CHAPTER 3 : EUCLIDEAN GEOMETRY

Def^n/ An isometry of $\mathbb{R}^3$ is a mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $d(F(p), F(q)) = d(p,q) \; \forall p, q \in \mathbb{R}^3$.

Note, $d(p,q) = \|p-q\|$ so the condition above yields $\|F(p)-F(q)\| = \|p-q\|$.

Ex (1) : TRANSLATION

$T(p) = p+a \quad \forall p \in \mathbb{R}^3$, clearly $T(p)-T(q) = (p+a)-(q+a) = p-q$ hence $\|T(p)-T(q)\| = \|p-q\|$.

Ex (2) : Rotation around coord. axis

In xy-plane by angle ϑ have $(p_1, p_2) \longmapsto (q_1, q_2)$ by:

$$q_1 = p_1 \cos\vartheta - p_2 \sin\vartheta$$
$$q_2 = p_1 \sin\vartheta + p_2 \cos\vartheta$$
$$q_3 = p_3 \quad \text{if extend to } \mathbb{R}^3$$

Rotation around z-axis given above $\ni C(p) = (p_1\cos\vartheta - p_2 \sin\vartheta, -, p_3)$

Lemma 1.3 : F, G isometrys of $\mathbb{R}^3 \Rightarrow G \circ F$ also an isometry.

Proof:
$$\begin{aligned} d(G(F(p)), G(F(q))) &= d(G(b), G(a)) \quad G \text{ an isom.} \\ &= d(b, a) \\ &= d(F(p), F(q)) \quad F \text{ an isom.} \\ &= d(p,q) \end{aligned}$$

Thus $G \circ F$ is an isometry.

Lemma 1.4 : (1.) If S, T are translations then $S \circ T = T \circ S$ is also a translation.
(2.) T a translation by a has $T^{-1}(p) = p-a$.
(3.) given $p, q \in \mathbb{R}^3$, $\exists!$ T translation for which $T(p) = q$.

Proof. (1), (2) are easy, well, so is (3) but: $T(p) = p+a$ for $a = q-p$ suffices.

Def^{n}/ Orthogonal Transformation $C: \mathbb{R}^3 \to \mathbb{R}^3$ is linear with $C(p) \cdot C(q) = p \cdot q \ \forall p, q$

Lemma 1.5: orthogonal transformations are isometries.

Proof:
$$\begin{aligned} d(C(p), C(q)) &= \|C(p) - C(q)\| \\ &= \|C(p-q)\| \\ &= \sqrt{C(p-q) \cdot C(p-q)} \\ &= \sqrt{(p-q)\cdot(p-q)} \\ &= d(p,q). // \end{aligned}$$

Lemma 1.6: If F is an isometry of $\mathbb{R}^3$ s.t. $F(0) = 0$ then F is an orthogonal transformation

Proof: $\|F(p) - F(0)\| = \|F(p)\| = d(p, 0) = \|p\|$ thus F preserves norms.

Consider, $d(F(p), F(q)) = d(p, q)$

$$\Rightarrow \|F(p) - F(q)\| = \|p - q\|$$

$$\Rightarrow (F(p) - F(q)) \cdot (F(p) - F(q)) = (p-q) \cdot (p-q)$$

$$\Rightarrow \|F(p)\|^2 - 2F(p) \cdot F(q) + \|F(q)\|^2 = \|p\|^2 - 2p \cdot q + \|q\|^2$$

But, we know $\|F(p)\| = \|p\|$ and $\|F(q)\| = \|q\|$, so

$$F(p) \cdot F(q) = p \cdot q$$

Hence F preserves dot-products. It remains to show F linear.

$$\begin{aligned} F(ap_1 + bp_2) \cdot F(q) &= (ap_1 + bp_2) \cdot q \\ &= a(p_1 \cdot q) + b(p_2 \cdot q) \\ &= \{a p_1\} \cdot q + b\, p_2 \cdot q \\ &= a F(p_1) \cdot F(q) + b F(p_2) \cdot F(q) = (aF(p_1) + bF(p_2)) \cdot q. \end{aligned}$$

* But, this holds $\forall F(q)$ where q is free to vary over $\mathbb{R}^3$. Thus $F(ap_1 + bp_2) = aF(p_1) + bF(p_2)$.

O'neil uses $F(p) = \sum [F(p) \cdot F(u_i)] F(u_i)$ to prove it

* cheated, let's see how O'neil does it,

$$F(P) = \sum' (F(P) \cdot F(U_i)) \, F(U_i)$$

$$F(P) \cdot F(U_i) = P \cdot U_i = P_i \quad \therefore \quad F(P) = \sum_i' P_i F(U_i).$$

Thus $F(aP + bq) = \sum_i' (aP+bq)_i F(U_i)$

$$= a \sum_i P_i F(U_i) + b \sum_i' q_i F(U_i)$$

$$= a F(P) + b F(q). //$$

<u>Remark</u>: that wasn't so bad.

$\boxed{\text{Th}^{m}/(1.7)\ \text{If } F \text{ is an isometry of } \mathbb{R}^3 \text{ then } \exists! \text{ translation } T \text{ and orthogonal transformation } C \text{ such that } F = TC.}$

<u>Proof</u>: Let $T(P) = P + F(0)$ then $T^{-1}(P) = P - F(0)$. Note $T^{-1}F$ is an isometry by Lemma 1.3. Moreover,

$$(T^{-1}F)(0) = T^{-1}(F(0)) = F(0) - F(0) = 0$$

Thus by Lemma 1.6, $\exists C$ orthogonal trans. such that $C = T^{-1}F$.
Consequently $F = TC$.

For uniqueness, Suppose $\exists \bar{T}, \bar{C}$ a trans, orthog. trans. such that $F = \bar{T}\bar{C}$. Note $F = F$ so $TC = \bar{T}\bar{C}$ thus

$C = T^{-1}\bar{T}\bar{C}$. Act on $P = 0$,

$$C(0) = T^{-1}\bar{T}\bar{C}(0) \Rightarrow 0 = T^{-1}\bar{T}(0)$$

$$\Rightarrow T(0) = \bar{T}(0)$$

$$\Rightarrow T(P) = \bar{T}(P) \quad \forall P \quad \text{as } T(P) = T(0) + P.$$

$$\Rightarrow T = \bar{T}$$

Hence $C = T^{-1}\bar{T}\bar{C} \Rightarrow C = \bar{C}$ thus uniqueness follows //

<u>Remark</u>: order matters $TC \neq CT$ generally, but upto this ordering, T, C uniquely describe a given isometry F.

MATRIX STRUCTURE OF ISOMETRY

1.) $T(P) = P + a$ is not matrix multiplication. It's not linear. We just add vector a.

2.) $C(P) \cdot C(q) = P \cdot q \quad \forall P, q$ is given by matrix multiplication. Since $C : \mathbb{R}^3 \to \mathbb{R}^3$ is linear $\exists [C]$ such that $C(P) = [C]P$ thus,

$$C(P) \cdot C(q) = P \cdot q \implies ([C]P) \cdot ([C]q) = P \cdot q$$

$$\Rightarrow P^T [C]^T [C] q = P^T q$$

used $v \cdot w = v^T w$ and socks-shoes $(AB)^T = B^T A^T$

But, this holds $\forall P, q \in \mathbb{R}^3$ and note
$u_i^T [C]^T [C] u_j = ([C]^T [C])_{ij}$ or
generally $u_i^T A u_j = A_{ij}$ so all components match $(I)_{ij} = \delta_{ij}$
and we conclude $([C]^T [C])_{ij} = u_i^T u_j = \delta_{ij} \Rightarrow \underline{[C]^T [C] = I}$ //

Thus, for $F = TC$ we may write

$$F(P) = T(C(P)) = T([C]P) = \underline{a + [C]P}.$$

If we instead wrote $F = CT$ then,

$$F(P) = C(T(P)) = C(a + P) = \underline{[C]a + [C]P}.$$

§ 3.2 THE TANGENT MAP OF AN ISOMETRY

The tangent map of $F : \mathbb{R}^3 \to \mathbb{R}^3$ is what I called the push-forward in Math 332, $F_* : T_p\mathbb{R}^3 \to T_{F(p)}\mathbb{R}^3$ or more descriptively, $(dF)_p : T_p\mathbb{R}^3 \to T_{F(p)}\mathbb{R}^3$. Let us recall the definition, well instead a prop. 7.5 in my notation:

$$(dF)_p(V_p) = \left(V_p[F_1], V_p[F_2], V_p[F_3]\right)_{F(p)}$$
$$= \sum_{i=1}^{3} V_p[F_i]\, U_i(F(p))$$
$$= \sum_{i=1}^{3} dF_i(V_p)\, U_i(F(p))$$

Alternatively, in terms of curves,

$$F_*(\alpha') = (F \circ \alpha)'$$

Or by Cor. 7.8,

$$F_*(U_j(p)) = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_j}(p)\, U_i(F(p)) \qquad (F_i = f_i \text{ here})$$

In math 332, I sell this as coord. change for $\frac{\partial}{\partial x}$, $\partial/\partial y$, $\partial/\partial z$ generalized.

Ex) $F(x,y,z) = (2x, 3y, 4z)$

$$F_1(x,y,z) = 2x$$
$$F_2(x,y,z) = 3y$$
$$F_3(x,y,z) = 4z$$

$$(dF_p)(U_1) = \sum_{i=1}^{3} \frac{\partial F}{\partial x_i} U_i = 2U_1$$
$$(dF_p)(U_2) = 3U_2$$
$$(dF_p)(U_3) = 4U_3$$

reverting to $V_p = \sum V_i \frac{\partial}{\partial x_i}\Big|_p$ notation momentarily (ignore if you like Steven)

If $u = 2x$, $v = 3y$, $w = 4z$

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} \quad \text{or} \quad \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial}{\partial w} = 2\frac{\partial}{\partial u}$$

Likewise $\frac{\partial}{\partial y} = 3\frac{\partial}{\partial v}$, $\frac{\partial}{\partial z} = 4\frac{\partial}{\partial w}$.

§ 3.2 continued:

Th^{m}/ Let F be an isometry of $\mathbb{R}^3$ with orthogonal part C then $\underbrace{F_*(V_p) = C(V)_{F(p)}}$

$$(dF_p)(V_p) = C(V)_{F(p)}$$

Proof: O'neil gives proof in terms of his def^{n} 7.4 of Chpt. 1. I'll use a coordinate-based proof instead.

$$F = TC$$

$$F(P) = a + C(P)$$

$$F_i = a_i + \text{row}_i(C) \cdot (x_1, x_2, x_3)$$

$$\frac{\partial F_k}{\partial x_i} = \frac{\partial}{\partial x_i}(a_k) + \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{3} C_{kj} x_j\right]$$

$$= 0 + \sum_{j=1}^{3} C_{kj} \frac{\partial x_j}{\partial x_i}$$

$$= C_{ki}$$

Thus, $F_*(U_j(P)) = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_j}(P)\, U_i(F(P))$

$$= \sum_{i=1}^{3} C_{ij}\, U_i(F(P))$$

$$= (C_{1j}, C_{2j}, C_{3j})_{F(P)}$$

$$= \text{col}_j([C]) \text{ at } F(P).$$

Thus, $F_* = C$ by linear algebra. Notice, we know $L : \mathbb{R}^3 \to \mathbb{R}^3$ is linear iff $\exists\, [L]$ for which $[L]P = L(P)$ $\forall P \in \mathbb{R}^3$. Moreover, $[L] = [L(U_1)|L(U_2)|L(U_3)]$. Maybe his proof is better. Bottom line, best linear approx. to the change in an affine map is the linear part of the map. This is merely a special case of that result. The fact C is orthog. trans. has little to do with it. //

$\underline{\text{Cor (2.2)}}$ $F_*(V_p) \cdot F_*(W_p) = V_p \cdot W_p$

that is; isometry F has tangent map which preserves dot-products on tangent space.

$\underline{\text{Proof}}$: $F_*(V_p) \cdot F_*(W_p) = C(V_p) \cdot C(W_p)$ C orthogonal transformation. //
$= V_p \cdot W_p$.

Oh, so, I've left ~~out~~ a step,

$$F_*(V_p) \cdot F_*(W_p) = (C(V))_{F(P)} \cdot (C(W))_{F(P)}$$
$$= C(V) \cdot C(W)$$
$$= V \cdot W$$
$$= V_p \cdot W_p$$

$\underline{\text{Comment}}$: isometries allow us to map frames to frames via the tangent map.

Th^{m} (2.3) Given any two frames on $\mathbb{R}^3$, e_1, e_2, e_3 at p and f_1, f_2, f_3 at q, $\exists!$ isometry F of $\mathbb{R}^3$ such that $F_*(e_i) = f_i$ for $i = 1, 2, 3$. $(dF_p(e_i) = f_i \in T_q\mathbb{R}^3)$
$(dF_p(e_i(p)) = f_i(q)$ fwiw$)$

$\underline{\text{Proof}}$: see pg. 109. Basically, $C(e_i) = f_i$ and extend linearly.

$\underline{\text{Calculational Scheme}}$:

$$A^T = [e_1 | e_2 | e_3] \quad \text{and} \quad B^T = [f_1 | f_2 | f_3]$$

By theorem $F_* = C$ has $Ce_i = f_i \rightarrow$
$$\rightarrow CA^T = [Ce_1 / Ce_2 / Ce_3] = [f_1 / f_2 / f_3] = B^T$$
$$\therefore \underline{C = B^T A}. \quad \text{as} \quad A^T A = I$$

The translation part is then simple to tack on.

- I'll try to work an example in class (today 1/27/14)