# §2.8 : THE STRUCTRAL EQUATIONS



In the previous section we saw for frame field $E_1, E_2, E_3$ the covariant derivative of the frame itself is given by the connection forms $\omega_{12}, \omega_{13}, \omega_{23}$:

$$\omega_{ij}(V) = (\nabla_V E_i) \cdot E_j(p)$$

$$\nabla_V E_i = \sum_j \omega_{ij}(V) E_j$$

essentially just the orthornormal basis components Th$^{\underline{m}}$: well, I'm not sure it's even a Th$^{\underline{m}}$...

Anyway, we saw, if

$$\begin{aligned} E_1 &= a_{11} U_1 + a_{12} U_2 + a_{13} U_3 \\ E_2 &= \vdots \; - - - - - - \\ E_3 &= - \; - \; - \quad a_{33} U_3 \end{aligned} \longrightarrow A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

So, if we use concatenation $A^T = \underbrace{[E_1 | E_2 | E_3]}$ just the vector, not the point of application.

well, really $(x_1, x_2, x_3) \longmapsto A(x_1, x_2, x_3)$ is the idea as we allow $p$ to vary. Recall,

Th$^{\underline{m}}$ (7.3) $\quad \omega = dA\, A^T$

In problem 5 of §2.7 hwk, pg. 93 we learn,

$$\boxed{\nabla_V W = \sum_j \Big[ V[f_j] + \sum_i f_i\, \omega_{ij}(V) \Big] E_j}$$

$$W = \sum_{i=1}^{3} f_i E_i$$

**Def$^{n}$/** $E_1, E_2, E_3$ a frame field then the dual forms $\theta_1, \theta_2, \theta_3$ of the frame field are 1-forms for which $\theta_i(V) = V \cdot E_i(P)$ for each $V \in T_p\mathbb{R}^3$

Alternatively and equivalently, say $\theta_i(E_j) = \delta_{ij}$ and extend linearly! Note $\theta_i(E_j) = E_j \cdot E_i = \delta_{ji}$ for all p.
The dual frame has already appeared

$$dx_i(U_j) = \delta_{ij} \qquad \underbrace{U_1, U_2, U_3}_{\text{frame}} \text{ has } \underbrace{dx_1, dx_2, dx_3}_{\text{dual frame.}}$$

**Lemma 8.2/** If $\phi \in \Lambda^1(\mathbb{R}^3)$ then $\phi = \sum_{i=1}^{3} \phi(E_i)\theta_i$
(given $E_1, E_2, E_3$ frame with dual frame $\theta_1, \theta_2, \theta_3$)

Proof: $\phi(E_j) = \underbrace{\left(\sum_{i=1}^{3} c_i\theta_i\right)}(E_j) = \sum_{i=1}^{3} c_i \underbrace{\theta_i(E_j)}_{\delta_{ij}} = c_j$

($\phi = \sum_{i=1}^{3} c_i\theta_i$ follows from fact $\theta_1, \theta_2, \theta_3$ span $\Lambda^1(\mathbb{R}^3)$, you can prove $\theta_1, \theta_2, \theta_3$ are LI and $\dim \Lambda^1(\mathbb{R}^3) = 3$ thus $\text{span}\{\theta_1, \theta_2, \theta_3\} = \Lambda^1(\mathbb{R}^3)$.)

O'neil just shows $\phi(V) = \sum_{i=1}^{3} \left(\phi(E_i)\,\theta_i\right)(V)$ directly.
also good.

$\therefore$

$$E_i = \sum_j{}' a_{ij} U_i$$

$$\theta_i = \sum{}' a_{ij}\, dx_j$$

Same coefficients. I'm tempted to derive this... will resist for moment.

Calculate, $\theta_i(U_j) = \sum_{k=1}^{3}{}' a_{ik} \underbrace{dx_k(U_j)}_{\delta_{kj}} = a_{ij}.$

**Th$^m$ (8.3) (CARTAN'S STRUCTURE EQUATIONS)**

Let $E_1, E_2, E_3$ be a frame field on $\mathbb{R}^3$ with dual forms $\theta_1, \theta_2, \theta_3$ and connection forms $\omega_{ij}$. Then,

(1) $d\theta_i = \sum_j{}' \omega_{ij} \wedge \theta_j$

(2.) $d\omega_{ij} = \sum_k{}' \omega_{ik} \wedge \omega_{kj}$

Proof: notation $\theta_i = \sum_j{}' a_{ij}\, dx_j \rightarrow \theta = A\, d\xi \leftarrow \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$

attitude matrix.

(1.) $d\theta = d(A\,d\xi) = dA \wedge d\xi + A \underbrace{d(d\xi)}_{0}$

$= \underbrace{dA\, A^T} \wedge \underbrace{A\,d\xi}$

$= \omega \wedge \theta \Rightarrow d\theta_i = \sum_j{}' \omega_{ij} \wedge \theta_j$ //

(2.) Note, $d(df\, g) = \underbrace{d(df)\, g - df \wedge dg}_{\text{graded Leibniz.}} = -df \wedge dg.$

(2) continued, we know $\omega = dA\,A^T$ and $A^TA = AA^T = I$
trying to calculate $d\omega$ via $d(df g) = -df \wedge dg$.

$$
\begin{aligned}
d\omega &= d(dA\,A^T) \\
&= -dA \wedge dA^T \\
&= -dA\,A^T \wedge A\,(dA)^T \\
&= -\omega \wedge (dA\,A^T)^T \\
&= -\omega \wedge \omega^T \\
&= -\omega \wedge -\omega \\
&= \omega \wedge \omega. \\
&= \omega\omega \quad \text{in his wedgeless notation.}
\end{aligned}
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Remark: O'neil asserts many properties of wedges & $d$ of matrix valued forms without so much as listing or proving them. If we have time 1/24/14 I'll see if we can prove a few.... whatever these props are...

Ex]

$$E_1 = \cos^2(x)\, U_1 + \cos(x)\sin(x)\, U_2 + \sin(x)\, U_3$$

$$E_2 = \sin x \cos x\, U_1 + \sin^2 x\, U_2 - \cos(x)\, U_3$$

$$E_3 = -\sin(x)\, U_1 + \cos(x)\, U_2$$

Give a weird frame (based on # 3 of § 2-7 for fun)

What is dual frame?

$$E_i = \sum a_{ij} U_j \quad \& \quad \theta_i = \sum_j a_{ij}\, dx_j$$

Same attitude.

$$\theta_1 = \cos^2(x)\, dx + \cos(x)\sin x\, dy + \sin x\, dz$$

$$\theta_2 = \sin x \cos x\, dx + \sin^2 x\, dy - \cos x\, dz$$

$$\theta_3 = -\sin x\, dx + \cos x\, dy$$

Can we calculate $\omega_{ij}$ from $d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$

$$d\theta_1 = \frac{\partial}{\partial x}(\cos x \sin x)\, dx \wedge dy + \cos x\, dx \wedge dz$$

Compare to read of $\omega_{ij}$ ...

$$\sum_{j=1}^{3} \omega_{1j} \wedge \theta_j = \omega_{12} \wedge \theta_2 + \omega_{13} \wedge \theta_3$$

$$= \omega_{12} \wedge (\sin x \cos x\, dx + \sin^2 x\, dy - \cos x\, dz)$$

$$+ \omega_{13} \wedge (-\sin x\, dx + \cos x\, dy)$$

Want to find a moderately nontrivial, nonstandard frame field example. We have cylindrical, spherical, cartesian.

Let $R(\theta)$ be a rotation, feed it $\theta = x + z^2$

$$E_1 = \mathrm{col}_1(R(x+z^2)), \quad E_2 = \mathrm{col}_2(R(x+z^2)), \quad E_3 = \mathrm{col}_3(R(x+z^2))$$

I believe $E_i \cdot E_j = \delta_{ij}$ and $E_1, E_2, E_3$ is smooth frame as $\theta$-dep. of $R$ is sine/cosine and $\theta = x + z^2$ is polynomial in $x, y, z$ coord. Note, $R = A$ in this case.

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$$\omega = dA\, A^T$$

$$= \begin{bmatrix} dA_{11} & dA_{12} & dA_{13} \\ dA_{21} & dA_{22} & dA_{23} \\ dA_{31} & dA_{32} & dA_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\Rightarrow \omega_{12} = dA_{11} A_{21} + dA_{12} A_{22} + dA_{13} A_{23}$$

$$\Rightarrow \omega_{13} = dA_{11} A_{31} + dA_{12} A_{32} + dA_{13} A_{33}$$

$$\Rightarrow \omega_{23} = dA_{31} A_{21} + d_{32} A_{22} + dA_{33} A_{23}$$

$$\omega_{12} = \left(\frac{\partial A_{11}}{\partial x}dx + \frac{\partial A_{11}}{\partial y}dy + \frac{\partial A_{11}}{\partial z}dz\right)A_{21} + \left(\frac{\partial A_{12}}{\partial x}dx + \frac{\partial A_{12}}{\partial y}dy + \frac{\partial A_{12}}{\partial z}dz\right)A_{22} + (dA_{13})A_{23}$$

$$= \left(\frac{\partial A_{11}}{\partial x}A_{21} + \frac{\partial A_{12}}{\partial x}A_{22} + \frac{\partial A_{13}}{\partial x}A_{23}\right)dx + \left(\frac{\partial A_{11}}{\partial z}A_{21} + \frac{\partial A_{12}}{\partial z}A_{22} + \frac{\partial A_{13}}{\partial z}A_{23}\right)dz$$

There would also be a $(\ )dy$ term if $\omega$ was built from $A$ with $y$-dependence. Really need explicit $A$ to appreciate what we're doing here..

$$d\theta_1 = \omega_{12} \wedge \theta_2 + \omega_{13} \wedge \theta_3$$
$$d\theta_2 = \omega_{21} \wedge \theta_1 + \omega_{23} \wedge \theta_3$$
$$d\theta_3 = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2$$

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Spherical Frame has:

$$\theta_1 = d\rho \qquad \Rightarrow d\theta_1 = 0$$
$$\theta_2 = \rho \cos\phi \, d\theta \qquad \Rightarrow d\theta_2 = \cos\phi \, d\rho \wedge d\theta - \rho \sin\phi \, d\phi \wedge d\theta$$
$$\theta_3 = \rho \, d\phi \qquad \Rightarrow d\theta_3 = d\rho \wedge d\phi$$

I wish to use Cartan's Eq$^{n}$'s to calculate $\omega_{ij}$ from $\theta_i$.

$$d\theta_1 = 0 \Rightarrow \omega_{12} \wedge \rho\cos\phi \, d\theta = -\omega_{13} \wedge \rho \, d\phi$$

$$d\theta_2 = \underbrace{\cos\phi \, d\rho \wedge d\theta}_{\text{I}} - \rho \underbrace{\sin\phi \, d\phi \wedge d\theta}_{\text{II}} = \underbrace{-\omega_{12} \wedge d\rho}_{\text{I}} + \underbrace{\omega_{23} \wedge \rho \, d\phi}_{\text{II}}$$

Observe I equate these to see $\boxed{\omega_{12} = \cos\phi \, d\theta}$ $(d\rho \wedge d\theta = -d\theta \wedge d\rho)$

Observe from II that $(\sin\phi \, d\theta \wedge \rho \, d\phi = \omega_{23} \wedge \rho \, d\phi)$

$$\boxed{\omega_{23} = \sin\phi \, d\theta}.$$

But, $\omega_{12} = \cos\phi \, d\theta \Rightarrow \omega_{12} \wedge \cos\phi \, d\theta = -\omega_{13} \wedge \rho \, d\phi$

$$\Rightarrow \sin\phi \, d\theta \wedge \cos\phi \, d\theta = -\omega_{13} \wedge \rho \, d\phi$$
$$\Rightarrow 0 = -\omega_{13} \wedge \rho \, d\phi$$
$$\Rightarrow \omega_{13} = \alpha \, d\phi$$

Returning to $d\theta_3 = d\rho \wedge d\phi = -\omega_{13} \wedge d\rho \Rightarrow \boxed{\omega_{13} = d\phi}$

We've calculated $\omega_{12}, \omega_{23}, \omega_{13}$ via the structure eq$^{n}$'s of CARTAN. I ASKED YOU to do a problem where we check the CARTAN EQ$^{n}$'s are correct, but, you should understand they can be used as the computational basis for the whole theory. This page just gives you a taste.