Printed Name: ________________________.

PHYSICS 331	MISSION 2: ELECTROSTATICS (37+3)PTS)

Problems mentioned refer to David Griffith's *Introduction to Electrodynamics*, 5th edition.

Problem 14 (2pt) Problem 2.2 (field from pair of point charges)

Problem 15 (3pt) Problem 2.3 (field from line-segment, at point on the end)

Problem 16 (2pt) Problem 2.4 (note: I would try to use Example 2.2 and superposition with the necessary geometry to solve this)

Problem 17 (3pt) Problem 2.12 (thank me for not assigning 2.7)

Problem 18 (4pts) Problems 2.16 & 2.24 about spherically symmetric shell charge

- **(a.)** find the electric field as described in 2.16
- **(b.)** find the potential as requested in 2.24

Problem 19 (4pts) Problems 2.14 & 2.23 about infinite line charge:

- **(a.)** find the electric field as described in 2.14
- **(b.)** find the potential as requested in 2.23

Problem 20 (3pts) Problem 2.18 (field for slab of charge)

Problem 21 (3pts) Problem 2.21 (which field is physically possible $\vec{E}$ and find potential for that field)

Problem 22 (4pts) Problems 2.22 & 2.29 (find potential for solid sphere)

- **(a.)** find the potential from the electric field as indicated by Problem 2.22
- **(b.)** find the potential from an integral over the charge distribution as requested in Problem 2.29

Problem 23 (3pts) Problem 2.30 (check on Poisson's equation)

Problem 24 (3pts) Problem 2.31 (check on boundary conditions)

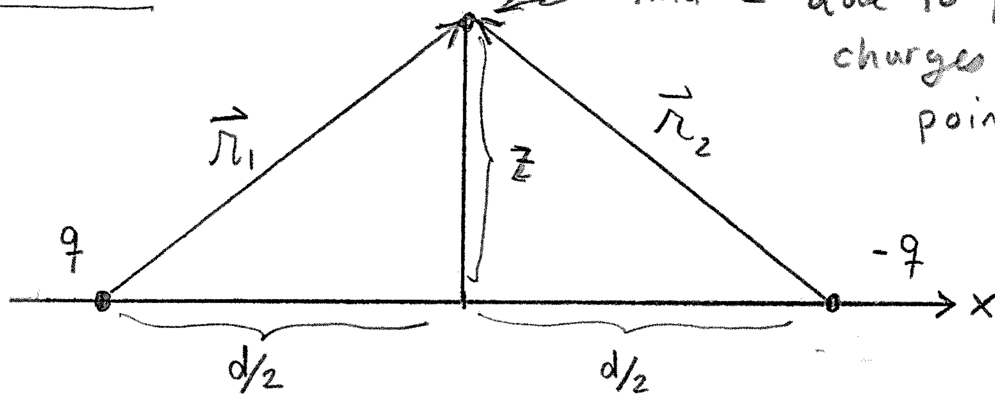
Problem 25 (3pts) Problem 2.32 (energy to assemble)

Problem 26 (3pts) Problem 2.37 (energy due to spherical configuration)

MISSION 2 SOLUTION

P14 Problem 2.2

find $\vec{E}$ due to pictured charges $\pm q$, at point $(0,0,z)$



$$\vec{r}_1 = \langle 0,0,z\rangle - \langle -d/2, 0, 0\rangle = \frac{d}{2}\hat{x} + z\hat{z}$$

$$\vec{r}_2 = \langle 0,0,z\rangle - \langle d/2, 0, 0\rangle = \frac{-d}{2}\hat{x} + z\hat{z}$$

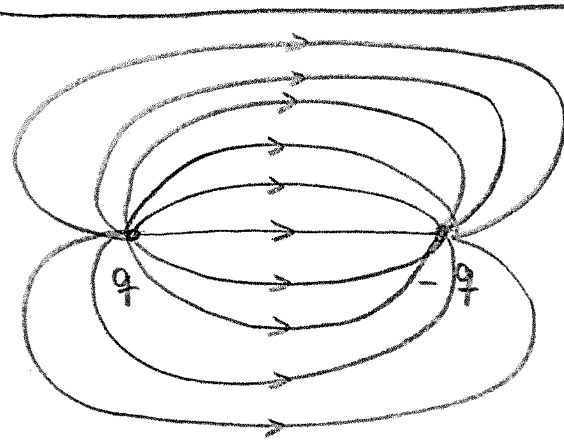
$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}_1}{r_1^3} - \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}_2}{r_2^3} \quad :$$

(field at $(0,0,z)$ is simply superposition of Coulomb fields of $\pm q$ as given)

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^3}\left[\vec{r}_1 - \vec{r}_2\right]$$ setting $r_1 = r_2 = r$.

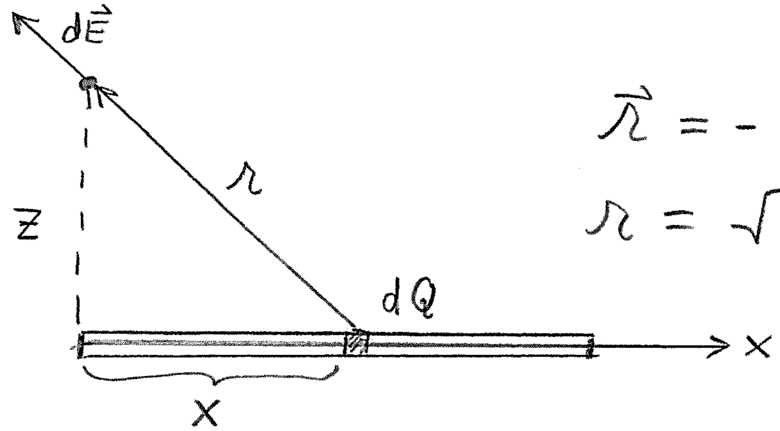
$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^3}\left[\frac{d}{2}\hat{x} + \underbrace{z\hat{z} - \left(-\frac{d}{2}\hat{x} + z\hat{z}\right)}_{\text{cancel}}\right]$$

$$\boxed{\vec{E} = \frac{qd\,\hat{x}}{4\pi\varepsilon_0\left(d^2/4 + z^2\right)^{3/2}}}$$



E-field lines point in $\hat{x}$ direction at $x = 0$.

P15 Find E-field distance z above straight line segment of length L with uniform charge density λ. Check limit $z \gg L$ (should get Coulomb field for $Q = \lambda L$)



$$\vec{\mathscr{r}} = -x\hat{x} + z\hat{z}$$

$$\mathscr{r} = \sqrt{x^2 + z^2}$$

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0}\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\lambda\, dx}{4\pi\varepsilon_0 (x^2+z^2)^{3/2}}(-x\hat{x} + z\hat{z})$$

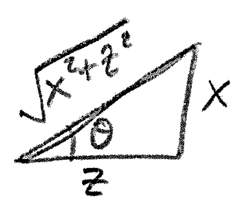
① $$\int_0^L \frac{x\,dx}{(x^2+z^2)^{3/2}} = -\sqrt{x^2+z^2}\,\Big|_0^L = \sqrt{L^2+z^2} - \sqrt{z^2}$$

② $$\int \frac{z\,dx}{(x^2+z^2)^{3/2}} = \int \frac{z^2\sec^2\theta\,d\theta}{(z^2\sec^2\theta)^{3/2}}$$

$$\begin{cases} x = z\tan\theta \\ x^2+z^2 = z^2\sec^2\theta \\ dx = z\sec^2\theta\,d\theta \end{cases}$$

$$= \int \frac{d\theta}{z\sec\theta}$$

$$= \frac{1}{z}\int \cos\theta\,d\theta$$



$$\tan\theta = \frac{x}{z}$$

$$\sin\theta = \frac{x}{\sqrt{x^2+z^2}}$$

$$= \frac{1}{z}\sin\theta + C$$

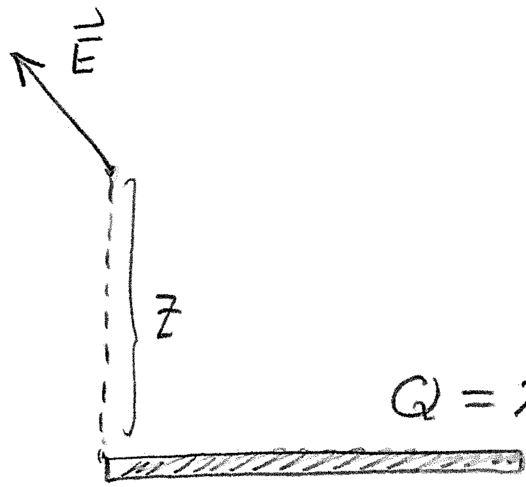
$$= \frac{x}{z\sqrt{x^2+z^2}} + C$$

③ $$\int_0^L \frac{z\,dx}{(x^2+z^2)^{3/2}} = \frac{L}{z\sqrt{L^2+z^2}}$$

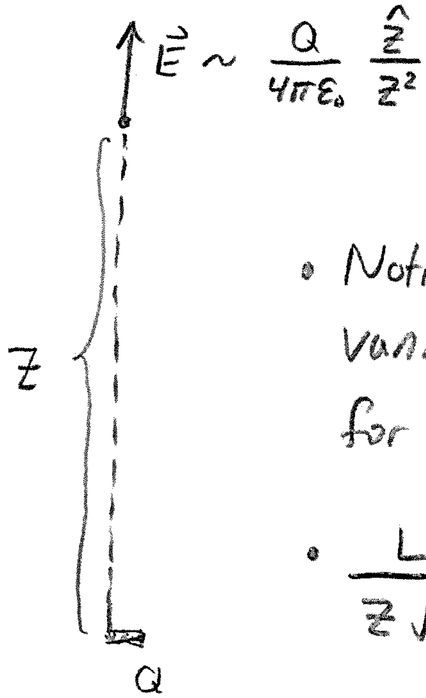
P15 continued

$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left(-\hat{x}\int_0^L \frac{x\,dx}{(x^2+z^2)^{3/2}} + \hat{z}\int_0^L \frac{z\,dx}{(x^2+z^2)^{3/2}}\right)$$

$$\boxed{\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\left(\sqrt{z^2} - \sqrt{z^2+L^2}\right)\hat{x} + \left(\frac{L}{z\sqrt{L^2+z^2}}\right)\hat{z}\right]}$$



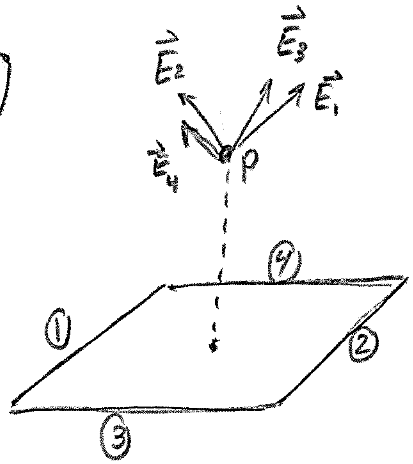
$Q = \lambda L$ uniformly distributed

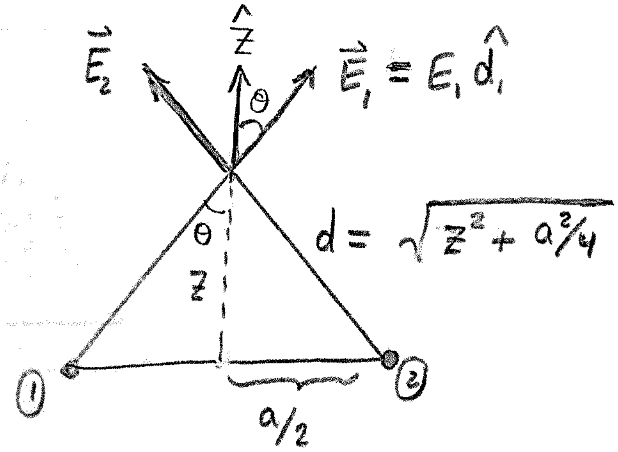


- Notice the $\hat{x}$-term in $\vec{E}$ vanishes as $z^2+L^2 \approx z^2$ for $z \gg L$.

- $$\frac{L\lambda}{z\sqrt{L^2+z^2}} \approx \frac{Q}{z\sqrt{z^2}} = \frac{Q}{z^2} \quad (z>0)$$

P16





By symmetry it is clear only the z-component of $\vec{E}$ is non-zero at P. Relative to each line segment P is distance d above the midpoint. Following Example 2.2 where

$$\vec{E} = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{2\lambda L\,\hat{z}}{z\sqrt{z^2+L^2}}}_{\text{for line segment length } 2L} \longrightarrow \vec{E} = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{\lambda a\,\hat{d}}{d\sqrt{d^2+a^2/4}}}_{\text{where } \hat{d} \text{ points from midpoint of line segment of length } a \text{ to point } P}$$

For one leg,

$$E_z = \vec{E}\cdot\hat{z} = \frac{\lambda a\,\hat{d}\cdot\hat{z}}{4\pi\varepsilon_0\, d\sqrt{d^2+a^2/4}} \qquad \hat{d}\cdot\hat{z} = \cos\theta = \frac{z}{d}$$

$$E_z = \frac{\lambda a z}{4\pi\varepsilon_0\, d^2\sqrt{d^2+a^2/4}} \quad \text{(from one leg)}$$

There are four line segments so we need

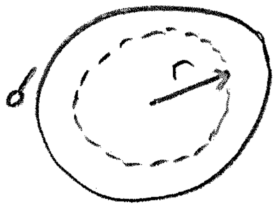
$$\boxed{\vec{E} = \left(\frac{4\lambda a z}{4\pi\varepsilon_0\,(z^2+a^2/4)\sqrt{z^2+a^2/2}}\right)\hat{z}}$$

For $z \gg a$ we find $E \approx \left(\frac{4\lambda a}{4\pi\varepsilon_0 z^2}\right)\hat{z} = \frac{4Q\hat{z}}{4\pi\varepsilon_0 z^2}$ which makes sense for charge $4Q$ where $Q = \lambda a$.

P17 GRIFFITHS 2.12

Spherical shell of radius R with uniform density σ. Find $\vec{E}$ inside and outside.

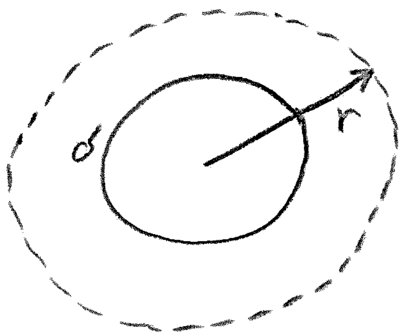
Inside:



$$\underbrace{\Phi_E}_{\text{spherical symmetry} \Rightarrow \vec{E} = E\hat{r}} = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} = 0$$

$$\underline{E = 0 \quad \text{for } 0 \le r < R}$$

Outside: once more, spherical symmetry gives $\vec{E} = E\hat{r}$ thus



$$\Phi_E = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} = \frac{4\pi R^2 \sigma}{\varepsilon_0}$$

$$\underline{E = \frac{4\pi R^2 \sigma}{4\pi \varepsilon_0 r^2} \quad \text{for } r > R}.$$

$$\boxed{\vec{E} = \begin{cases} 0 & : \text{for } 0 \le r < R \\ \left(\dfrac{R^2 \sigma}{\varepsilon_0 r^2}\right)\hat{r} & : \text{for } r > R \end{cases}}$$

Remark: as $r \longrightarrow R^+$ we see $\vec{E} \longrightarrow \frac{\sigma}{\varepsilon_0}\hat{r}$ in good agreement with our study of $\vec{E}$ field near surface... well actually... it's kind of interesting in that for plane σ get $\frac{\sigma}{2\varepsilon_0} = E$.

P18 (a.) Griffiths 2.16

A thick spherical shell carries charge density

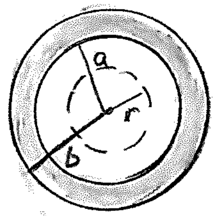
$$\rho = \frac{k}{r^2} \quad \text{for } a \le r \le b$$

Find the electric field in cases

(i.) $r<a$, (ii) $a<r<b$, (iii) $r>b$

Plot $|\vec{E}|$ as function of r in case $b=2a$.

(i.) $r<a$ If we consider Gaussian sphere at $r<a$ by spherical symmetry $\vec{E} = E\hat{r}$ where $E=E(r)$



$$\Phi_E = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow 4\pi r^2 E = 0$$

$$\Rightarrow \boxed{\vec{E} = 0 \quad \text{for } r<a}$$

(ii.) $a<r<b$ Suppose Gaussian sphere at $a<R<b$

$$Q_{enc} = \int \rho \, d\tau = \int_0^{2\pi}\int_0^{\pi}\int_a^{R} \frac{k}{r^2} r^2 \sin\theta \, dr \, d\theta \, d\phi = 4\pi k (R-a)$$

$$\Phi_E = 4\pi R^2 E = \frac{Q_{enc}}{\varepsilon_0} = \frac{4\pi k (R-a)}{\varepsilon_0} \Rightarrow \boxed{\vec{E} = \left(\frac{k(r-a)}{r^2 \varepsilon_0}\right)\hat{r} \quad \text{for } a \le r \le b}$$

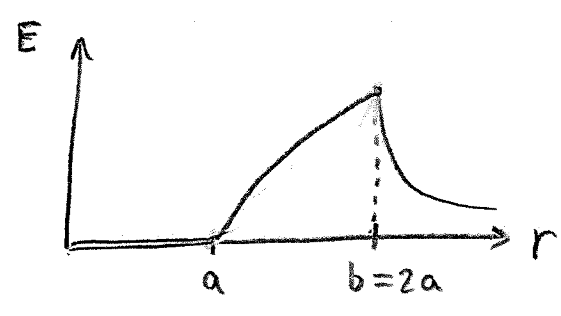
(iii.) $r>b$ Gaussian sphere at $r>b$ encloses charge $Q_{enc} = 4\pi k (b-a)$ by calculations in ii.)

$$\Phi = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \frac{4\pi k (b-a)}{\varepsilon_0} = 4\pi r^2 E$$

$$\boxed{\vec{E} = \frac{k(b-a)}{\varepsilon_0 r^2}\hat{r}}$$

(for $r \ge b$.)

Rough Plot,



P18 continued

(b.) 2.24, Find potential for spherical shell of 2.16, using $V(\infty)=0$

We found $\vec{E} = E\hat{r}$ where

$$E_r = \begin{cases} 0 & : 0 \le r \le a \\ \frac{k}{\varepsilon_0}\left[\frac{1}{r} - \frac{a}{r^2}\right] & : a \le r \le b \\ \frac{k}{\varepsilon_0}(b-a)\frac{1}{r^2} & : r \ge b \end{cases}$$

To find potential V such that $\vec{E} = -\nabla V$ we need to solve $\frac{dV}{dr} = -E_r$ to obtain continuous V subject the condition $V(\infty)=0$.

① for $r \ge b$ set $\boxed{V(r) = \frac{k(b-a)}{\varepsilon_0}\frac{1}{r}}$

② for $a \le r \le b$,

$$V = \frac{-k}{\varepsilon_0}\int\left(\frac{1}{r} - \frac{a}{r^2}\right)dr$$

$$V(r) = \frac{-k}{\varepsilon_0}\left(\ln(r) + \frac{a}{r}\right) + V_1$$

continuity for V at $r=b$.

$$V(b) = \frac{-k}{\varepsilon_0}\left(\ln(b) + \frac{a}{b}\right) + V_1 = \frac{k(b-a)}{\varepsilon_0}\frac{1}{b}$$

$$V_1 = \frac{k}{\varepsilon_0}\left[\frac{b-a}{b} + \ln b + \frac{a}{b}\right] = \frac{k}{\varepsilon_0}\left[1 + \ln(b)\right]$$

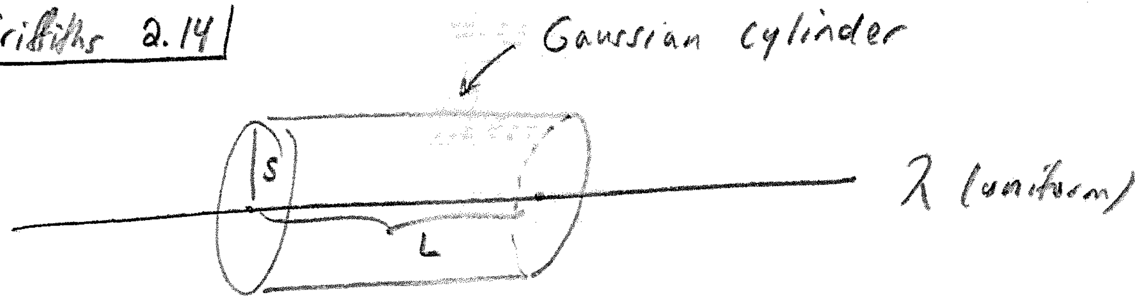
$$\Rightarrow V(r) = \frac{-k}{\varepsilon_0}\left[\ln(r) + \frac{a}{r} - 1 - \ln b\right]$$

$$\boxed{V(r) = \frac{k}{\varepsilon_0}\left[\ln\left(\frac{b}{r}\right) + 1 - \frac{a}{r}\right] \quad : a \le r \le b}$$

③ for $0 \le r \le a$ the potential has $\frac{dV}{dr} = 0$ hence V constant.

$$\Rightarrow \boxed{V(r) = V(a) = \frac{k}{\varepsilon_0}\ln(b/a) \quad \text{for } 0 \le r \le a.}$$

P19 (a.) Griffiths 2.14



cylindrical symmetry $\Rightarrow$ $\vec{E} = E(s)\hat{s}$ $\Rightarrow$ $\Phi_E = 2\pi s L E$ (no caps)

Then $\frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0} = 2\pi s L E \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s}$

$$\therefore \quad \boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s}}$$

(b.) Griffiths 2.23

Find potential for uniform λ analyzed above.

$$\vec{E} = -\nabla V = -\hat{s}\frac{\partial V}{\partial s} - \frac{1}{s}\hat{\phi}\frac{\partial V}{\partial \phi} - \hat{z}\frac{\partial V}{\partial z} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s}$$

We find V is function of s-alone,

$$\frac{dV}{ds} = \frac{-\lambda}{2\pi\varepsilon_0 s}$$

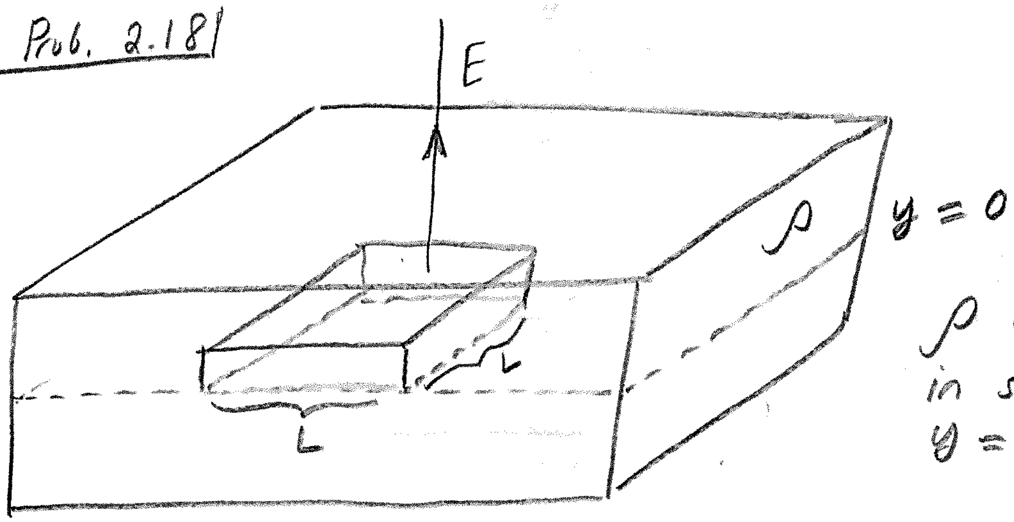
$$\boxed{V(s) = \frac{-\lambda}{2\pi\varepsilon_0}\ln(s) = \frac{\lambda}{2\pi\varepsilon_0}\ln(1/s)}$$

(my choice of potential gives $V(s) = 0$ when $s = 1$)

alternatively,

$$\underbrace{V(s) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(s_0/s\right)}_{\text{you can check, } -\nabla V = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s} \ \checkmark}$$ sets $V(s) = 0$ at $s = s_0$.

P20 Prob. 2.18

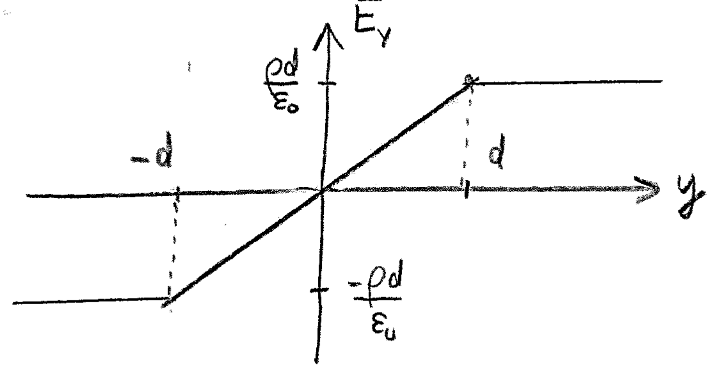


ρ constant in slab from $y = -d$ to $y = d$

Consider square box of side length L of height $y < d$. By symmetry $E = 0$ at $y = 0$ and $E = 0$ through sides of the box. Only the top at y gives flux,

$$\Phi_E = L^2 E = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\rho y L^2 \Rightarrow \boxed{\vec{E} = \frac{\rho y}{\varepsilon_0}\hat{y}}$$

The same story plays out for $-d < y < 0$ where the $\vec{E}$ field points downward for $y < 0$, still the formula boxed above holds true. (by up/down symmetry along y)



$$\sigma = \frac{\Delta Q}{\Delta A} = \frac{\rho(2d)L^2}{L^2} = 2\rho d$$

$$\underbrace{\pm\frac{\rho d}{\varepsilon_0} = \pm\frac{\sigma}{2\varepsilon_0}}_{}$$

makes sense given our discussion of planar charge.

P21 Griffiths 2.21

(a.) $\vec{E} = k\langle xy,\ 2yz,\ 3xz\rangle$

(b.) $\vec{E} = k\langle y^2,\ 2xy+z^2,\ 2yz\rangle$

Given $k \neq 0$ is constant. Determine which $\vec{E}$ is physically possible and find V for that one. Use origin as reference for the potential. I set $k=1$

(a.) $\nabla\times\vec{E} = \langle \partial_y(3xz) - \partial_z(2yz),\ \partial_z(xy) - \partial_x(3xz),\ \partial_x(2yz) - \partial_y(xy)\rangle$

$= \langle -2y,\ -3z,\ -x\rangle \neq 0 \Rightarrow \underline{\vec{E} \neq -\nabla V}$.

(b.) $\nabla\times\vec{E} = \langle \partial_y(2yz) - \partial_z(2xy+z^2),\ \partial_z y^2 - \partial_x(2yz),\ \partial_x(2xy+z^2) - \partial_y y^2\rangle$

$= \langle 2z - 2z,\ 0,\ 2y - 2y\rangle$

$= \underline{\langle 0, 0, 0\rangle}$.

$$V(x,y,z) = -\int_{(0,0,0)}^{(x,y,z)} y^2\,dx + (2xy+z^2)\,dy + 2yz\,dz$$

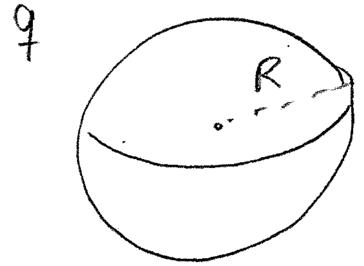
$$= -\int_{(0,0,0)}^{(x,y,z)} d(xy^2 + yz^2)$$

$$= -(xy^2 + yz^2)$$

Remark: sorry, I have ignored text instruction here.

$$\boxed{V(x,y,z) = -xy^2 - yz^2}$$

$\boxed{P22}$ (a.) Griffiths 2.22



We derived in Lecture $\vec{E} = E_r \hat{r}$ and,

$$E_r = \begin{cases} \dfrac{q r}{4\pi\varepsilon_0 R^3} & : \text{for } 0 \le r \le R \\ \dfrac{q}{4\pi\varepsilon_0 r^2} & : \text{for } r \ge R \end{cases}$$

Then $V(r) = \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{r}$ has $\vec{E} = -\nabla V$ and $V(\infty) = 0$

in the case $r \ge R$. Next, $\dfrac{dV}{dr} = -E_r = \dfrac{-qr}{4\pi\varepsilon_0 R^3}$

yields $V(r) = \dfrac{-qr^2}{8\pi\varepsilon_0 R^3} + V_1$ for $0 \le r \le R$

continuity at $r = R \Rightarrow \dfrac{-qR^2}{8\pi\varepsilon_0 R^3} + V_1 = \dfrac{q}{4\pi\varepsilon_0 R}$

Hence $V_1 = \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{R}\left[1 + \dfrac{1}{2}\right] = \dfrac{3q}{8\pi\varepsilon_0 R}$

$$\boxed{\begin{aligned} V(r) &= \frac{q}{4\pi\varepsilon_0 R}\left[\frac{3}{2} - \frac{r^2}{2R^3}\right] \quad \text{for } 0 \le r \le R \\ V(r) &= \frac{q}{4\pi\varepsilon_0}\frac{1}{r} \quad \text{for } r \ge R \end{aligned}}$$

<u>Remark</u>: $\boxed{E7}$ in Lecture 6 7 shows we can also calculate $V(r)$ by $-\int \vec{E}\cdot d\vec{\ell}$. I usually prefer the method I showed above.

P22 continued

(b.) Griffiths 2.29

use Eq. 2.29 to find $V(r)$ within uniformly charged sphere of radius R and total charge q

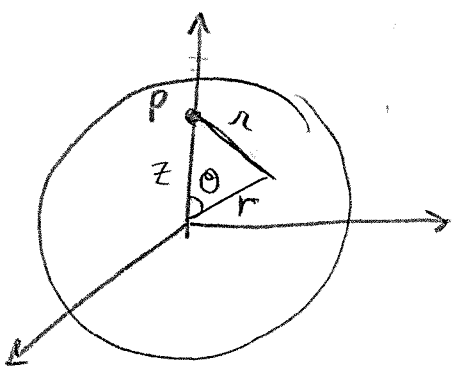
$$\rho(r) = \begin{cases} \dfrac{q}{\frac{4}{3}\pi R^3} = \dfrac{3q}{4\pi R^3} & \text{for } 0 \le r \le R \\ 0 & \text{for } r > R \end{cases}$$

Equation 2.29

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{\mathscr{r}}\,d\tau'$$

$$V(P) = \frac{\rho}{4\pi\varepsilon_0}\int \frac{d\tau}{\mathscr{r}}$$

place P at $(0,0,z)$ to avoid primes on source points and to simplify computation



$$\mathscr{r}^2 = z^2 + r^2 - 2rz\cos\theta$$

$$\mathscr{r} = \sqrt{z^2+r^2-2rz\cos\theta}$$

$$V = \frac{\rho}{4\pi\varepsilon_0}\int \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{z^2+r^2-2rz\cos\theta}}$$ (*), here z is constant in the integration.

$$\int_0^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{z^2+r^2-2rz\cos\theta}} = \frac{1}{rz}\sqrt{z^2+r^2-2rz\cos\theta}\,\Big|_0^{\pi}$$

$$= \frac{1}{rz}\left(\sqrt{z^2+r^2+2rz} - \sqrt{z^2+r^2-2rz}\right)$$

$$= \frac{1}{rz}\left[\,|z+r| - |z-r|\,\right]$$

P22 continued Since $z+r>0$,

$$\int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{z^2+r^2-2rz\cos\theta}} = \frac{1}{rz}\left[z+r-|z-r| \right]$$

$$= \frac{1}{rz}\begin{cases} z+r-(z-r) & : z>r \\ z+r-(r-z) & : z<r \end{cases}$$

$$= \begin{cases} 2/z & : r<z \\ 2/r & : z<r\leq R \end{cases}$$

Thus, returning to * from last page, note $\int_0^{2\pi} d\phi = 2\pi$ hence all that remains is radial integration, notice the r^2

$$V = \frac{2\pi\rho}{4\pi\varepsilon_0}\left[\int_0^z \frac{2r^2}{z}dr + \int_z^R \frac{2r^2}{r}dr \right]$$

$$= \frac{\rho}{\varepsilon_0}\left[\frac{r^3}{3z}\Big|_0^z + \frac{r^2}{2}\Big|_z^R \right]$$

$$= \frac{\rho}{\varepsilon_0}\left[\frac{1}{3}z^2 + \frac{1}{2}(R^2-z^2) \right]$$

$$= \frac{\rho}{2\varepsilon_0}\left[R^2 - \frac{1}{3}z^2 \right]$$

But, $\rho = \dfrac{3q}{4\pi R^3}$ $\therefore$ $V(r) = \dfrac{3q}{2\varepsilon_0(4\pi R^3)}\left[R^2 - \frac{1}{3}z^2 \right]$

$$\Rightarrow \quad V = \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{z^2}{R^2}\right)$$

$$\Rightarrow \quad \boxed{V(r) = \frac{q}{4\pi\varepsilon_0 R}\left[\frac{3}{2} - \frac{r^2}{2R^3} \right]}$$

for $0 \leq r \leq R$

P23 Problem 2.30

Check that $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}} d\tau'$

satisfies Poisson's equation by applying the Laplacian and using $\nabla^2\left(\frac{1}{\mathscr{r}}\right) = -4\pi\,\delta^3(\vec{\mathscr{r}})$

$$\nabla^2 V = \nabla^2\left(\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}} d\tau'\right)$$

$\underbrace{\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2}$ independent from x', y', z' and hence $\vec{r}'$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\,\nabla^2\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\left(-4\pi\,\delta^3(\vec{\mathscr{r}})\right) d\tau'$$

$$= \frac{-1}{\varepsilon_0}\int \rho(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')\, d\tau'$$

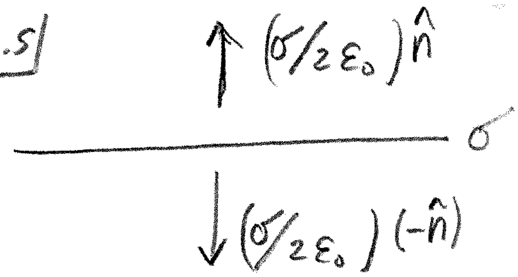
$$= \frac{-1}{\varepsilon_0}\rho(\vec{r})$$

$$\therefore \quad \nabla^2 V = -\rho/\varepsilon_0 \quad \text{Poisson's Eq}^{\text{n}}!$$

P24 Problem 2.31

(a.) Check results of Examples 2.5, 2.6 and Problem 2.12 are consistent with Eq^{n} 2.33 $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$

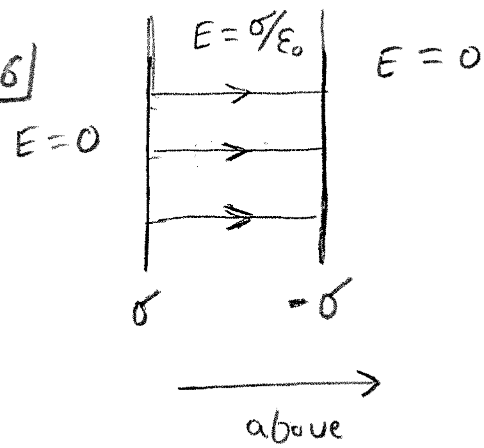
Ex. 2.5



$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{2\varepsilon_0}\hat{n} - \left(\frac{-\sigma}{2\varepsilon_0}\hat{n}\right)$$

$$= \frac{\sigma}{\varepsilon_0}\hat{n}$$ ✓

Ex 2.6



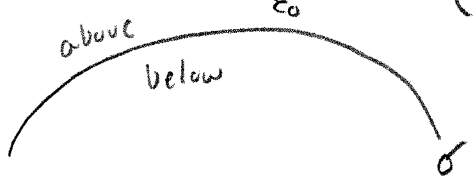
$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$ between σ & $-\sigma$

$$\vec{E}_{above} - \vec{E}_{below} = \underbrace{0 - \frac{\sigma}{\varepsilon_0}\hat{n}}_{\text{for } -\sigma}$$

$$\vec{E}_{above} - \vec{E}_{below} = \underbrace{\frac{\sigma}{\varepsilon_0}\hat{n} - 0}_{\text{for } \sigma}$$

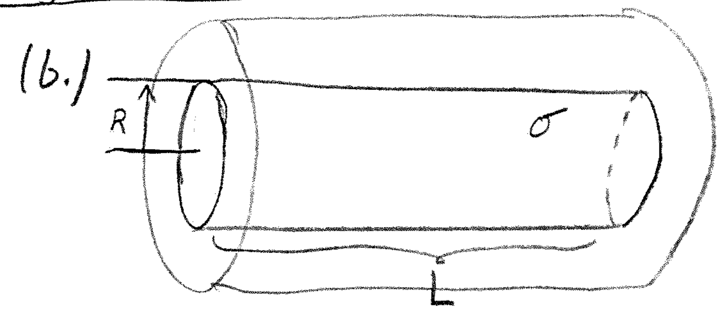
Problem 2.12

$E = \frac{\sigma}{\varepsilon_0}\hat{r}$ (checks, see $\boxed{P17}$)



P24 continued

(b.)



$$Q_{enc} = \sigma(2\pi R L)$$

$$\Phi_E = 2\pi s L E$$

$$\Phi_E = \frac{Q_{enc}}{\epsilon_0} \longrightarrow 2\pi s L E = \frac{\sigma(2\pi R L)}{\epsilon_0}$$

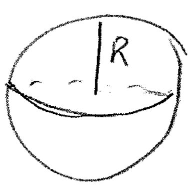
$$\underline{\vec{E} = \frac{R\sigma}{\epsilon_0}\frac{\hat{s}}{s} \text{ for } s > R}.$$

$$\vec{E} = 0 \text{ for } s < R$$

at $s = R$,

$$\underline{\vec{E}_{out} - \vec{E}_{in} = \frac{\sigma}{\epsilon_0}\hat{s}}. \checkmark \quad \left(s = R \text{ makes } \frac{R\sigma}{\epsilon_0 s} = \frac{\sigma}{\epsilon_0}\right)$$

(c.) Example 2.8



$$V_{out} = \frac{q}{4\pi\epsilon_0}\frac{1}{r}, \quad r \geq R$$

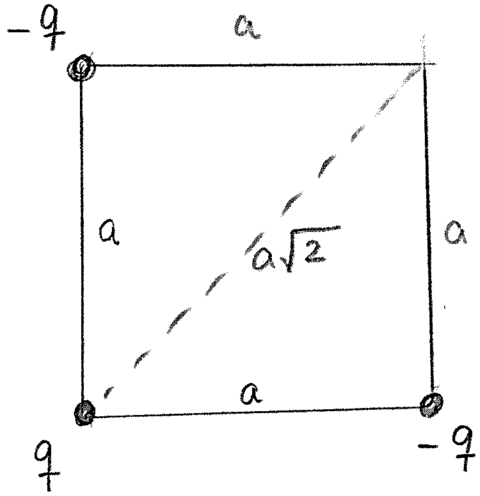
$$V_{in} = \frac{q}{4\pi\epsilon_0}\frac{1}{R}, \quad r \leq R$$

$$\frac{\partial V_{out}}{\partial r} = \frac{-q}{4\pi\epsilon_0 r^2}, \quad \frac{\partial V_{in}}{\partial r} = 0$$

$$\vec{E}_{out} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{R^2}, \quad \vec{E}_{in} = 0 \quad (\text{at } r = R)$$

$$\underline{\vec{E}_{out} - \vec{E}_{in} = \frac{\sigma}{\epsilon_0}\hat{r} \text{ where } \sigma = \frac{q}{4\pi R^2}}. \checkmark$$

P25 Problem 2.32



(a.) to add 4^{th} charge q in upper right corner we do work to build potential energy w.r.t. given charges, minimal work needed is,

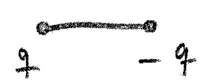
$$W = \frac{-qq}{4\pi\epsilon_0 a} - \frac{qq}{4\pi\epsilon_0 a} + \frac{qq}{4\pi\epsilon_0 a\sqrt{2}}$$

$$\boxed{W = \frac{q^2}{4\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 2\right)}$$

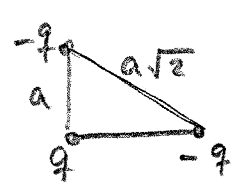
(b.) How much work to assemble all 4?

1^{st} one free,

2^{nd} one adds $\frac{-q^2}{4\pi\epsilon_0 a}$



3^{rd} one takes additional



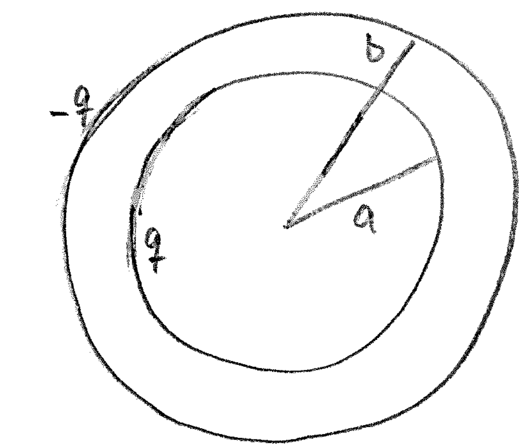
$\frac{-q^2}{4\pi\epsilon_0 a} + \frac{(-q)(-q)}{4\pi\epsilon_0 a\sqrt{2}}$

Thus, in total

$$W = \frac{q^2}{4\pi\epsilon_0 a}\left(-1 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 2\right)$$

$$= \frac{q^2}{4\pi\epsilon_0 a}\left(-4 + \frac{2}{\sqrt{2}}\right) = \boxed{\frac{q^2}{4\pi\epsilon_0 a}\left(\sqrt{2} - 4\right)}$$

$\boxed{P26}$ PROBLEM 2.37



spherical shells, with uniform charge

charge q at $r = a$

charge $-q$ at $r = b$

Calculate energy of the configuration

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \quad \text{for } a < r < b$$

$\vec{E} = 0$ elsewhere.

(except $r = a, b$ where E undefined)

(a.)
$$W = \frac{\epsilon_0}{2}\int E^2 \, d\tau$$
$$= \frac{\epsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{q^2}{(4\pi\epsilon_0)^2}\frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{r^4}$$
$$= \frac{q^2}{8\pi\epsilon_0}\left(\frac{-1}{r}\Big|_a^b\right) = \boxed{\frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

(b.)
$$W_{tot} = W_1 + W_2 + \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau$$
$$\vec{E}_1 = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \quad \text{and} \quad \vec{E}_2 = \frac{-q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \qquad \left(\begin{matrix}\vec{E}_1 = 0 & 0 < r < a \\ \vec{E}_2 = 0 & 0 < r < b\end{matrix}\right)$$
$$W_1 = \frac{\epsilon_0}{2}\int E_1^2 \, d\tau = \frac{4\pi\epsilon_0 q^2}{2(4\pi\epsilon_0)^2}\frac{-1}{r}\Big|_a^{\infty} = \frac{q^2}{8\pi\epsilon_0 a}$$
$$W_2 = \frac{\epsilon_0}{2}\int E_2^2 \, d\tau = \frac{q^2}{8\pi\epsilon_0 b}$$
$$\vec{E}_1 \cdot \vec{E}_2 = -\frac{q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4} \quad \text{for } r > b$$
$$\epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau = \frac{-q^2\epsilon_0}{(4\pi\epsilon_0)^2}\int_0^{2\pi}\int_0^{\pi}\int_b^{\infty}\frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{r^4}$$
$$= \frac{-q^2}{4\pi\epsilon_0}\left(\frac{-1}{r}\Big|_b^{\infty}\right) = \frac{-q^2}{4\pi\epsilon_0 b}$$
$$\therefore \quad W_{total} = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b}\right) = \text{same as (a.)}$$