Printed Name: ______________________.

PHYSICS 331	MISSION 1: VECTORS AND CALCULUS (37+3)PTS)

Problems mentioned refer to David Griffith's *Introduction to Electrodynamics*, 5th edition. Note, I have many problems solved from the 3rd edition, but the numbering sometimes differs. Furthermore, I should warn, I have learned easier ways to solve many of those problems. For example, in this assignment the use of index notation can save much calculation. For a vector, or vector field, $\vec{F} = \langle F_1, F_2, F_3\rangle$ we write

$$\vec{F} = F_1 \underbrace{\langle 1,0,0\rangle}_{\widehat{x_1}} + F_2 \underbrace{\langle 0,1,0\rangle}_{\widehat{x_2}} + F_3 \underbrace{\langle 0,0,1\rangle}_{\widehat{x_3}} = F_1\widehat{x_1} + F_2\widehat{x_2} + F_3\widehat{x_3} = \sum_{i=1}^{3} F_i\widehat{x_i}.$$

In this notation vector addition is defined by $(\vec{A}+\vec{B})_i = A_i + B_i$ and scalar multiplication is defined by $(c\vec{A})_i = cA_i$ for all $i = 1,2,3$. Given $\vec{A}, \vec{B}$ we have $\vec{A} = \sum_{i=1}^{3} A_i\widehat{x_i}$ and $\vec{B} = \sum_{i=1}^{3} B_i\widehat{x_i}$. Moreover,

$$\vec{A}\cdot\vec{B} = \sum_{i=1}^{3} A_iB_i \qquad \& \qquad \vec{A}\times\vec{B} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}A_iB_j\widehat{x_k}.$$

Gradient of f, divergence of $\vec{F}$ and curl of $\vec{F}$ are likewise defined by

$$\nabla f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\widehat{x_i} \qquad \& \qquad \nabla\cdot\vec{F} = \sum_{i=1}^{3}\frac{\partial F_i}{\partial x_i} \qquad \& \qquad \nabla\times\vec{F} = \sum_{i,j,k=1}^{3}\epsilon_{ijk}\frac{\partial F_j}{\partial x_i}\widehat{x_k}.$$

Or, using the notation $\frac{\partial}{\partial x_i} = \partial_i$ we have $\nabla f = \sum_{i=1}^{3}(\partial_i f)\widehat{x_i}$ and $\nabla\cdot\vec{F} = \sum_{i=1}^{3}\partial_i F_i$ and $\nabla\times\vec{F} = \sum_{i,j,k=1}^{3}\epsilon_{ijk}(\partial_i F_j)\widehat{x_k}$. I should mention, when faced with ϵ_{ijk} symbols contracted against one another there is a well-known[1] identity which is exceedingly helpful: $\sum_k \epsilon_{mnk}\epsilon_{ijk} = \delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}$.

Advice: these problems are not to be done in one day. I strongly recommend reading all of them, or at least keep reading until you see one that you feel confident you can solve. Work that problem. Work problems on separate pages so you have no issue of turning them in order. Please only work on one side for the sake of my grading. Then, go on, find another problem you can devote to destruction. Continue, until you find the core problems which mock and malign your attacks. Sleep on those. Try them again. Rinse, repeat. Of course, you are welcome to ask for hints at any time. I will try to build hints into lecture as well. This particular Mission is more broad than most so the difficulty is probably not indicative of later problem sets. It is both harder and easier than those. Finally, keep in mind the solutions I have posted, there are solutions to some of these problems. I also don't mind you looking up solutions to the Griffith's problems, it's not difficult to find instructors solution manuals. The point of these is for you to relearn Calculus III, how to use vector calculus and the very new parts of curvelinear vector calculus and the Dirac Delta function.

Problem 1 (3pts) This problem is based in part on Problems 1.5 & 1.6.

(a.) Show $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\cdot\vec{C}\right)\vec{B} - \left(\vec{A}\cdot\vec{B}\right)\vec{C}$. Note: this is often known as the *BAC-CAB* identity since that is how it is written in the front-cover of the text.

[1] within the context of people who face such problems

(b.) Show $\vec{A}\times\left(\vec{B}\times\vec{C}\right)+\vec{B}\times\left(\vec{C}\times\vec{A}\right)+\vec{C}\times\left(\vec{A}\times\vec{B}\right)=0$. Note: this is the Jacobi identity, the fact that the cross-product satisfies this rule paired with it's linearity shows $\mathbb{R}^3$ paired with $\times$ forms a *Lie Algebra*. Lie algebras are of central importance to modern particle physics.

(c.) When is it true that $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\times\vec{B}\right)\times\vec{C}$?

Problem 2 (4pts) This problem based in part on Griffith's Problem 1.21 & 1.24. I give a bit more guidance.

(a.) Let $\vec{A},\vec{B}$ and f be smooth. Show,

(i.) $\nabla\cdot(f\vec{A})=(\nabla f)\cdot\vec{A}+f(\nabla\cdot\vec{A})$

(ii.) $\nabla\times(f\vec{A})=(\nabla f)\times\vec{A}+f(\nabla\times\vec{A})$

(b.) Let g be smooth and nonzero and let $\vec{B}=\frac{\vec{A}}{g}$ hence $\vec{A}=g\vec{B}$.

(i.) Derive the quotient rule for divergence from $\nabla\cdot\vec{A}=\nabla\cdot(g\vec{B})$.

(ii.) Derive the quotient rule for curl from $\nabla\times\vec{A}=\nabla\times(g\vec{B})$.

Problem 3 (3pts) Problem 1.13 (calculus of the displacement vector). We define $\vec{\mathscr{r}}=\vec{r}-\vec{r}\,'$ where $\vec{r}=(x,y,z)$ and $\vec{r}\,'=(x',y',z')$. Here we consider $\vec{r}$ as a variable point and $\vec{r}\,'$ as fixed.

(a.) Show that: $\nabla(\mathscr{r}^2)=2\vec{\mathscr{r}}$

(b.) Show that: $\nabla\left(\frac{1}{\mathscr{r}}\right)=\frac{-\hat{\mathscr{r}}}{\mathscr{r}^2}$

(c.) Find the formula for $\nabla(\mathscr{r}^n)$

Problem 4 (2pts) Problem 1.36. I diverge notationally from Griffith's in that I prefer to use ∂S to denote the positively oriented boundary of the surface S and $\partial\mathcal{V}$ to denote the positively oriented boundary of the volume $\mathcal{V}$.

(a.) Show that $\displaystyle\int_S f(\nabla\times\vec{A})\cdot d\vec{a}=\int_S\left[\vec{A}\times(\nabla f)\right]\cdot d\vec{a}+\oint_{\partial S} f\vec{A}\cdot d\vec{l}$

(b.) Show that $\displaystyle\int_{\mathcal{V}}\vec{B}\cdot(\nabla\times\vec{A})d\tau=\int_{\mathcal{V}}\vec{A}\cdot(\nabla\times\vec{B})d\tau+\oint_{\partial\mathcal{V}}(\vec{A}\times\vec{B})\cdot d\vec{a}$.

Problem 5 (3pt) Problem 1.38

Problem 6 (3pt) Problem 1.39

Problem 7 (3pt) Problem 1.44a & 1.45a & 1.46

Problem 8 (3pt) Problem 1.47

Problem 9 (3pt) Problem 1.50

Problem 10 (3pt) Problem 1.54

Problem 11 (3pt) Problem 1.59

Problem 12 (4pt) Problem 1.61

Problem 13 (3pt) Problem 1.63

MISSION 1 SOLUTION

$\boxed{P1}$ Note $\sum_k \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ is nice to know.

(a.) $$\vec{A}\times(\vec{B}\times\vec{C}) = \sum_{i,j,k} \epsilon_{ijk} A_i (\vec{B}\times\vec{C})_j \hat{x}_k$$
$$= \sum_{i,j,k} \epsilon_{ijk} A_i \Big(\sum_{l,m} \epsilon_{lmj} B_l C_m\Big)\hat{x}_k$$
$$= \sum_{i,k,l,m}\sum_j \epsilon_{kij}\epsilon_{lmj} A_i B_l C_m \hat{x}_k$$
$$= \sum_{i,k,l,m} (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) A_i B_l C_m \hat{x}_k$$
$$= \sum_{i,k} B_k A_i C_i \hat{x}_k - \sum_{i,k} C_k A_i B_i \hat{x}_k$$
$$= \Big(\sum_i A_i C_i\Big)\Big(\sum_k B_k \hat{x}_k\Big) - \Big(\sum_i A_i B_i\Big)\Big(\sum_k C_k \hat{x}_k\Big)$$
$$= \underline{(\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}.$$

(b.) $$\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = ?$$
$$= \cancel{(\vec{A}\cdot\vec{C})}\vec{B} - \cancel{(\vec{A}\cdot\vec{B})}\vec{C} + \cancel{(\vec{B}\cdot\vec{A})}\vec{C} - \cancel{(\vec{B}\cdot\vec{C})}\vec{A} + \cancel{(\vec{C}\cdot\vec{B})}\vec{A} - \cancel{(\vec{C}\cdot\vec{A})}\vec{B}$$
1 2 2 3 3 1
$$= 0$$ (I used $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ to make above cancellations.

(c.) $$\underline{\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\times\vec{C}}\,* \iff \left(\begin{matrix}\text{ONE OF}\\ \text{THEM}\\ \text{ZERO}\end{matrix},\ \text{ALL}\perp,\ \vec{A},\vec{C}\text{ colinear}\right)$$
$$= -\vec{C}\times(\vec{A}\times\vec{B})$$
$$= \vec{C}\times(\vec{B}\times\vec{A}) \iff (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = (\vec{C}\cdot\vec{A})\vec{B} - (\vec{C}\cdot\vec{B})\vec{A}$$
$$\iff \underline{(\vec{A}\cdot\vec{B})\vec{C} = (\vec{B}\cdot\vec{C})\vec{A}}.$$

If any of $\vec{A},\vec{B},\vec{C}$ are zero then * clearly holds true. For $\vec{A},\vec{B},\vec{C}$ nonzero, if $\vec{A}\cdot\vec{B} = 0$ then $\vec{B}\cdot\vec{C} = 0$ must also hold for * hence $\vec{A}\perp\vec{B}\perp\vec{C}$ makes * true. Alternatively, $\vec{A}\cdot\vec{B}\neq 0$ and $\vec{B}\cdot\vec{C}\neq 0$ in which case $\vec{C}$ and $\vec{A}$ are <u>colinear</u>.

PROBLEM 2

(a.) (i.) $\nabla\cdot(f\vec{A}) = \sum_i \partial_i (f A_i)$

$$= \sum_i \left((\partial_i f) A_i + f \partial_i A_i\right)$$

$$= \sum_i (\partial_i f) A_i + f \sum_i \partial_i A_i$$

$$= \underline{(\nabla f)\cdot\vec{A} + f\,\nabla\cdot\vec{A}}\,.//$$

(ii.) $\left(\nabla\times(f\vec{A})\right)_k = \sum_{i,j} \epsilon_{ijk}\, \partial_i (f A_j)$

$$= \sum_{i,j} \epsilon_{ijk}\left[(\partial_i f) A_j + f\,\partial_i A_j\right]$$

$$= \sum_{i,j} \epsilon_{ijk}\, (\nabla f)_i A_j + f \sum_{i,j} \epsilon_{ijk}\, \partial_i A_j$$

$$= \underline{\left((\nabla f)\times\vec{A} + f\,\nabla\times\vec{A}\right)_k}\,.$$

Thus $\underline{\nabla\times(f\vec{A}) = (\nabla f)\times\vec{A} + f\,\nabla\times\vec{A}}\,.//$

(b.) Consider $\vec{B} = \frac{1}{g}\vec{A}$ and $\vec{A} = g\vec{B}$,

(i.) $\nabla\cdot\vec{A} = \nabla\cdot(g\vec{B}) = (\nabla g)\cdot\vec{B} + g\,\nabla\cdot\vec{B}$

$\Rightarrow \nabla\cdot\vec{B} = \frac{1}{g}\left(\nabla\cdot\vec{A} - (\nabla g)\cdot\vec{B}\right)$ since $g\vec{B} = \vec{A}$

$$\therefore\ \underline{\nabla\cdot\left(\frac{1}{g}\vec{A}\right) = \frac{(\nabla\cdot\vec{A})g - (\nabla g)\cdot\vec{A}}{g^2}}\,//$$

(ii.) $\nabla\times\vec{A} = \nabla\times(g\vec{B}) = (\nabla g)\times\vec{B} + g\,\nabla\times\vec{B}$

$$\Rightarrow \nabla\times\left(\frac{1}{g}\vec{A}\right) = \left(\nabla\times\vec{A} - \nabla g\times\frac{1}{g}\vec{A}\right)\frac{1}{g}$$

$$\therefore\ \underline{\nabla\times\left(\frac{1}{g}\vec{A}\right) = \frac{1}{g^2}\left((\nabla\times\vec{A})g - (\nabla g)\times\vec{A}\right)}\,.//$$

PROBLEM 3 Let $\vec{r} = (x, y, z)$ and $\vec{r}' = (x', y', z')$ and define $\vec{\mathscr{r}} = \langle x - x', y - y', z - z' \rangle = \vec{r} - \vec{r}'$.

(a.) $$\nabla(\mathscr{r}^2) = \nabla\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]$$
$$= \langle 2(x-x'), 2(y-y'), 2(z-z') \rangle$$
$$= \underline{2\vec{\mathscr{r}}}.$$

(b.) $$\nabla\left(\frac{1}{\mathscr{r}}\right) = \frac{-1}{\mathscr{r}^2} \nabla \mathscr{r} = \underline{\frac{-\hat{\mathscr{r}}}{\mathscr{r}^2}}.$$

Note: $$\nabla \mathscr{r} = \nabla \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$
$$= \frac{1}{2} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \langle 2(x-x'), 2(y-y'), 2(z-z') \rangle$$
$$= \frac{\langle x-x', y-y', z-z' \rangle}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} = \frac{\vec{\mathscr{r}}}{\|\vec{\mathscr{r}}\|} = \hat{\mathscr{r}}.$$

(c.) $$\nabla(\mathscr{r}^n) = n\mathscr{r}^{n-1} \nabla \mathscr{r} = \underline{n\mathscr{r}^{n-1}\hat{\mathscr{r}}}.$$

PROBLEM 4

(a.) $\nabla \times (f\vec{A}) = (\nabla f) \times \vec{A} + f(\nabla \times \vec{A})$

$$\int_S f(\nabla \times \vec{A}) \cdot d\vec{a} = \int_S \left[\nabla \times (f\vec{A}) - (\nabla f) \times \vec{A} \right] \cdot d\vec{a}$$

$$= \oint_{\partial S} f\vec{A} \cdot d\vec{l} - \int_S ((\nabla f) \times \vec{A}) \cdot d\vec{a}$$

$$= \underline{\int_S (\vec{A} \times \nabla f) \cdot d\vec{a} + \oint_{\partial S} f\vec{A} \cdot d\vec{l}}$$ //

(b.) $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

$$\int_{\mathcal{V}} \vec{B} \cdot (\nabla \times \vec{A}) d\tau = \int_{\mathcal{V}} \left[\nabla \cdot (\vec{A} \times \vec{B}) + \vec{A} \cdot (\nabla \times \vec{B}) \right] d\tau$$

$$= \underline{\oint_{\partial \mathcal{V}} (\vec{A} \times \vec{B}) \cdot d\vec{a} + \int_{\mathcal{V}} \vec{A} \cdot (\nabla \times \vec{B}) d\tau}$$ //

$\boxed{\text{PROBLEM 5}}$ (1.38)

Express $\hat{r}, \hat{\theta}, \hat{\phi}$ in terms of $\hat{x}, \hat{y}, \hat{z}$ and vice-versa.

In lecture 2 I derived via calculus,

$$\hat{r} = (\cos\varphi \sin\theta)\hat{x} + (\sin\varphi \sin\theta)\hat{y} + (\cos\theta)\hat{z}$$

$$\hat{\theta} = (\cos\theta\cos\varphi)\hat{x} + (\cos\theta\sin\varphi)\hat{y} - (\sin\theta)\hat{z}$$

$$\hat{\varphi} = -(\sin\varphi)\hat{x} + (\cos\varphi)\hat{y}$$

Notice $\hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{\varphi} = \hat{\theta}\cdot\hat{\varphi} = 0$ and $||\hat{r}|| = ||\hat{\theta}|| = ||\hat{\varphi}|| = 1$ thus, $\hat{r}, \hat{\theta}, \hat{\varphi}$ is orthonormal frame and hence

$$\hat{x} = (\hat{x}\cdot\hat{r})\hat{r} + (\hat{x}\cdot\hat{\theta})\hat{\theta} + (\hat{x}\cdot\hat{\phi})\hat{\varphi}$$

$$\therefore \underline{\hat{x} = (\cos\varphi\sin\theta)\hat{r} + (\cos\theta\cos\varphi)\hat{\theta} - (\sin\varphi)\hat{\varphi}}.$$

Likewise,

$$\hat{y} = (\hat{y}\cdot\hat{r})\hat{r} + (\hat{y}\cdot\hat{\theta})\hat{\theta} + (\hat{y}\cdot\hat{\varphi})\hat{\varphi}$$

$$\therefore \underline{\hat{y} = (\sin\varphi\sin\theta)\hat{r} + (\cos\theta\sin\varphi)\hat{\theta} + (\cos\varphi)\hat{\varphi}}.$$

and,

$$\hat{z} = (\hat{z}\cdot\hat{r})\hat{r} + (\hat{z}\cdot\hat{\theta})\hat{\theta} + (\hat{z}\cdot\hat{\varphi})\hat{\varphi}$$

$$\underline{\hat{z} = (\cos\theta)\hat{r} - (\sin\theta)\hat{\theta}}.$$

$\boxed{\text{PROBLEM 6}}$ 1.39 *: both cases I use formula from cover of Griffiths.

(a.) consider $\vec{V}_1 = r^2\hat{r}$, check divergence Th$^{\underline{m}}$ for B_R with $\partial B_R = S_R$. Use formula for divergence in sphericals

$$\nabla\cdot\vec{V}_1 = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\cdot r^2\right]}_{*} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^4\right] = \frac{1}{r^2}4r^3 = 4r$$

Then,

$$\int_{B_R}(\nabla\cdot\vec{V}_1)\,d\tau = \int_0^R\int_0^{2\pi}\int_0^{\pi}(4r)(r^2\sin\theta\,d\theta\,d\phi\,dr)$$

$$= \int_0^R 4r^3dr\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta$$

$$= \underline{4\pi R^4}.$$

$$\int_{\partial B_R}\vec{V}_1\cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi}(R^2\hat{r})\cdot(R^2\sin\theta\,d\theta\,d\phi\,\hat{r})$$

$$= R^3\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta$$

$$= \underline{4\pi R^3}. \qquad \therefore \int_{B_R}(\nabla\cdot\vec{V}_1)\,d\tau = \int_{\partial B_R}\vec{V}_1\cdot d\vec{a}$$

(b.) consider $\vec{V}_2 = \frac{1}{r^2}\hat{r}$, does div. Th$^{\underline{m}}$ work?

$$\nabla\cdot\vec{V}_2 = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{1}{r^2}\right]}_{*} = 0 \quad \text{for} \quad r\neq 0$$

$$\int_{\partial B_R}\vec{V}_2\cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2\sin\theta\,d\theta\,d\phi\,\hat{r}\right) = \int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\,d\theta = \underline{4\pi}.$$

Of course, we know how to "fix" this, $\underline{\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})}$.

Then $$\int_{B_R}(\nabla\cdot\vec{V}_2)\,d\tau = \int_{B_R}\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau = \int_{B_R}4\pi\,\delta^3(\vec{r})\,d\tau = 4\pi.$$

PROBLEM 7

(1.44a) $\int_2^6 (3x^2-2x-1)\delta(x-3)dx = 3(3)^2-2(3)-1 = \boxed{20}$

(1.45a) $\int_{-2}^{2}(2x+3)\delta(3x)dx = \int_{-2}^{2}(2x+3)\frac{1}{3}\delta(x)dx = \frac{2(0)+3}{3} = \boxed{1}$

(Eq^ 1.94 as we proved in Lecture 3)

(1.46) Show that

(a.) $x\frac{d}{dx}(\delta(x)) = -\delta(x)$

Notice, for any ordinary function f,

$$\frac{d}{dx}[xf\delta] = f\delta + x\frac{df}{dx}\delta + xf\frac{d}{dx}[\delta(x)]$$

Hence,

$$\int_{-\infty}^{\infty} -xf\frac{d}{dx}[\delta(x)]dx = \int_{-\infty}^{\infty}\left(f(x) + x\frac{df}{dx}\right)\delta(x) - \int_{-\infty}^{\infty}\frac{d}{dx}[xf\delta]dx$$

$$= f(0) + 0f'(0) - xf(x)\delta(x)\Big|_{-\infty}^{\infty}$$

$$= f(0)$$

Then $\int_{-\infty}^{\infty} f\,\delta(x)dx = f(0)$ $\therefore$ $-x\frac{d}{dx}[\delta(x)] = \delta(x)$

$$\Rightarrow \underline{x\frac{d}{dx}[\delta(x)] = -\delta(x)}.$$

(b.) Define $\Theta(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x\le 0\end{cases}$, show $\frac{d\Theta}{dx} = \delta(x)$

Note, $\frac{d\Theta}{dx} = \begin{cases}\infty & \text{for } x=0 \\ 0 & \text{for } x\neq 0\end{cases}$

and,

$$\int_{-\infty}^{\infty}\frac{d\Theta}{dx}dx = \Theta(\infty)-\Theta(-\infty) = 1$$

$$\int_{-\infty}^{\infty}\delta(x)dx = 1$$

$\therefore$ $\underline{\frac{d\Theta}{dx} = \delta(x)}.$

P8 (1.47)

(a.) charge density for point charge q at $\vec{r}'$

$$\boxed{\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{r}')}$$

$$\int_{\mathcal{V}} \rho(\vec{r})\,d\tau = \int_{\mathcal{V}} q\,\delta^3(\vec{r}-\vec{r}')\,d\tau = \begin{cases} q & \text{if } \vec{r}' \in \mathcal{V} \\ 0 & \text{if } \vec{r}' \notin \mathcal{V} \end{cases}$$

(b.) write charge density for dipole with $-q$ at origin and pt. charge $+q$ at $\vec{a}$.

$$\boxed{\rho(\vec{r}) = -q\,\delta^3(\vec{r}) + q\,\delta^3(\vec{r}-\vec{a})}$$

(c.) find $\rho = \frac{dQ}{dV}$ for uniform spherical shell of radius R and total charge Q.

$$\boxed{\rho(\vec{r}) = \frac{Q}{4\pi R^2}\,\delta(r-R)} = \begin{cases} 0 & \text{if } r \neq R \\ \infty & \text{if } r = R \end{cases}$$

Then, if $r = R$ is within $\mathcal{V}$ then,

$$\int_{\mathcal{V}} \rho(\vec{r})\,d\tau = \int_{B_R} \frac{Q}{4\pi R^2}\,\delta(r-R)\,d\tau$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{Q}{4\pi R^2}\,\delta(r-R)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{Q}{4\pi R^2}\,R^2\sin\theta\,d\theta\,d\phi = \frac{QR^2}{4\pi R^2}(2\pi)(2) = Q.$$

P9 1.50

(a.) $\vec{F_1} = x^2\hat{z}$ and $\vec{F_2} = \langle x, y, z\rangle$

Calculate curl and divergence of $\vec{F_1}$ & $\vec{F_2}$.

Which can be written as grad. of scalar?

Which can be written as curl of a vector?

Find the scalar & vector potentials.

$$\nabla\cdot\vec{F_1} = 0 \quad \text{and} \quad \nabla\cdot\vec{F_2} = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 3.$$

$$\nabla\times\vec{F_1} = \left(\hat{x}\,\partial_x + \hat{y}\,\partial_y + \hat{z}\,\partial_z\right)\times(x^2\hat{z}) \qquad \underline{\nabla\times\vec{F_2} = 0}.$$

$$= \hat{x}\times\hat{z}\,\partial_x(x^2)$$

$$= 2x(-\hat{y})$$

$$= \underline{-2x\hat{y}}.$$

Thus, $\exists f:\mathbb{R}^3\to\mathbb{R}$ with $\vec{F_2} = \nabla f$, indeed

$\frac{\partial f}{\partial x} = x$, $\frac{\partial f}{\partial y} = y$, $\frac{\partial f}{\partial z} = z$ has solution

$$\boxed{f(x,y,z) = \tfrac{1}{2}(x^2+y^2+z^2) \quad \text{with} \quad \nabla f = \vec{F_2}}$$

Likewise, we wish to find $\vec{A} = \langle a,b,c\rangle$ s.t.

$$\nabla\times\vec{A} = \vec{F_1} = \langle 0,0,x^2\rangle$$

$$\langle \partial_y c - \partial_z b,\ \partial_z a - \partial_x c,\ \partial_x b - \partial_y a\rangle = \langle 0,0,x^2\rangle$$

adhoc, many other choices! } Setting $b = c = 0$ makes it easy, just need to solve $\frac{\partial a}{\partial z} = 0$ and $-\frac{\partial a}{\partial y} = x^2 \Rightarrow \underline{a = -x^2y}$.

$$\boxed{\nabla\times\vec{A} = \vec{F_1} \text{ if we use } \vec{A} = \langle -x^2y, 0, 0\rangle}$$

P9 continued / 1.50

(b.) $\vec{F}_3 = yz\hat{x} + zx\hat{y} + xy\hat{z}$ can be written both as ∇g and $\nabla \times \vec{B}$ for appropriate g and $\vec{B}$.

$$\nabla(xyz) = \langle yz, xz, xy \rangle = \vec{F}_3 \quad \therefore \quad \underline{g = xyz}.$$

If I was to guess, given the symmetry of $\vec{F}_3$ perhaps $\vec{B} = \langle xyz, xyz, xyz \rangle$ will do.

$$\nabla \times \vec{B} = \langle (\partial_y - \partial_z)(xyz), (\partial_y - \partial_z)(xyz), (\partial_x - \partial_y)(xyz) \rangle$$

$$= \langle xz - xy, xz - xy, yz - xz \rangle$$

well, nope.

ok, I'll try to solve these w/o just guessing,

$$\nabla \times \vec{B} = \langle yz, xz, xy \rangle$$

$$\partial_y B_3 - \partial_z B_2 = yz$$
$$\partial_z B_1 - \partial_x B_3 = xz$$
$$\partial_x B_2 - \partial_y B_1 = xy$$

If $B_2 = 0$ then $\partial_y B_3 = yz \;\therefore\; B_3 = \frac{1}{2}y^2 z + C_1(x,y)$

Need $\underbrace{\frac{\partial B_1}{\partial z} - \frac{\partial C_1}{\partial x} = xz}_{B_1 = \frac{1}{2}xz^2 + C_2(x,y)}$ and $\underbrace{\frac{-\partial B_1}{\partial y} = xy}_{B_1 = -\frac{1}{2}xy^2 + C_3(x,z)}$

Thus, setting $C_1 = 0$, (there are many other solutions)

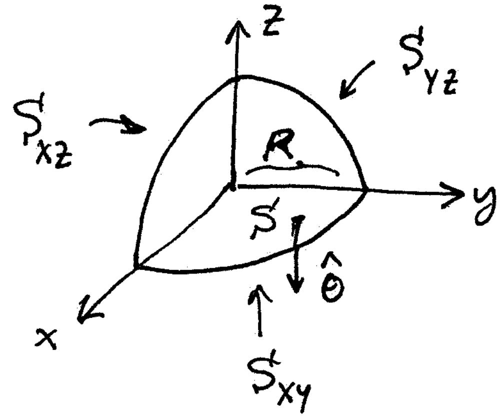
$$\boxed{\vec{B} = \langle \tfrac{1}{2}x(z^2 - y^2), 0, \tfrac{1}{2}y^2 z \rangle \text{ has } \nabla \times \vec{B} = \vec{F}_3}$$

P10 (1.54)

Check divergence Th^m for vector field

$$\vec{V} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$$

using $\mathcal{V}$ as an octant of $\mathcal{B}_R$



$$d\vec{a}_1 = (R^2\sin\theta\,d\phi\,d\theta)\,\hat{r}$$
$$d\vec{a}_{xz} = (dx\,dz)(-\hat{y})$$
$$d\vec{a}_{xy} = (-dx\,dy)\,\hat{z}$$
$$d\vec{a}_{yz} = (-dy\,dz)\,\hat{x}$$

Probably should rewrite $d\vec{a}_{xz}$, $d\vec{a}_{xy}$, $d\vec{a}_{yz}$ in sphericals.

$$\int_{S} \vec{V}\cdot d\vec{a}_1 = \int_0^{\pi/2}\int_0^{\pi/2} (R^2\cos\theta)(R^2\sin\theta\,d\phi\,d\theta) = R^4\left(\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4} \quad \text{I}$$

$$\int_{S_{xy}} \vec{V}\cdot d\vec{a}_{xy} = \int_{S_{xy}} \vec{V}\cdot(da\,\hat{\theta})$$

since cos

$$da = r\,d\phi\,dr$$
$$d\vec{a}_{xy} = r\,d\phi\,dr\,\hat{\theta}$$

$$= \int_0^R\int_0^{\pi/2} (r^2\cos\phi)(r\,d\phi\,dr)$$

$$= \left(\frac{r^4}{4}\Big|_0^R\right)\left(+\sin\phi\Big|_0^{\pi/2}\right) = R^4/4 \quad \text{II}$$

$$\int_{S_{yz}} \vec{V}\cdot d\vec{a}_{yz} = \int_{S_{xy}} \vec{V}\cdot(r\,d\theta\,dr\,\hat{\varphi})$$

$$= \int_0^R\int_0^{\pi/2} \underbrace{(-r^2\cos\theta\sin\phi)(r\,d\theta\,dr)}_{\text{at } \phi = \pi/2} = -\int_0^R r^3 dr\int_0^{\pi/2}\cos\theta\,d\theta = \frac{-R^4}{4} \quad \text{III}$$

P10 continued

$$\int_{S_{xz}} \vec{V}\cdot d\vec{a}_{xz} = \int_{S_{xz}} \vec{V}\cdot \left(r\,d\theta\,dr(-\hat{\phi})\right)$$

$$= \int_0^R \int_0^{\pi/2} \left(-r^2\cos\theta\,\underbrace{\sin\phi}_{0}\,\hat{\phi}\right)\cdot\left(r\,d\theta\,dr\,(-\hat{\phi})\right)$$

subject $\phi = 0$

$$= \underline{0} \quad \text{IV}$$

Consequently, I + II + III + IV yields $\boxed{\int_{\partial\mathcal{V}} \vec{V}\cdot d\vec{a} = \frac{\pi R^4}{4}}$ as II and III cancel out.

It remains to calculate $\int_{\mathcal{V}} (\nabla\cdot\vec{V})\,d\tau$, using formula for div,

$$\nabla\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^4\cos\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\, r^2\cos\phi\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[-r^2\cos\theta\sin\phi\right]$$

$$= 4r\cos\theta + \underbrace{\frac{r\cos\theta\cos\phi}{\sin\theta} - \frac{r\cos\theta\cos\phi}{\sin\theta}}_{\text{nice.}}$$

$$= \underline{4r\cos\theta}$$

$$\int_{\mathcal{V}} (\nabla\cdot\vec{V})\,d\tau = \int_0^R \int_0^{\pi/2} \int_0^{\pi/2} (4r\cos\theta)\, r^2\sin\theta\, d\theta\, d\phi\, dr$$

$$= \int_0^R 4r^3\,dr \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \int_0^{\pi/2} d\phi$$

$$= (R^4)\left(\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}\right)\left(\frac{\pi}{2}\right)$$

$$= \boxed{\frac{\pi R^4}{4}}$$

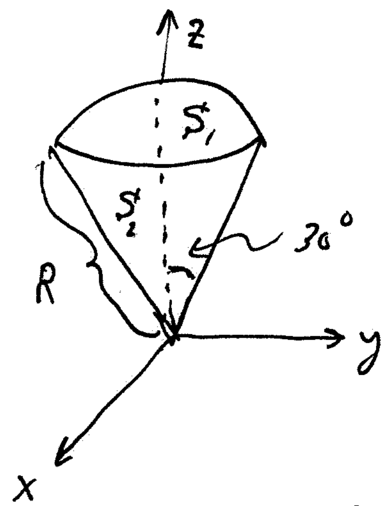
verified.

[P11] P.1.59

Check divergence Th^m for vector field

$$\vec{V} = r^2 \sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r^2\tan\theta\,\hat{\phi}$$

w.r.t ice cream cone:



$$d\vec{a}_1 = (\overbrace{R^2\sin\theta\, d\theta\, d\phi}^{dl_\phi\, dl_\theta})\,\hat{r}$$

$$d\vec{a}_2 = (\underbrace{r\sin\theta\, d\phi\, dr}_{dl_\phi\, dl_r})\,\hat{\theta}$$

$$\int_{S_1} \vec{V}\cdot d\vec{a}_1 = \int_0^{\pi/6}\int_0^{2\pi} (R^2\sin\theta\,\hat{r})\cdot(R^2\sin\theta\, d\phi\, d\theta\,\hat{r})$$

$$= \left(\int_0^{\pi/6} \sin^2\theta\, d\theta\right)\left(R^4\int_0^{2\pi} d\phi\right)$$

$$= \left(\int_0^{\pi/6} \tfrac{1}{2}(1-\cos(2\theta))\, d\theta\right)(2\pi R^4)$$

$$= \tfrac{1}{2}\left(\theta - \tfrac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/6} (2\pi R^4)$$

$$= \pi R^4\left[\frac{\pi}{6} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right] = \underline{\pi R^4\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)}\ \text{(I)}$$

$$\int_{S_2} \vec{V}\cdot d\vec{a}_2 = \int_0^R\int_0^{2\pi} (4r^2\cos\theta)(r\sin\theta\, d\phi\, dr) \quad (\text{with } \theta = \pi/6)$$

$$= \left(\frac{\sqrt{3}}{4}\right)\int_0^R 4r^3 dr \int_0^{2\pi} d\phi$$

$$= \underline{\left(\frac{\sqrt{3}}{4}\right)(R^4)(2\pi)}\ \text{(II)}.$$

P11 P.1.59 continued

$$\int_{\partial V} \vec{v}\cdot d\vec{a} = \pi R^4\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)+\pi R^4\left(\frac{\sqrt{3}}{2}\right)$$

$$= \underline{\frac{\pi R^4}{12}\left[2\pi + 3\sqrt{3}\right]}.$$

Calculate the divergence,

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 v_r\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\, v_\theta\right] + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^4\sin\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[4r^2\sin\theta\cos\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[r^2\tan\theta\right]$$

$$= 4r\sin\theta + \frac{4r^2}{r\sin\theta}\left(\cos^2\theta - \sin^2\theta\right) + \frac{r^2}{r\sin\theta}(0)$$

$$= 4r\sin\theta + 4r\left(\frac{\cos^2\theta}{\sin\theta} - \sin\theta\right)$$

$$= \frac{4r\cos^2\theta}{\sin\theta}$$

Hence,

$$\int_V (\nabla\cdot\vec{v})\,d\tau = \int_0^R\int_0^{2\pi}\int_0^{\pi/6} \frac{4r\cos^2\theta}{\sin\theta}\, r^2\sin\theta\, d\theta\, d\phi\, dr$$

$$= \int_0^R 4r^3 dr \int_0^{2\pi} d\phi \int_0^{\pi/6}\cos^2\theta\, d\theta \qquad \cos^2\theta = \tfrac{1}{2}(1+\cos(2\theta))$$

$$= 2\pi R^4\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right)\Big|_0^{\pi/6}$$

$$= 2\pi R^4\left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)$$

$$= \underline{\frac{\pi R^4}{12}\left[2\pi + 3\sqrt{3}\right]}.$$

Verified ✓

P.12 1.61

(a.) Show $\int_{\mathcal{V}} (\nabla T) d\tau = \oint_{\partial \mathcal{V}} T\, d\vec{a}$, hint: consider $\vec{V} = \vec{c}\, T$ for $\vec{c}$ constant

$$\int_{\mathcal{V}} \nabla \cdot (\vec{c}\, T) d\tau = \int_{\partial \mathcal{V}} (\vec{c}\, T) \cdot d\vec{a} = \vec{c} \cdot \int_{\partial \mathcal{V}} T\, d\vec{a} \quad \text{I}$$

However, $\nabla \cdot (\vec{c}\, T) = \cancel{(\nabla \cdot \vec{c})}^{\,0}\, T + \vec{c} \cdot \nabla T = \vec{c} \cdot \nabla T$ hence,

$$\int_{\mathcal{V}} \nabla \cdot (\vec{c}\, T) d\tau = \int_{\mathcal{V}} (\vec{c} \cdot \nabla T)\, d\tau = \vec{c} \cdot \int_{\mathcal{V}} (\nabla T)\, d\tau \quad \text{II}$$

Consequently, $\vec{c} \cdot \int_{\partial \mathcal{V}} T\, d\vec{a} = \vec{c} \cdot \int_{\mathcal{V}} (\nabla T)\, d\tau$ for all $\vec{c}$.

Thus $\underline{\int_{\mathcal{V}} (\nabla T) d\tau = \oint_{\partial \mathcal{V}} T\, d\vec{a}}$ .//

<u>Remark</u>: to understand (a.) we extended integration to vector fields, $\int_{\mathcal{V}} \langle V_1, V_2, V_3 \rangle d\tau = \langle \int_{\mathcal{V}} V_1\, d\tau, \int_{\mathcal{V}} V_2\, d\tau, \int_{\mathcal{V}} V_3\, d\tau \rangle$

or, equivalently, $\int_{\mathcal{V}} \sum_{i=1}^{3} V_i \hat{x}_i\, d\tau = \sum_{i=1}^{3} \left(\int_{\mathcal{V}} V_i\, d\tau \right) \hat{x}_i$. Likewise,

$$\oint_{\partial \mathcal{V}} f\, d\vec{a} = \left\langle \oint_{\partial \mathcal{V}} f\, da_1, \oint_{\partial \mathcal{V}} f\, da_2, \oint_{\partial \mathcal{V}} f\, da_3 \right\rangle \text{ for } d\vec{a} = \langle da_1, da_2, da_3 \rangle$$

and writing $d\vec{a} = (da)\hat{n}$ we have $\oint_{\partial \mathcal{V}} f\, d\vec{a} = \oint_{\partial \mathcal{V}} \hat{n} f\, da$, but $\hat{n}$ is not generally constant so we cannot factor $\hat{n}$ out of the integral.

$\boxed{P12}$ continued

(b.) Let us consider $\vec{V}\times\vec{c}$ for $\vec{c}$ constant,

$$\underbrace{\int_{\mathcal{V}} \nabla\cdot(\vec{V}\times\vec{c})\,d\tau}_{\int_{\mathcal{V}} \left(\vec{c}\cdot(\nabla\times\vec{V}) - \vec{V}\cdot(\cancelto{0}{\nabla\times\vec{c}})\right)d\tau} = \oint_{\partial\mathcal{V}} (\vec{V}\times\vec{c})\cdot d\vec{a}$$

$$\int_{\mathcal{V}} \left(\vec{c}\cdot(\nabla\times\vec{V}) - \vec{V}\cdot(\cancelto{0}{\nabla\times\vec{c}})\right)d\tau = \oint_{\partial\mathcal{V}} (\vec{V}\times\vec{c})\cdot \hat{n}\,da$$

$$\int_{\mathcal{V}} \vec{c}\cdot(\nabla\times\vec{V})\,d\tau = \oint_{\partial\mathcal{V}} (\hat{n}\times\vec{V})\cdot\vec{c}\,da$$ by cyclicity of triple product.

$$\Rightarrow \quad \vec{c}\cdot\int_{\mathcal{V}}(\nabla\times\vec{V})\,d\tau = \vec{c}\cdot\oint_{\partial\mathcal{V}}(\hat{n}\,da)\times\vec{V}$$

$$\Rightarrow \quad \underline{\int_{\mathcal{V}}(\nabla\times\vec{V})\,d\tau = \oint_{\partial\mathcal{V}} d\vec{a}\times\vec{V} = -\oint_{\partial\mathcal{V}}\vec{V}\times d\vec{a}}.$$

(c.)
$$\underbrace{\int_{\mathcal{V}} \nabla\cdot(T\nabla U)\,d\tau}_{\int_{\mathcal{V}}(\nabla T\cdot\nabla U + T\,\nabla\cdot(\nabla U))\,d\tau} = \oint_{\partial\mathcal{V}}(T\nabla U)\cdot d\vec{a}$$

$$\int_{\mathcal{V}}(\nabla T\cdot\nabla U + T\,\nabla\cdot(\nabla U))\,d\tau = \underline{\int_{\mathcal{V}}(\nabla T\cdot\nabla U + T\nabla^2 U)\,d\tau = \oint_{\partial\mathcal{V}}(T\nabla U)\cdot d\vec{a}}.$$

(d.) from (c.) we likewise obtain $\int_{\mathcal{V}}(\nabla U\cdot\nabla T + U\nabla^2 T)\,d\tau = \oint_{\partial\mathcal{V}}(U\nabla T)\cdot d\vec{a}$ then taking the difference yields (since $\nabla U\cdot\nabla T = \nabla T\cdot\nabla U$),

$$\underline{\int_{\mathcal{V}}(T\nabla^2 U - U\nabla^2 T)\,d\tau = \oint_{\partial\mathcal{V}}(T\nabla U - U\nabla T)\cdot d\vec{a}}.$$

$\boxed{P12}$ continued,

(e.)
$$\int_S (\nabla \times (\vec{c}\,T)) \cdot d\vec{a} = \oint_{\partial S} (\vec{c}\,T) \cdot d\vec{l}$$

$$\Rightarrow \int_S \left(\cancelto{0}{(\nabla \times \vec{c})} T - \vec{c} \times \nabla T \right) \cdot d\vec{a} = \oint_{\partial S} (\vec{c}\,T) \cdot d\vec{l}$$

$$\Rightarrow -\int_S (\vec{c} \times \nabla T) \cdot \hat{n}\, da = \vec{c} \cdot \oint T\, d\vec{l}$$

$$\Rightarrow -\int_S (\nabla T \times \hat{n}) \cdot \vec{c}\, da = \vec{c} \cdot \oint_{\partial S} T\, d\vec{l}$$

$$\vec{c} \cdot \int_S -\nabla T \times (\hat{n}\, da) = \vec{c} \cdot \oint_{\partial S} T\, d\vec{l}$$

$$\boxed{\int_S \nabla T \times d\vec{a} = -\oint_{\partial S} T\, d\vec{l}}$$

(f.)
$$\int_S (\nabla \times (\vec{v} \times \vec{c})) \cdot d\vec{a} = \int_{\partial S} (\vec{v} \times \vec{c}) \cdot d\vec{l} = \int (\vec{v} \times \vec{c}) \cdot (dl\,\hat{T})$$

$$\int_S \Big((\vec{c} \cdot \nabla)\vec{v} - \underbrace{(\vec{v} \cdot \nabla)\vec{c} + \vec{v}(\nabla \cdot \vec{c})}_{0} - \vec{c}(\nabla \cdot \vec{v}) \Big) \cdot d\vec{a} = \int \vec{c} \cdot (dl\,\hat{T} \times \vec{v})$$

<u>Ok</u>... what gives...

need deeper magic

(I didn't quite get (f.))

P1a continued

$$\int_S \nabla \times (\vec{V} \times \vec{c}) \cdot d\vec{a} = \int_{\partial S} (\vec{V} \times \vec{c}) \cdot d\vec{l} = \int_{\partial S} (d\vec{l} \times \vec{V}) \cdot \vec{c}$$

$$\nabla \times (\vec{V} \times \vec{c}) = (\nabla \times \vec{V}) \times \vec{c} \quad : \text{ since } \vec{c} \text{ constant.}$$

$$((\nabla \times \vec{V}) \times \vec{c}) \cdot d\vec{a} = d\vec{a} \times (\nabla \times \vec{V}) \cdot \vec{c}$$

Therefore,

$$\int_S d\vec{a} \times (\nabla \times \vec{V}) \cdot \vec{c} = \oint_{\partial S} (d\vec{l} \times \vec{V}) \cdot \vec{c} = -\oint_{\partial S} (\vec{V} \times d\vec{l}) \cdot \vec{c}$$

and as the above holds $\forall \vec{c}$ we find

$$\int_S d\vec{a} \times (\nabla \times \vec{V}) = -\oint_S \vec{V} \times d\vec{l}$$

almost ☺

P13 1.63

(a.) find divergence of $\vec{v} = \frac{\hat{r}}{r}$. Using front cover,

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{1}{r}\right] + 0 + 0 = \boxed{\frac{1}{r^2}}$$

No, there is no δ-function at origin since

$$\int_{B_R} (\nabla \cdot \vec{v}) \cdot d\tau = \oint_{\partial B_R} \vec{v} \cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi} \left(\frac{\hat{r}}{R}\right)(R^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$$

$$\| \qquad\qquad = 4\pi R$$

$$\int_{B_R} \frac{1}{r^2} r^2 \sin\theta \, d\theta \, d\phi \, dr = 4\pi R$$

nothing to "repair"

Continuing,

$$\nabla \cdot (r^n \hat{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 r^n\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^{n+2}\right] = \underline{(n+a)r^{n-1}}$$

However, as we've discussed, $\underline{\nabla \cdot \left(\frac{1}{r^2}\hat{r}\right) = 4\pi\delta^3(\vec{r}) \ (n=-2)}$.

and as discussed in Lecture 3, $\nabla \cdot (r^n \hat{r})$ undefined for $n < -2$.

(b.) $$\nabla \times (r^n \hat{r}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left[r^n\right]\hat{\theta} - \frac{1}{r}\frac{\partial}{\partial \theta}\left[r^n\right]\hat{\phi} = \boxed{0}$$

$$\int_{\mathcal{V}} (\nabla \times \vec{v}) d\tau = -\oint_{\partial \mathcal{V}} \vec{v} \times d\vec{a} \qquad (1.61 \text{ (b)})$$

$$= -\oint_{\partial \mathcal{V}} \underbrace{(r^n \hat{r}) \times (da \, \hat{r})}_{0} \quad \text{for } \partial\mathcal{V} = S_R$$

$$= \boxed{0}.$$