\vec{R}$ with the path $t \mapsto \vec{\gamma}(t) = \langle R\cos(t), R\sin(t)\rangle$ we obtain an ellipse on the plane:*

$$\vec{r}(t) = (\vec{R} \circ \vec{\gamma})(t) = \vec{r}_o + R\cos(t)\vec{A} + R\sin(t)\vec{B}$$

Of course, we could also put a spiral on a plane through much the same device:





The idea of the last example can be used to create many interesting examples. These should suffice for our purposes here. I really just want you to think about what a parametrization does. Moreover, I hope you can find it in your heart to regard the parametric viewpoint as primary. Notice that any curves in three dimensions would require two independent equations in x, y, z. We saw how much of a hassel this was for something as simple as a line. I'd rather not attempt a general treatment of the purely cartesian description of the curves in this section[15] I instead offer a pair of examples to give you a flavor:

Example 1.4.34. *Suppose $x^2 + y^2 + z^2 = 4$ and $x = \sqrt{2}$ then the solution set of this pair of equations defines a curve in $\mathbb{R}^3$. Substutiting $x = \sqrt{3}$ into $x^2 + y^2 + z^2 = 4$ gives $y^2 + z^2 = 1$. The solution set is just a unit-circle in the yz-coordinates placed at $x = \sqrt{3}$. We can parametrize it via:*

$$\vec{r}(t) = \langle \sqrt{3}, \cos t, \sin t\rangle.$$

Example 1.4.35. *Suppose $z = x^2 - y^2$ and $z = 2x$. The solution set is once more a curve in $\mathbb{R}^3$. We can substitute $z = 2x$ into $z = x^2 - y^2$ to obtain $x^2 - y^2 = 2x$ hence $x^2 - 2x - y^2 = 0$ and completing the square reveals $(x-1)^2 - y^2 = 1$. This is the equation of a hyperbola. A natural parametrization is given by $x = 1 + \cosh t$ and $y = \sinh t$ then since $z = 2x$ we have $z = 2 + 2\cosh t$. In total,*

$$\vec{r}(t) = \langle 1 + \cosh t, \sinh t, 2 + 2\cosh t\rangle$$

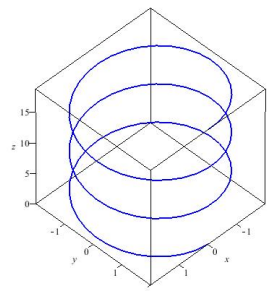
We'll explain the geometry of these calculations in the next section. Basically the idea is just that when two surfaces intersect in $\mathbb{R}^3$ we may obtain a curve.

[15] which is not to say it hasn't been done, in fact, viewing curves as solutions to equations is also a powerful technique, but we focus our efforts in the parametric setting.

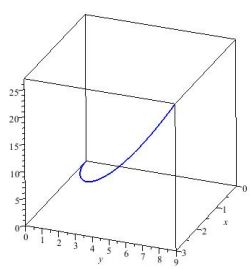
Example 1.4.36. *A helix of radius R which wraps around the z-axis and has a slope of m is given by:*

$$\vec{r}(t) = \langle R\cos(t), R\sin(t), mt\rangle$$

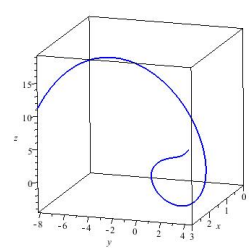
for $t \in [0, \infty)$.



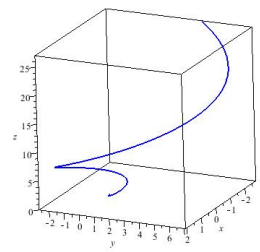
Example 1.4.37. *The curve parametrized by $\vec{r}(t) = \langle t, t^2, t^3$ for $t \geq 0$ has scalar parametric equations $x = t, y = t^2, z = t^3$ and a graph*



Example 1.4.38. *The curve parametrized by $\vec{r}(t) = \langle t, t^2\cos(3t), t^3\sin(3t)\rangle$ for $t \geq 0$ has scalar parametric equations $x = t, y = t^2\cos(t), z = t^3\sin(t)$ and a graph*



Example 1.4.39. *The curve parametrized by $\vec{r}(t) = \langle t\cos(3t), t^2\sin(3t), t^3\rangle$ for $t \geq 0$ has scalar parametric equations $x = t\cos(3t), y = t^2\sin(3t), z = t^3$ and a graph*



We will explore the geometry of curves in the next chapter. We'll find relatively simple calculations which allow us to test how a curve bends within it's plane of motion and bend off it's plane of motion. In other words, we'll find a way to test if a curve lies in a plane and also how it curves away from it's tangential direction. These quantities are called *torsion* and *curvature*. It turns out that these two quantities often classify a curve upto congruence in the sense of high-school geometry. In other words, there is just one circle of radius 1 and we can rotate it and translate it throughout $\mathbb{R}^3$. In this sense all circles in $\mathbb{R}^3$ are the same. We've already seen in this section that parametrization alone does not capture this concept. Why? Well there are many parametrizations of a circle. Are those different circles? I would say not. I say there is a circle and there are many pictures of the circle, some CW, some CCW, but so long as those pictures cover the same curve then they are merely differing perspectives on the same object. That said, these differing pictures are different. They are unique in their assignments of points to parameter values. The problem of the differential geometry of curves is to extract from this infinity of parametrizations some universal data. One seeks a few constants which invariantly characterize the curve independent of the perspective a particular geometer has used to capture it. More generally this is the problem of geometry. How can we classify spaces? What constants can we label a space with unambiguously?

1.5 surfaces

A surface in $\mathbb{R}^3$ is a subset which usually looks two dimensional. There are three main viewpoints; graphs, parametrizations or patches, and level-surfaces. As usual, the parametric view naturally generalizes to surfaces in $\mathbb{R}^n$ for $n > 3$ with relatively little qualification. That said, we almost without exception focus on surfaces in $\mathbb{R}^3$ in this course so I focus our efforts in that direction. This section is introductory in nature, basically this is just show and tell with a little algebra. Your goal should be to learn the names of these surfaces and more importantly to gain a conceptual foothold on the different ways to look at two-dimensional subsets of $\mathbb{R}^3$. Many of the diagrams in this section were created with Maple, others perhaps Mathematica. Ask if interested.

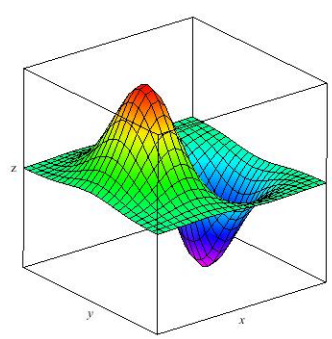
1.5.1 surfaces as graphs

Given a function of two variables it is natural to graph such a function in three-dimensions. In particular, we define:

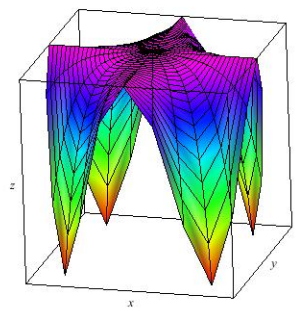
Definition 1.5.1. *graph of a function of two variables.*

Suppose $f : dom(f) \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a function then the set of all (x, y, z) such that $z = f(x, y)$ for some $(x, y) \in dom(f)$ is called the **graph** of f. Moreover, we denote $graph(f) = \{(x, y, f(x, y)) \mid (x, y) \in dom(f)\}$.

Example 1.5.2. *Let $f(x, y) = xe^{-x^2-y^2}$. The graph looks something like:*



Example 1.5.3. *Let $f(x, y) = -cosh(xy)$. The graph looks something like:*

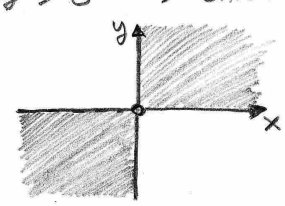


What is f? Well, many interpretations exist. For example, f could represent the temperature at (x, y). Or, f could represent the mass per unit area close to (x, y), this would make f a mass density function. More generally, if you have a variable which depends by some single-valued rule to another pair of variables then you can find a function in that application. Sometimes college algebra students will ask, but what *is* a function really? With a little imagination the answer is most anything. It could be that f is the cost for making x widgets and y gadgets. Or, perhaps $f(x, y)$ is the grade of a class as a function of x males and y females. Enough. Let's get back to the math, I'll generally avoid cluttering these notes with these silly comments, however, you are free to ask in office hours. Not all such discussion is without merit. Application is important, but is not at all necessary for good mathematics.

We can add, multiple and divide functions of two variables in the same way we did for functions of one variable. Natural domains are also implicit within formulas and points are excluded for much the same reason as in single-variable calculus; we cannot divide by zero, take an even root of a negative number or take a logarithm of a non-positive quantity if we wish to maintain a real output[16]. A typical example is given below:

Example 1.5.4. .

$\boxed{E18}$ $f(x,y) = \sqrt{xy}/(x^2+y^2)$ find dom(f). So we have to throw out the origin to avoid % by zero. Then we need $xy > 0$ $\Rightarrow$ either $x>0$ and $y>0$ OR $x<0$ and $y<0$.



the dom(f) consists of two disconnected parts.

[16]complex variables do give meaning to even roots and logarithms of negative numbers however, division by zero and logarithm of zero continue to lack an arithmetical interpretation.

1.5.2 parametrized surfaces

Definition 1.5.5. *vector-valued functions of two real variables, parametrized surfaces.*

A vector valued function of a two real variables is an assignment of a vector for each pair of real numbers in some domain D of $\mathbb{R}^2$. It's a mapping $(u,v) \mapsto \vec{F}(u,v) = \big\langle F_1(u,v), F_2(u,v), \ldots, F_n(u,v)\big\rangle$ for each $(u,v) \in D \subset \mathbb{R}^2$. We say $F_j : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the j-th component function of $\vec{F}$. Let $S = \vec{F}(D)$ then S is said to be a **surface parametrized by** $\vec{F}$. Equivalently, but not so usefully, we can write the scalar parametric equations for S above as

$$x_1 = F_1(u,v), \quad x_2 = F_2(u,v), \ \ldots, \ x_n = F_n(u,v)$$

for all $(u,v) \in D$. We call $\vec{F}$ a **patch** on S.

When we define a parametrization of a surface it is important to give the <u>formula</u> for the patch **and** the <u>domain</u> D of the parameters. We call D the **parameter space**. Usually we are interested in the case of a surface which is embedded in $\mathbb{R}^3$ so I will focus the examples in that direction. Note however that the parametric equation for a plane actually embeds the plane in $\mathbb{R}^n$ for whatever n you wish, there is nothing particular to three dimensions for the construction of the line or plane parametrizations.

Example 1.5.6. *Suppose $S = \{(x,y,z) \mid z = f(x,y)\}$ where f is a function. Naturally we parametrize this graph via the choice $x = u, y = v$,*

$$\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$$

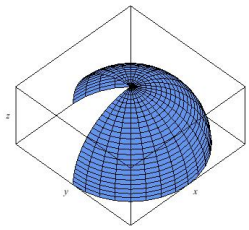
for $(u,v) \in dom(f)$.

As I discussed in the plane section, a graph is given in terms of cartesian coordinates. In the case of surfaces in $\mathbb{R}^3$ you'll often encounter the presentation $z = f(x,y)$ for some function f. This is an important class of examples, however, the criteria that f be a function is quite limiting.

Example 1.5.7. *Let $\vec{r}(u,v) = \langle R\cos(u)\sin(v), R\sin(u)\sin(v), R\cos(v)\rangle$ for $(u,v) \in [0,2\pi] \times [0,\pi]$. In this case we have scalar equations:*

$$x = R\cos(u)\sin(v), \quad y = R\sin(u)\sin(v), \quad z = R\cos(v).$$

It's easy to show $x^2 + y^2 + z^2 = R^2$ and we should recognize that these are the parametric equations which force $\vec{r}(u,v)$ to land a distance of R away from the origin for each choice of (u,v). Let $S = \vec{r}(D)$ and recognize S is a **sphere** *of radius R centered at the origin. If we restrict the domain of $\vec{r}$ to $0 \leq u \leq \frac{3\pi}{2}$ and $0 \leq v \leq \frac{\pi}{2}$ then we select just a portion of the sphere:*

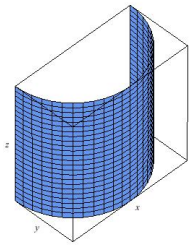


Notice that we could cover the whole sphere with a single patch. We cannot do that with a graph. This is the same story we saw in the two-dimensional case in calculus II. Parametrized curves are not limited by the vertical line test. Graphs are terribly boring in comparison to the geometrical richness of the parametric curve. As an exotic example from 1890, Peano constructed a continuous[17] path from $[0,1]$ which covers all of $[0,1]\times[0,1]$. Think about that. Such curves are called *space filling curves.* There are textbooks devoted to the study of just those curves. For example, see Hans Sagan's *Space Filling Curves.*

Example 1.5.8. *Let $\vec{r}(u,v) = \langle R\cos(u), R\sin(u), v\rangle$ for $(u,v)\in[0,2\pi]\times\mathbb{R}$. In this case we have scalar equations:*

$$x = R\cos(u), \quad y = R\sin(u), \quad z = v.$$

It's easy to show $x^2+y^2=R^2$ and z is free to range over all values. This surface is a circle at each possible z. We should recognize that these are the parametric equations which force $\vec{r}(u,v)$ to land on a cylinder of raduius R centered on the z-axis. If we restrict the domain of $\vec{r}$ to $0\leq u\leq\pi$ and $0\leq v\leq 2$ then we select a finite half-cylinder:

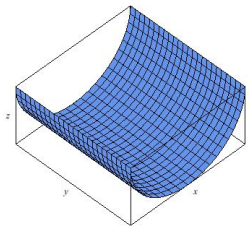


Example 1.5.9. *Let $\vec{r}(u,v) = \langle a\cos(u), v, b\sin(u)\rangle$ for $(u,v)\in[0,2\pi]\times\mathbb{R}$. In this case we have scalar equations:*

$$x = a\cos(u), \quad y = v, \quad z = b\sin(u).$$

It's easy to show $x^2/a^2+z^2/b^2=1$ and y is free to range over all values. This surface is a ellipse at each possible y. We should recognize that these are the parametric equations which force $\vec{r}(u,v)$ to land on an elliptical cylinder centered on the y-axis. If we restrict the domain of $\vec{r}$ to $0\leq u\leq\pi$ and $0\leq v\leq 2$ then we select a finite half-cylinder:

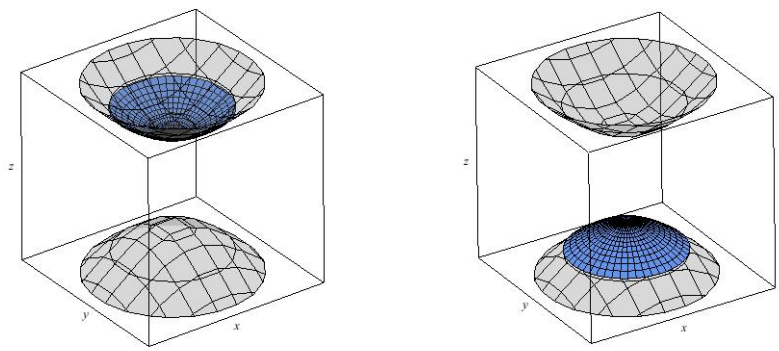
[17] we will define this carefully in a future chapter



Example 1.5.10. *Let* $\vec{r}(u,v) = \langle R\cos(u)\sinh(v), R\sin(u)\sinh(v), R\cosh(v)\rangle$ *for* $(u,v) \in [0,2\pi] \times \mathbb{R}$. *In this case we have scalar equations:*

$$x = R\cos(u)\sinh(v), \quad y = R\sin(u)\sinh(v), \quad z = R\cosh(v).$$

It's easy to show $-x^2 - y^2 + z^2 = R^2$. *If we restrict the domain of* $\vec{r}$ *to* $0 \leq u \leq 2\pi$ *and* $-2 \leq v \leq 2$ *then we select a portion of the upper branch:*

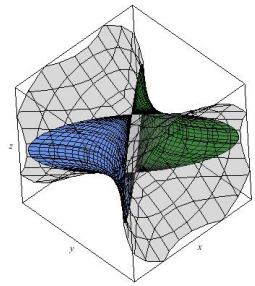


The part of the lower branch which is graphed above is covered by the mapping Let $\vec{r}(u,v) = \langle R\cos(u)\sinh(v), R\sin(u)\sinh(v), -R\cosh(v)\rangle$ *for* $(u,v) \in [0,2\pi] \times [-2,2]$. *The grey shape is where the parametrization will cover if we enlarge the domain of the parameterizations.*

Example 1.5.11. *Let* $\vec{r}(u,v) = \langle R\cosh(u)\sin(v), R\sinh(u)\sin(v), R\cos(v)\rangle$ *for* $(u,v) \in \mathbb{R} \times [0,2\pi]$. *In this case we have scalar equations:*

$$x = R\cosh(u)\sin(v), \quad y = R\sinh(u)\sin(v), \quad z = R\cos(v).$$

It's easy to show $x^2 - y^2 + z^2 = R^2$. *I've plotted* $\vec{r}$ *with domain restricted to* $dom(\vec{r}) = [-1.3, 1.3] \times [0,\pi]$ *in blue and* $dom(\vec{r}) = [-1.3, 1.3] \times [\pi, 2\pi]$ *in green. The grey shape is where the parametrization will go if we enlarge the domain.*



1.5.3 surfaces as level sets

Unlike curves, we do not need two equations to fix a surface in $\mathbb{R}^3$. In three dimensional space[18] if we have just one equation in x, y, z that should suffice to leave just two free variables. In a nutshell that is what a surface is. It is a space which has two degrees of freedom. In the parametric set-up we declare those freedoms explicitly from the outset by the construction of the patch in terms of the parameters. In the level set formulation we focus the attention on an equation which defines the surface of interest. We already saw this for a plane; the solutions of $ax + by + cz = d$ fill out a plane with normal $\langle a, b, c\rangle$.

Definition 1.5.12. *level surface in three dimensional space*

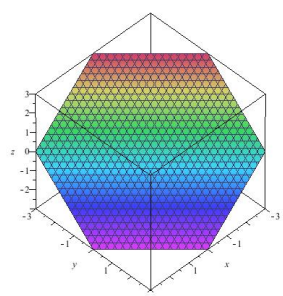
Suppose $F : dom(F) \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a function. Let

$$S = \{(x, y, z) \mid F(x, y, z) = k\}.$$

We say that S is a **level surface** of level k with level function F. In other words, $S = F^{-1}\{k\}$ is a level surface.

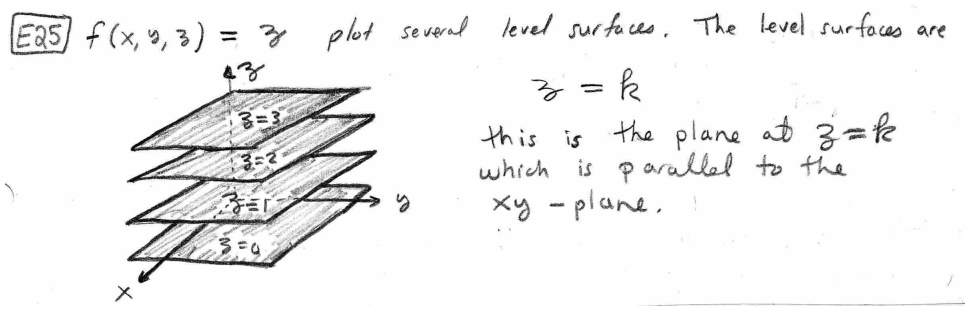
A level surface is a *fiber* of a real-valued function on $\mathbb{R}^3$.

Example 1.5.13. *Let $F(x, y, z) = a(x - x_o) + b(y - y_o) + c(z - z_o)$. Recognize that the solution set of $F(x, y, z) = 0$ is the plane with base-point (x_o, y_o, z_o) and normal $\langle a, b, c\rangle$.*



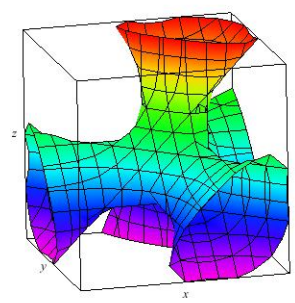
[18] to pick out a two-dimensional surface in $\mathbb{R}^4$ it would take two equations in t, x, y, z, but, we really only care about $\mathbb{R}^3$ so, I'll behave and stick with that case.

Example 1.5.14. *some level surfaces I can plot without fancy CAS programs:*



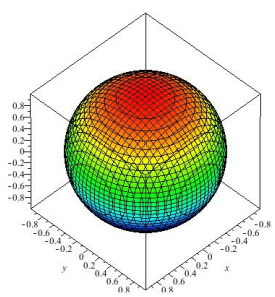
The example below is not such a case:

Example 1.5.15. *This surface has four holes. I have an animation on my website, check it out.*



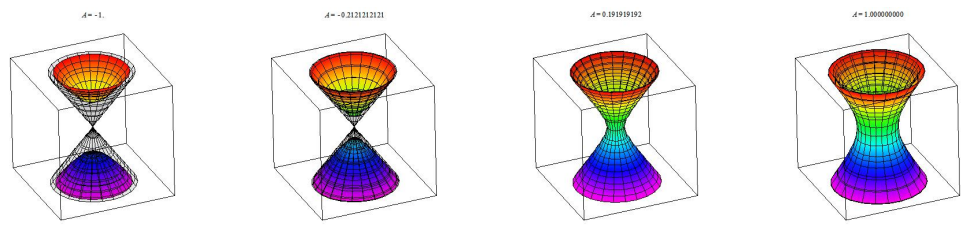
In fact, I don't think I want to parametrize this beast. Wait, I have students, isn't this what homework is for?

Example 1.5.16. *Let $F(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2$. The solution set of $F(x, y, z) = 1$ is called an* **ellipsoid** *centered at the origin. In the special case $a = b = c = R$ the ellipsoid is a sphere of radius R. Here's a special case, $a = b = c = 1$ the unit-sphere:*



Example 1.5.17. *Let $F(x, y, z) = x^2 + y^2 - z^2$. The solution set of $F(x, y, z) = 0$ is called an* **cone** *through the origin. However, the solution set of $F(x, y, z) = k$ for $k \neq 0$ forms a* **hyperboloid of**

one-sheet *for* $k > 0$ *and a* **hyperboloid of two-sheets** *for* $k < 0$. *The hyperboloids approach the cone as the distance from the origin grows. I plot a few representative cases:*



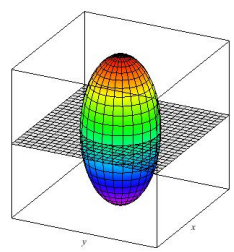
There is an animation on my webage, take a look.

Some of the examples above fall under the general category of a **quadratic surface**. Suppose

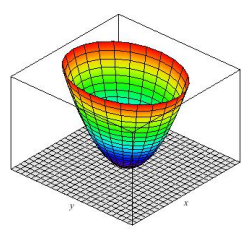
$$Q(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz.$$

For any particular nontrivial selection of constants $a, b, \ldots, h, i$ we say the solution of $Q(x, y, z) = k$ is a **quadratic surface**. For future reference let me list the proper terminology. We'd like to get comfortable with these terms.

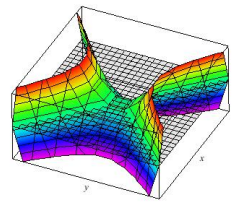
1. a standard **ellipsoid** is the solution set of $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. If $a = b = c$ then we say the ellipsoid is a **sphere**.



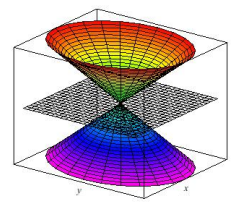
2. a standard **elliptic paraboloid** is the solution set of $z/c = x^2/a^2 + y^2/b^2$. If $a = b$ then we say the paraboloid is **circular**.



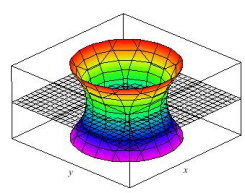
3. a standard **hyperbolic paraboloids** is the solution set of $z/c = y^2/b^2 - x^2/a^2$.



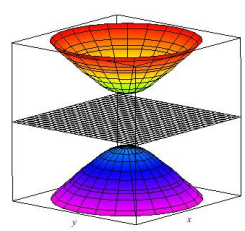
4. a standard **elliptic cone** is the solution set of $z^2/c^2 = x^2/a^2 + y^2/b^2$. If $a = b$ then we say the cone is **circular**.



5. a standard **hyperboloid of one sheet** is the solution set of $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$.



6. a standard **hyperboloid of two sheets** is the solution set of $x^2/a^2 + y^2/b^2 - z^2/c^2 = -1$.



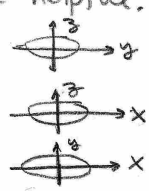
If you study the formulas above you'll notice the absense of certain terms in the general quadratic form: terms such as dxy, exz, fyz, gx, hy are absent. Inclusion of these terms will either shift or rotate the standard equations. However, we need linear algebra to properly construct the rotations from the eigenvectors of the quadratic form. I leave that for Math 321 where we have more toys to play with. You'll have to be content with the standard examples for the majority of this course.

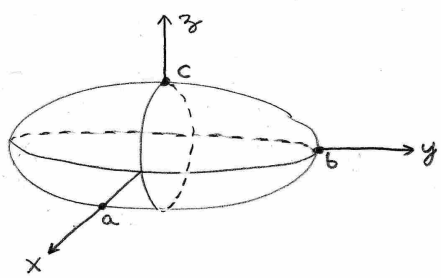
I've inserted the term *standard* because I don't mean to say that every elliptic cone has the same equation as I give. I expect you can translate the standard examples up to an interchange of coordinates, that ought not be too hard to understand. For example, $y = x^2 + 2z^2$ is clearly an elliptical cone. Or $y = x^2 - z^2$ is clearly a hyperbolic paraboloid. Or $x^2 + z^2 - y^2 = 1$ is clearly a hyperboloid of one sheet whereas $-x^2 - z^2 + y^2 = 1$ is a hyperboloid of two sheets. These are the possibilities we ought to anticipate when faced with the level set of some quadratic form. I don't try to memorize all of these, I use the method sketched in the next pair of examples. Basically the idea is simply to slice the graph into planes where we find either circles, hyperbolas, lines, ellipses or perhaps nothing at all. Then we take a few such slices and extrapolate the graph. Often the slices $x = 0$, $y = 0$ and $z = 0$ are very helpful.

Example 1.5.18. .

$\boxed{E21}$ Ellipsoid. The name is appropriate, anyway we slice it we'll get an ellipse. I usually look at the coordinate planes then go from there to gather whatever other data might appear to be helpful.

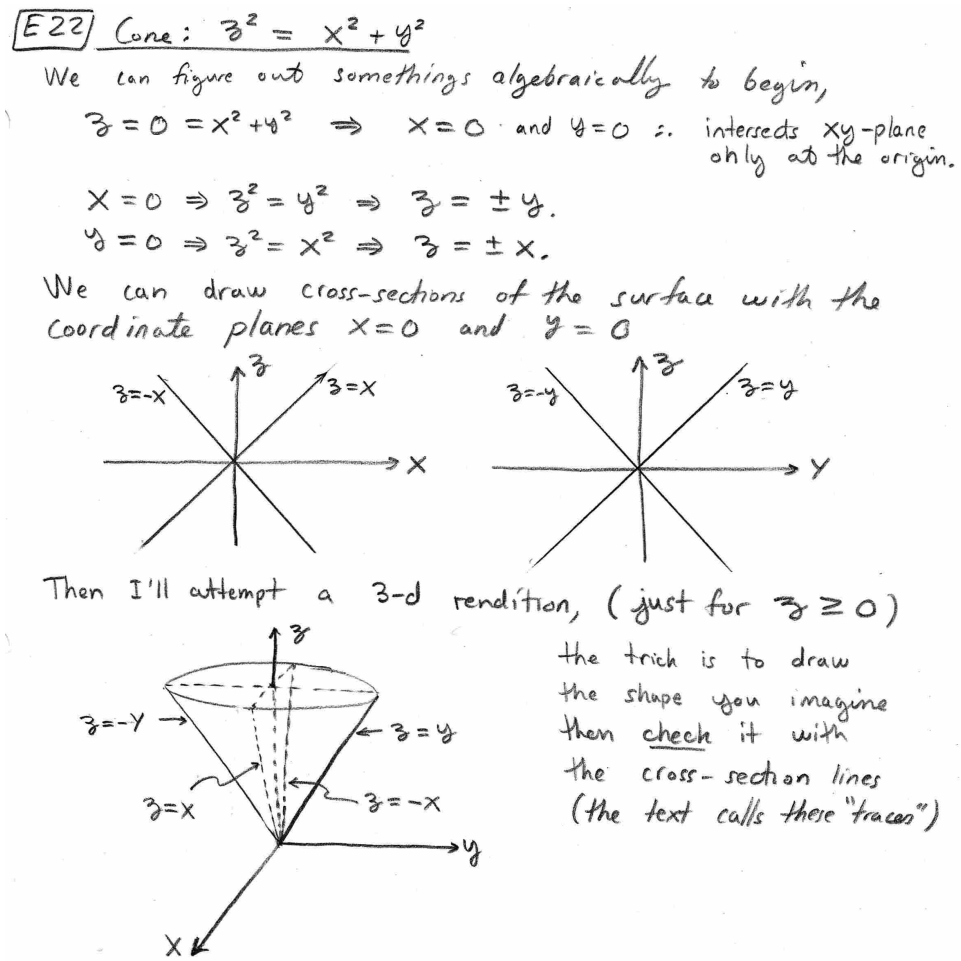
$$x = 0 \Rightarrow y^2/b^2 + z^2/c^2 = 1$$
$$y = 0 \Rightarrow x^2/a^2 + z^2/c^2 = 1$$
$$z = 0 \Rightarrow x^2/a^2 + y^2/b^2 = 1$$





Assuming $a, b, c > 0$ it looks something like this. Notice the idea is to use the cross-sections to get an idea where the surface is.

Example 1.5.19. .



Example 1.5.20. *Let $F(x, y, z) = x^2 + y^2 - z$. The solution set of $F(x, y, z) = k$ sometimes called a paraboloid. Notice that if we fix a value for z say $z = c$ then $x^2 + y^2 - c = k$ reduces to $x^2 + y^2 = k + c$. If $k + c \geq 0$ then the solution in the $z = c$ plane is a circle or a point. In other words, all horizontal cross-sections of this shape are circles and if $z < -k$ there is no solution. This surface opens up from the vertex at $(0, 0, -k)$.*

What about the geometry of surfaces? How can we classify surfaces? What constants can we label a surface with unambiguously? Here's a simple example: $x^2 + y^2 + z^2 = 1$ and $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 1$ define the same shape at different points in $\mathbb{R}^3$. The problem of the differential geometry of surfaces is to find general invariants which discover this equivalence through mathematical calculation. This is a more difficult problem and we will not treat it in this course. It turns out this geometry begins to provide the concepts needed for Einstein's General Relativity. In any event, we do not currently have a course at LU which does this topic justice. I know of

at least two professors who will happily conduct an independent study on this topic once you've mastered linear algebra.

1.5.4 combined concept examples

Example 1.5.21. .

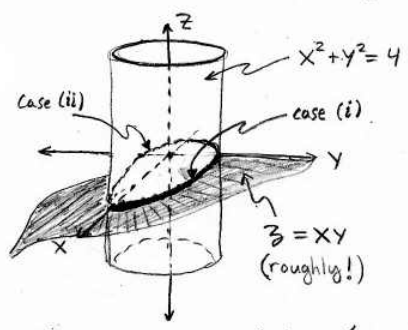
At what point does $r(t) = \langle t, 0, 2t - t^2 \rangle$ intersect the paraboloid $z = x^2 + y^2$? As usual we find intersection point by assuming both eq^{n}'s hold,

$$z = x^2 + y^2 \Rightarrow 2t - t^2 = t^2 + 0^2$$
$$\Rightarrow 2t - 2t^2 = 2t(1-t) = 0$$
$$\Rightarrow \underline{t = 0 \quad \text{OR} \quad t = 1}$$

the points of intersection are $\boxed{(0,0,0) \text{ and } (1,0,1).}$

Example 1.5.22. .

Find vector function which represents the intersection of the surfaces $x^2 + y^2 = 4$ and $z = xy$. Lets use x as the parameter. Then



$$y = \pm\sqrt{4 - x^2}$$
$$z = x(\pm\sqrt{4-x^2})$$

the question then is (+) or (−) when and where? I'll break it up into cases.

(i) $y \geq 0 \Rightarrow r(x) = \langle x, \sqrt{4-x^2}, x\sqrt{4-x^2} \rangle$, $-2 \leq x \leq 2$.
(ii) $y \leq 0 \Rightarrow r(x) = \langle x, -\sqrt{4-x^2}, -x\sqrt{4-x^2} \rangle$, $-2 \leq x \leq 2$.

I suppose we could paste these together by shifting the parameter on either (i) or (ii). There are other ways, for example,

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 2\sin(2t), \quad 0 \leq t \leq 2\pi$$

Example 1.5.23. .

find curve of intersection of $z = \sqrt{x^2+y^2}$ and the plane $z = 1+y$. Again use x as parameter,

note $y = z-1$

$$\begin{aligned}
&\Rightarrow z = \sqrt{x^2+(z-1)^2} \\
&\Rightarrow z^2 = x^2 + (z-1)^2 \\
&\Rightarrow z^2 = x^2 + z^2 - 2z + 1 \\
&\Rightarrow 2z = x^2+1 \\
&\Rightarrow \underline{z = \tfrac{1}{2}(x^2+1)} \\
&\qquad y = z-1 = \tfrac{1}{2}x^2 + \tfrac{1}{2} - 1 = \underline{\tfrac{1}{2}(x^2-1) = y}
\end{aligned}$$

So if you wish we may introduce t as a parameter then

$\boxed{r(t) = \langle t, \tfrac{1}{2}(t^2-1), \tfrac{1}{2}(t^2+1)\rangle}$ (there are other answers.)

Example 1.5.24. .

Consider the following trajectories, do they collide? For $t \geq 0$

$r_1(t) = \langle t^2, 7t-12, t^2\rangle$ and $r_2(t) = \langle 4t-3, t^2, 5t-6\rangle$

For vector functions to be equal we need each component to match with the corresponding component. That is,

$$\begin{aligned}
x_1 = x_2 &\Rightarrow t^2 = 4t-3 \Rightarrow t^2-4t+3 = (t-1)(t-3) = 0 \quad \therefore \boxed{t=1 \ \& \ t=3} \\
y_1 = y_2 &\Rightarrow 7t-12 = t^2 \Rightarrow t^2-7t+12 = (t-3)(t-4) = 0 \quad \therefore \boxed{t=3 \ \& \ t=4} \\
z_1 = z_2 &\Rightarrow t^2 = 5t-6 \Rightarrow t^2-5t+6 = (t-3)(t-2) = 0 \quad \therefore \boxed{t=3 \ \& \ t=2}
\end{aligned}$$

We find $x_1 = x_2$ at $t=1$, $y_1 = y_2$ at $t=4$ and $z_1 = z_2$ at $t=2$.

But only at $t=3$ do we get $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$, this means the $\boxed{\text{particles will collide at } t=3}$

Example 1.5.25. .

the question of collision is $r_1(t) \stackrel{?}{=} r_2(t)$ for some $t \geq 0$?. I leave that to you. The question of intersection is a bit different we should study $r_1(t) \stackrel{?}{=} r_2(s)$, can we find $s, t \geq 0$ so that the positions match-up (for possibly different times)

$$\left.\begin{aligned}
x&: \ t = 1+2s \\
y&: \ t^2 = 1+6s \\
z&: \ t^3 = 1+14s
\end{aligned}\right\} \underline{1 = t-2s = t^2-6s = t^3-14s}$$

note $s=0$ works if we make $t=1$.

$\boxed{\text{So yes the paths intersect at } r_1(1) = r_2(0) = \langle 1,1,1\rangle.}$

there is one other place they intersect, can you find it?

1.6 curvelinear coordinates

Cartesian coordinates are a nice starting point, but they make many simple problems needlessly complex. If a two-dimensional problem has a quantity which only depends on distance from the central point then probably polar coordinates will simplify the equations of the problem. Similarly, if a three dimensional problem possesses a cylindrical symmetry then use cylindrical coordinates. If a three dimensional problem has spherical symmetry then use spherical coordinates.

A coordinate system is called **right-handed** if the unit-vectors which point in the direction of increasing coordinates at each point are related to each other by the right-hand-rule just like the xyz-coordinates. We call this set of unit-vectors the **frame** of the coordinate system. Generally a frame in $\mathbb{R}^n$ is just an assignment of n-vectors at each point in $\mathbb{R}^n$. In linear-algebra langauge, a frame is an assignment of a basis at each point of $\mathbb{R}^n$. Dimensions $n = 2$ and $n = 3$ suffice for our purposes. If y_1, y_2, y_3 denote coordinates with unit-vectors $\widehat{u}_1, \widehat{u}_2, \widehat{u}_3$ in the direction of increasing y_1, y_2, y_3 respective then we say the coordinate system is **right-handed** iff

$$\widehat{u}_1 \times \widehat{u}_2 = \widehat{u}_3, \quad \widehat{u}_2 \times \widehat{u}_3 = \widehat{u}_1, \quad \widehat{u}_3 \times \widehat{u}_1 = \widehat{u}_2.$$

In contrast with the constant frame $\{\widehat{x}, \widehat{y}, \widehat{z}\}$ the frame $\{\widehat{u}_1, \widehat{u}_2, \widehat{u}_3\}$ is usually point-dependent. An assignment of a vector to each point in some space is called a **vector field**. A frame is actually a triple of vector fields which is given over a space. Enough terminology, the equations speak for themselves soon enough.

1.6.1 polar coordinates

Polar coordinates (r, θ) are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

In quadrants I and IV (regions with $x > 0$) we also have

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}\left[\frac{y}{x}\right].$$

In quadrants II and III (regions with $x < 0$) we have

$$r^2 = x^2 + y^2, \qquad \theta = \pi + \tan^{-1}\left[\frac{y}{x}\right].$$

Geometrically it is clear that we can label any point in $\mathbb{R}^2$ either by cartesian coordinates (x, y) or by polar coordinates (r, θ). We may view equations in cartesian or polar form.

Example 1.6.1. *The circle $x^2 + y^2 = R^2$ has polar equation $r = R$.*

Typically in the polar context the angle plays the role of the independent variable. In the same way it is usually customary to write $y = f(x)$ for a graph we try to write $r = f(\theta)$.

Example 1.6.2. *The line $y = mx + b$ has polar equation $r\sin\theta = mr\cos\theta + b$ hence*

$$r = \frac{b}{\sin\theta - m\cos\theta}.$$

Example 1.6.3. *The polar equation $\theta = \pi/4$ translates to $y = x$ for $x > 0$. The reason is that*

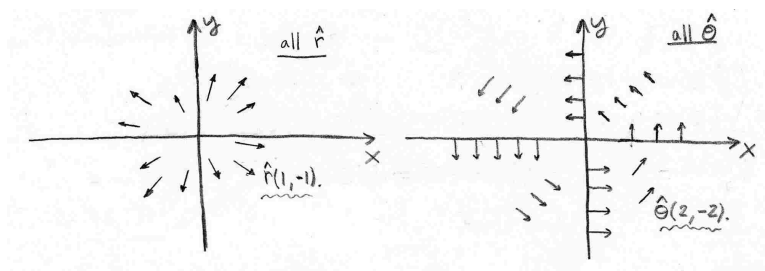
$$\frac{\pi}{4} = \tan^{-1}\left[\frac{y}{x}\right] \quad \Rightarrow \quad \tan\frac{\pi}{4} = \frac{y}{x} \quad \Rightarrow \quad 1 = \frac{y}{x} \quad \Rightarrow \quad y = x$$

and the ray with $\theta = \pi/4$ is found in quadrant I where $x > 0$.

Let us denote unit vectors in the direction of increasing r, θ by $\widehat{r}$, $\widehat{\theta}$ respective. You can derive by geometry alone that

$$\begin{aligned} \widehat{r} &= \cos(\theta)\,\widehat{x} + \sin(\theta)\,\widehat{y} \\ \widehat{\theta} &= -\sin(\theta)\,\widehat{x} + \cos(\theta)\,\widehat{y}. \end{aligned} \tag{1.5}$$

We call $\{\widehat{r},\ \widehat{\theta}\}$ the **frame** of polar coordinates. Notice that these are perpendicular at each point; $\widehat{r} \bullet \widehat{\theta} = 0$.



Example 1.6.4. *If we want to assign a vector to each point on the unit circle such that the vector is tangent and pointing in the counter-clockwise (CCW) direction then a natural choice is $\widehat{\theta}$.*

Example 1.6.5. *If we want to assign a vector to each point on the unit circle such that the vector is pointing radially out from the center then a natural choice is $\widehat{r}$.*

Suppose you have a perfectly flat floor and you pour paint slowly in a perfect even stream then in principle you'd expect it would spread out on the floor in the $\widehat{r}$ direction if we take the spill spot as the origin and the floor as the xy-plane.

1.6.2 cylindrical coordinates

Cylindrical coordinates (r, θ, z) are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

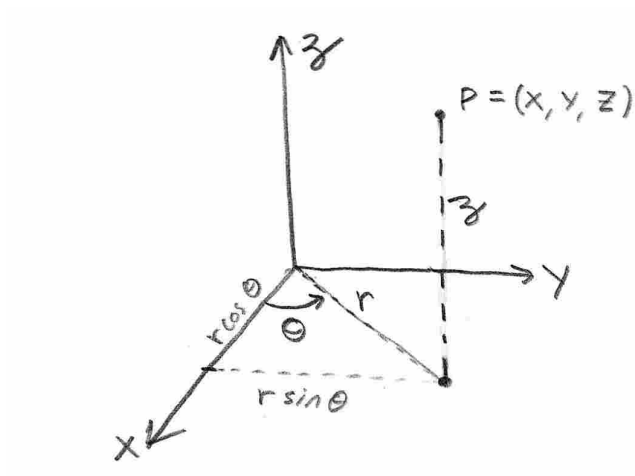
In quadrants I and IV (regions with $x > 0$) we also have

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}\left[\frac{y}{x}\right].$$

In quadrants II and III (regions with $x < 0$) we have

$$r^2 = x^2 + y^2, \qquad \theta = \pi + \tan^{-1}\left[\frac{y}{x}\right].$$

Geometrically it is clear that we can label any point in $\mathbb{R}^3$ either by cartesian coordinates (x, y, z) or by cylindrical coordinates (r, θ, z).



We may view equations in cartesian or cylindrical form.

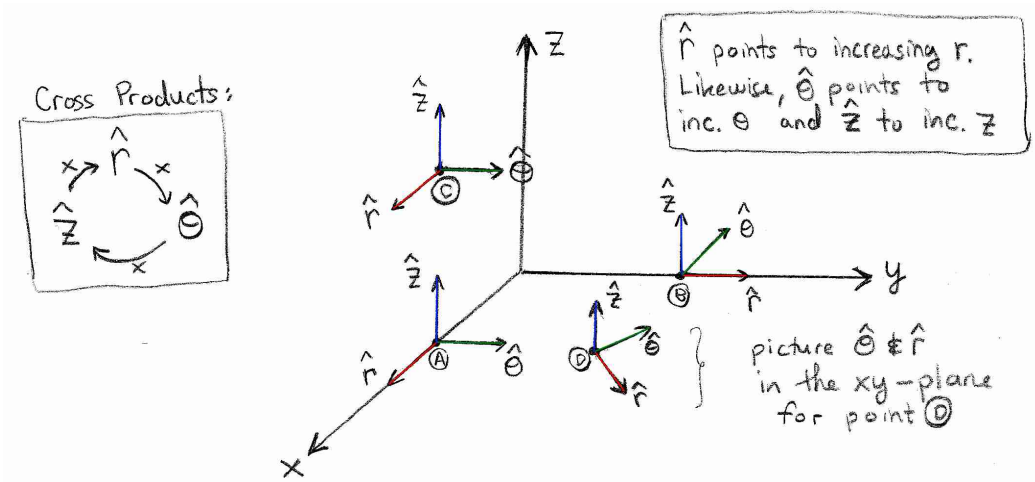
Example 1.6.6. *In cylindrical coordinates the equation $r = 1$ is a cylinder since the z-variable is free. If we denote the unit-circle by $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ then the solution set of $r = 1$ has the form $S_1 \times \mathbb{R}$. At each z we get a copy of the circle S_1.*

Example 1.6.7. *The equation $\theta = \pi/4$ is a half-plane which has equation $y = x$ subject to the condition $x > 0$.*

Let us denote unit vectors in the direction of increasing $r,\ \theta,\ z$ by $\widehat{r},\ \widehat{\theta},\ \widehat{z}$ respectively. You can derive by geometry alone that

$$\begin{aligned} \widehat{r} &= \cos(\theta)\,\widehat{x} + \sin(\theta)\,\widehat{y} \\ \widehat{\theta} &= -\sin(\theta)\,\widehat{x} + \cos(\theta)\,\widehat{y} \\ \widehat{z} &= \langle 0, 0, 1\rangle. \end{aligned} \tag{1.6}$$

We call $\{\widehat{r},\ \widehat{\theta},\ \widehat{z}\}$ the **unit-frame** of cylindrical coordinates.



Example 1.6.8. *Suppose we have a line of electric charge smeared along the z-axis with charge density λ. One can easily derive from Gauss' Law that the electric field has the form:*

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_o r}\widehat{r}.$$

Example 1.6.9. *If we have a uniform current $I\,\widehat{z}$ flowing along the z-axis then the magnetic field can be derived from Ampere's Law and has the simple form:*

$$\vec{B} = \frac{\mu_o I}{2\pi r}\widehat{\theta}$$

Trust me when I tell you that the formulas in terms of cartesian coordinates are not nearly as clean.

If we fix our attention to a particular point the cylindrical frame has the same structure as the cartesian fram $\{\widehat{x}, \widehat{y}, \widehat{z}\}$. In particular, we can show that

$$\widehat{r}\bullet\widehat{r} = 1, \quad \widehat{\theta}\bullet\widehat{\theta} = 1, \quad \widehat{z}\bullet\widehat{z} = 1$$

$$\widehat{\theta}\bullet\widehat{r} = 0, \quad \widehat{\theta}\bullet\widehat{z} = 0, \quad \widehat{z}\bullet\widehat{r} = 0.$$

We can also calculate either algebraically or geometrically that:

$$\widehat{r}\times\widehat{\theta} = \widehat{z}, \quad \widehat{\theta}\times\widehat{z} = \widehat{r}, \quad \widehat{z}\times\widehat{r} = \widehat{\theta}$$

Therefore, the cylindrical coordinate system (r, θ, z) is a **right-handed** coordinate system since it provides a right-handed basis of unit-vectors at each point. We can summarize these relations compactly with the notation $\widehat{u}_1 = \widehat{r},\ \widehat{u}_2 = \widehat{\theta},\ \widehat{u}_3 = \widehat{z}$ whence:

$$\widehat{u}_i\bullet\widehat{u}_j = \delta_{ij}, \qquad \widehat{u}_i\times\widehat{u}_j = \sum_{k=1}^{3}\epsilon_{ijk}\widehat{u}_k$$

this is the same pattern we saw for the cartesian unit vectors.

1.6.3 spherical coordinates

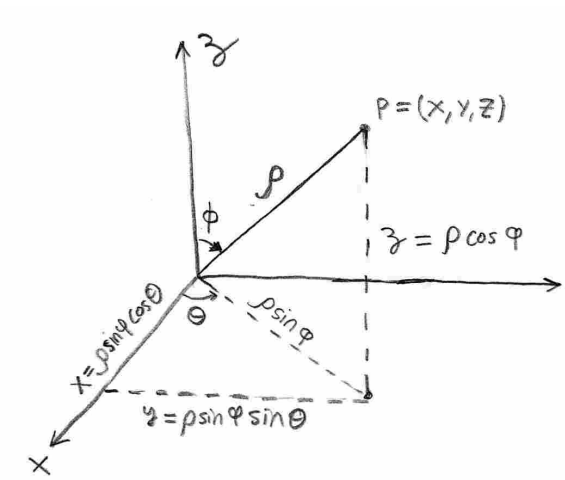
Spherical coordinates[19] (ρ, ϕ, θ) relate to Cartesian coordinates as follows

$$\begin{aligned} x &= \rho\cos(\theta)\sin(\phi) \\ y &= \rho\sin(\theta)\sin(\phi) \\ z &= \rho\cos(\phi) \end{aligned} \tag{1.7}$$

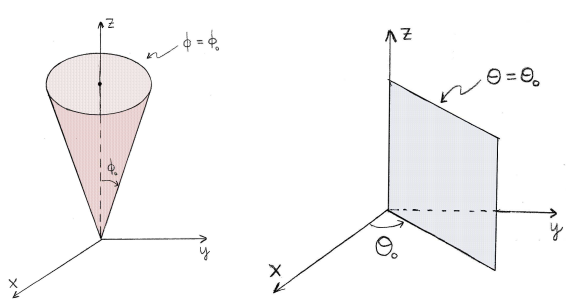
where $\rho > 0$ and $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. We can derive,

$$\begin{aligned} \rho^2 &= x^2 + y^2 + z^2 \\ \tan(\phi) &= \sqrt{x^2+y^2}/z \\ \tan(\theta) &= y/x. \end{aligned} \tag{1.8}$$

It is clear that any point in $\mathbb{R}^3$ is labeled both by cartesian coordinates (x, y, z) or spherical coordinates (ρ, ϕ, θ).



Also, it is important to distinguish between the geometry of the **polar angle** θ and the **azmuthial angle** ϕ



[19] I'll use notation which is consistent with Stewart, but beware there is a better notation used in physics and engineering where the meaning of ϕ and θ are switched and the spherical radius ρ is instead denoted by r

Example 1.6.10. *The equation* $\sqrt{x^2+y^2+z^2} = R$ *is written as* $\rho = R$ *in spherical coordinates.*

Example 1.6.11. *The plane* $a(x-1)+b(y-2)+c(z-3)=0$ *has a much uglier form in spherical coordinates. Its:* $a(\rho\cos(\theta)\sin(\phi)-1)+b(\rho\sin(\theta)\sin(\phi)-2)+c(\rho\cos(\phi)-3)=0$ *hence*

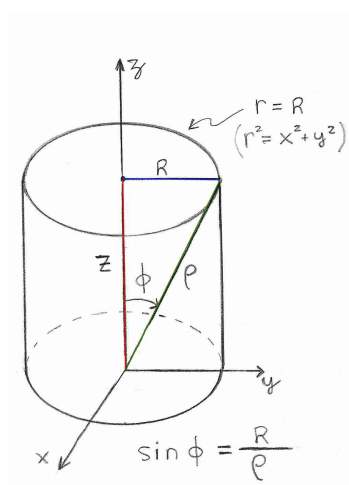
$$\rho = \frac{a+2b+3c}{a\cos(\theta)\sin(\phi)+b\sin(\theta)\sin(\phi)+c\cos(\phi)}.$$

Example 1.6.12. *The equation of a cylinder is* $r = R$ *in cylindrical coordinates. In spherical coordinates it is not as pretty. Note that* $r = R$ *gives* $x^2+y^2 = R^2$ *and*

$$x^2+y^2 = \rho^2\cos^2(\theta)\sin^2(\phi)+\rho^2\cos^2(\theta)\sin^2(\phi) = \rho^2\sin^2(\phi)$$

Thus, the equation of a cylinder in spherical coordinates is $R = \rho\sin(\phi)$.

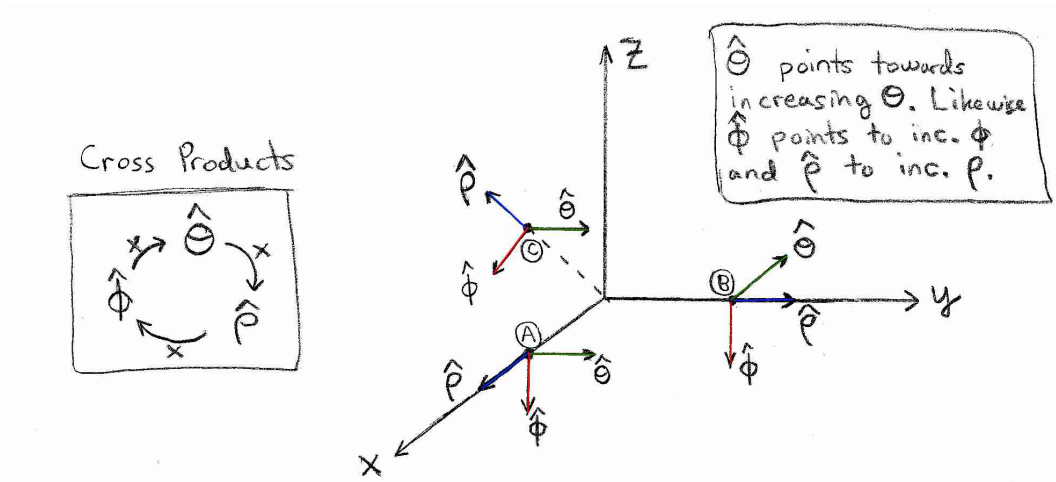
You might notice that the formula above is easily derived geometrically. If you picture a cylinder and draw a rectangle as shown below it is clear that $\sin(\phi) = \frac{R}{\rho}$.



It is important to be proficient in both visualization and calculation. They work together to solve problems in this course, if you get stuck in one direction sometimes the other will help you get free. Let us denote unit vectors in the direction of increasing ρ, ϕ, θ by $\widehat{\rho}$, $\widehat{\phi}$, $\widehat{\theta}$ respectively. You can derive by geometry alone that

$$\begin{aligned}
\widehat{\rho} &= \sin(\phi)\cos(\theta)\,\widehat{x} + \sin(\phi)\sin(\theta)\,\widehat{y} + \cos(\phi)\,\widehat{z} \\
\widehat{\phi} &= -\cos(\phi)\cos(\theta)\,\widehat{x} - \cos(\phi)\sin(\theta)\,\widehat{y} + \sin(\phi)\,\widehat{z} \\
\widehat{\theta} &= -\sin(\theta)\,\widehat{x} + \cos(\theta)\,\widehat{y}.
\end{aligned} \tag{1.9}$$

We call $\{\widehat{\rho},\ \widehat{\phi},\ \widehat{\theta}\}$ the **frame** of spherical coordinates. At each point these unit-vectors point in particular direction.



In constrast to the cartesian frame which is constant[20] over all of $\mathbb{R}^3$.

Example 1.6.13. *Suppose we a point charge q is placed at the origin then by Gauss' Law we can derive*

$$\vec{E} = \frac{q}{4\pi\epsilon_o\rho^2}\widehat{\rho}.$$

This formula makes manifest the spherical direction of the electric field, the absence of the angular unit-vectors says the field has no angular dependence and hence its values depend only on the spherical radius ρ. This is called the **Coulomb field** *or monopole field. Almost the same math applies to gravity. If M is placed at the origin then*

$$\vec{F} = \frac{GmM}{\rho^2}(-\widehat{\rho}).$$

gives the gravitational force $\vec{F}$ of M on m at distance ρ from the origin. The direction of the gravitational field is $-\widehat{\rho}$ which simply says the field points radially inward.

The spherical frame gives us a basis of vectors to build vectors at each point in $\mathbb{R}^3$. More than that, the spherical frame is an orthonormal frame since at any particular point the frame provides an orthonormal set of vectors. In particular, we can show that

$$\widehat{\rho}\bullet\widehat{\rho} = 1, \quad \widehat{\phi}\bullet\widehat{\phi} = 1, \quad \widehat{\theta}\bullet\widehat{\theta} = 1$$

$$\widehat{\phi}\bullet\widehat{\rho} = 0, \quad \widehat{\theta}\bullet\widehat{\rho} = 0, \quad \widehat{\phi}\bullet\widehat{\theta} = 0.$$

We can also calculate either algebraically or geometrically that:

$$\widehat{\theta}\times\widehat{\rho} = \widehat{\phi}, \quad \widehat{\rho}\times\widehat{\phi} = \widehat{\theta}, \quad \widehat{\phi}\times\widehat{\theta} = \widehat{\rho}$$

[20] How do I know the cartesian frame is unchanging? It's not complicated really; $\widehat{x} = \langle 1,0,0\rangle$, $\widehat{y} = \langle 0,1,0\rangle$ and $\widehat{z} = \langle 0,0,1\rangle$.

Therefore, the spherical coordinate system (ρ, ϕ, θ) is a **right-handed** coordinate system since it provides a right-handed basis of unit-vectors at each point. We can summarize these relations compactly with the notation $\widehat{u}_1 = \widehat{\rho},\ \widehat{u}_2 = \widehat{\phi},\ \widehat{u}_3 = \widehat{\theta}$ whence:

$$\widehat{u}_i \bullet \widehat{u}_j = \delta_{ij}, \qquad \widehat{u}_i \times \widehat{u}_j = \sum_{k=1}^{3} \epsilon_{ijk} \widehat{u}_k$$

this is the same pattern we saw for the cartesian unit vectors.

We will return to the polar, cylindrical and spherical coordinate systems as the course progresses. Even now we could consider a multitude of problems based on the combination of the material covered thus-far and it's intersection with curvelinear coordinates. There are other curved coordinate systems beyond these standard three, but I leave those to your imagination for the time being. I do discuss a more general concept of coordinate system in the advanced calculus course. In manifold theory the concept of a coordinate system is refined in considerable depth. We have no need of such abstraction here so I'll behave[21].

[21] I'd guess most calculus text editors would say this whole paragraph is misbehaving, but I have no editor so ha.