Notes on Chapter 1 of Elementary Differential Geometry by Oneil, 2nd Ed.

- p, q are points in $\mathbb{R}^3$ typically
- He identifies p and $\vec{p}$, we can add, subtract and scalar multiply points.
- $x(p) = (p_1, p_2, p_3)$ if $p = (p_1, p_2, p_3)$

$$x_1(p) = p_1$$
$$x_2(p) = p_2$$
$$x_3(p) = p_3$$
$$\underbrace{x_1, x_2, x_3}_{\text{coordinate functions.}} : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

- a <u>function</u> will usually be assumed to be a real-valued, differentiable function. See pg. 4–5 for nuances.
- <u>Exercises on pg. 5</u> (I think you can do these)

Def^{n}/ 2.1 A <u>tangent</u> <u>vector</u> V_p to $\mathbb{R}^3$ consists of two points: its vector part V and its point of application P. Note $\underbrace{V_p = W_q \iff p = q \ \& \ V = W}$

I used (p, v) to capture this idea for V_p in Math332

$V_p = T_p\mathbb{R}^3$

Def^n/ A vector field V on $\mathbb{R}^3$ is a function $p \xmapsto{V} V_p \in T_p\mathbb{R}^3$.

can add, subtract and scalar multiply by functions

$$(fV)(p) = f(p)\,V(p)$$

Def^n/ U_1, U_2, U_3 are vector fields on $\mathbb{R}^3$ which are constant

$$\underbrace{U_1(p) = (1,0,0)_p,\ U_2(p) = (0,1,0)_p,\ U_3(p) = (0,0,1)_p.}_{\text{natural frame field on } \mathbb{R}^3}$$

Lemma: given $p \longmapsto V(p)$ a vector field on $\mathbb{R}^3$ there exist $v_1, v_2, v_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (functions) s.t.

$$V = v_1 U_1 + v_2 U_2 + v_3 U_3$$

v_1, v_2, v_3 are the Euclidean Coordinate Functions of V

Exercises on pg. 11: (these are probably worth completing to better assimilate notation)

Def^n/ A tangent vector $V_p \in T_p\mathbb{R}^3$ may act on a function via the directional derivative;

$$V_p[f] = \frac{d}{dt}\Big(f(p+tv)\Big)\Big|_{t=0}$$

$$= \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p)$$

Lemma 3.2

Th^m/(3.3)

$$\left.\begin{aligned} (aV_p + bW_p)[f] &= aV_p[f] + b\,W_p[f] \\ V_p[af+bg] &= aV_p[f] + b\,V_p[g] \\ V_p[fg] &= V_p[f]\,g(p) + f(p)\,V_p[g] \end{aligned}\right\} \text{linear \& Leibniz.}$$

vectors at a point act on functions to give #'s
vector fields act on functions to give functions, see
Cor 3.4 to see properties of linear & Leibniz
transfer to vector fields.

Example: $U_1[f] = \frac{\partial f}{\partial x_1}$ why?

$$U_1(p)[f] = \frac{d}{dt}\left(f(p+tU_1)\right)\Big|_{t=0} = \frac{\partial f}{\partial x_1}(p) \quad \forall p \in \mathbb{R}^3$$

$$\Rightarrow \quad U_1[f] = \frac{\partial f}{\partial x_1}$$

Likewise, $U_2[f] = \frac{\partial f}{\partial x_2}$, $U_3[f] = \frac{\partial f}{\partial x_3}$. These formulas are nice to remember for later.

Example: $f = x^2 y + z^3$

$$V = xU_1 - y^2 U_3$$

$$\begin{aligned} V[f] &= (xU_1 - y^2 U_3)[x^2y + z^3] \\ &= xU_1(x^2y + z^3) - y^2 U_3[x^2y + z^3] \\ &= \underline{2x^2y - 3y^2z^2}. \end{aligned}$$

Technically, this is an abuse of language, $f(x,y,z) = x^2y + z^3$ not f and the abuse extends to $V[f]$ which should be $V[f](p_1,p_2,p_3)$ etc.
or, maybe, it is rigorous if x, y, z are <u>functions</u>
hmm....

$$f(p_1, p_2, p_3) = p_1^2 p_2 + p_3^3$$

$$(x^2y + z^3)(p_1,p_2,p_3) = (x(p_1,p_2,p_3))^2 y(p_1,p_2,p_3) + (z(p_1,p_2,p_3))^3$$

etc...

<u>Exercises on p. 15</u> (look fun.)

path: $\alpha : I \longrightarrow \mathbb{R}^3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$

curve $C : f = a$

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t) \right)_{\alpha(t)}$$

$$= \sum_{i=1}^{3} \frac{d\alpha_i}{dt}(t)\, U(\alpha(t))$$

- Please read about reparametrization

Exercises pg. 22: (9 problems)

One - Forms

$$\phi(fV + gW) = f\phi(V) + g\phi(W)$$
$$(f\phi + g\psi)(V) = f\phi(V) + g\psi(V)$$

Def^{n}/ The differential df of $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a 1-form where $df(V_p) = V_p[f]$

Lemma 5.4 $\phi = \sum f_i\, dx^i$ then $f_i = \phi(U_i)$

Euclidean coord. function of ϕ

Exercises 27: (11 problems, fun.)

Differential Forms (§1.6) please read. We've done it all and more before. I will use $\wedge$ everywhere, I'm not fond of notation on pg. 29 Ex. 6.1. Likewis §1.7 is about the push-forward in an altered notation.

Def^n/ Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be mapping. If $V_p \in T_p\mathbb{R}^n$ then let $F_*(V_p)$ be the initial velocity of the curve $t \longmapsto F(P+tV)$. The function F_{*p} sends $V_p \in T_p\mathbb{R}^n$ to $F_{*p}(V_p) \in T_{F(P)}\mathbb{R}^m$ and is called the "tangent map"

Prop: $F_{*p}(V) = \left(V[f_1], \ldots, V[f_m]\right)_{F(P)}$

Cor 7.6 F_{*p} is linear transformation.

Cor. 7.8 $F_*(U_j(P)) = \sum_{i=1}^{m} \underbrace{\frac{\partial f_i}{\partial x_j}}_{\substack{\text{components of} \\ \text{Jacobian matrix}}} \overline{U}_i(F(P))$

Def^n/ $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is <u>REGULAR</u> provided F_{*p} is one-one at each $p \in \mathbb{R}^3$.

- Again, please read § 1.7 & 1.8 to remind results from MATH 332.