Printed Name: ______________________________.

PHYSICS 331	MISSION 3: ELECTROSTATICS (37+3)PTS)

Problems mentioned refer to David Griffith's *Introduction to Electrodynamics*, 5th edition.

Problem 27 (4pts) Suppose two infinite conducting half-planes meet at the origin. One points along the positive x-axis and the other along the positive y-axis. If a charge Q is placed at (a, b) then find the potential at (x, y, z) where $x, y > 0$.

Problem 28 (3pt) Problem 3.3 (Laplace solution in sphericals)

Problem 29 (3pt) Problem 3.7 (method of images, plane with few charges)

Problem 30 (3pt) Problem 3.8 (method of images, grounded sphere with near point charge)

Problem 31 (3pt) Problem 3.12 (method of images)

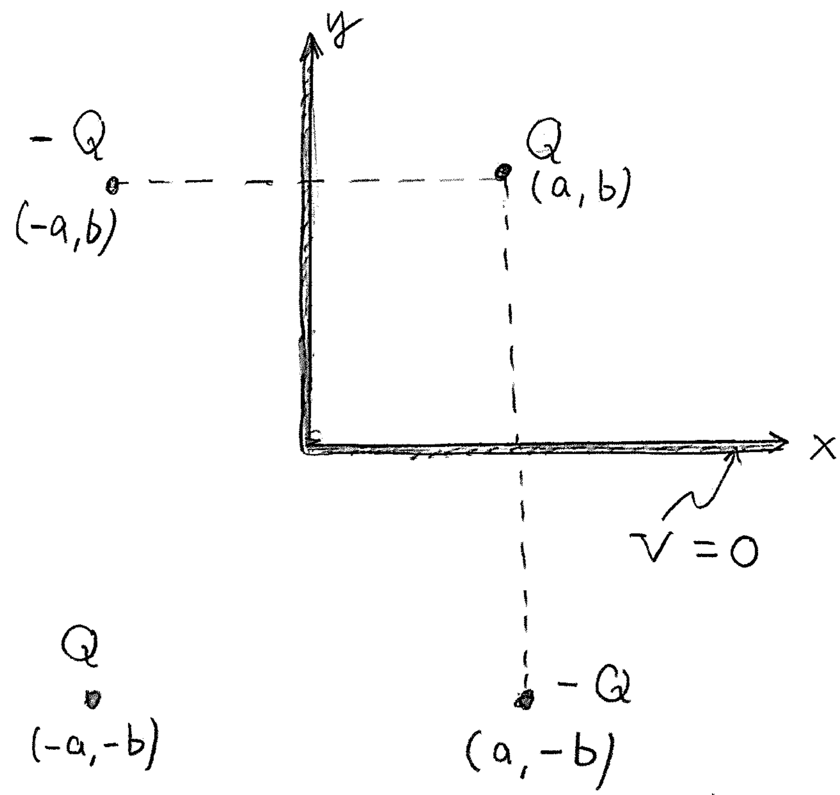
Problem 32 (8pt) Problem 3.15 (Cartesian geometry problem, separation of variable with Fourier technique)

Problem 33 (8pt) Problem 3.17 (Cartesian geometry problem, separation of variable with Fourier technique)

Problem 34 (8pt) Problem 3.21 (spherical potential, requires Legendre polynomial technique)

Mission 3 Solution

$\boxed{\text{P27}}$ Consider two conducting half-planes which meet at the origin (xz-plane and yz-plane with $x, y \geq 0$) (Suppress z)



We suspect placing image charges as pictured may produce $V = 0$ along $x = 0$, $y > 0$ or $y = 0$, $x > 0$. Let's check on my claim,

$$\boxed{V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right]}$$

$$V(0,y,z) = \frac{Q}{4\pi\varepsilon_0}\left[\cancel{\frac{1}{\sqrt{a^2+(y-b)^2+z^2}}} + \cancel{\frac{1}{\sqrt{a^2+(y+b)^2+z^2}}} - \cancel{\frac{1}{\sqrt{a^2+(y-b)^2+z^2}}} - \cancel{\frac{1}{\sqrt{a^2+(y+b)^2+z^2}}}\right]$$

Thus $V(0,y,z) = 0$ for all points on yz-plane.

P27 continued

Nearly the same calculation for xz-plane ($y=0$).

$$V(x,0,z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+b^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+b^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+b^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+b^2+z^2}}\right]$$

thus $V(x,0,z) = 0$ for all points on yz-plane.

Hence, by uniqueness Th$^{\underline{m}}$ applied to region

$$\mathcal{V} = \{(x,y,z) \mid x, y > 0\}$$

we find the potential constructed by the method of images produced the correct potential for $\mathcal{V}$ given the charge Q at (a,b) and the grounded half-planes at $x=0$, $y=0$.

P28 GRIFFITHS 3.3

Find general solution to Laplace's Equation for
(a.) when V depends only on r (spherical)
(b.) when V depends only on s (cylindrical)

(a.) $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] \quad \text{since } \frac{\partial V}{\partial \theta} = 0 \text{ and } \frac{\partial V}{\partial \phi} = 0$$

But, $V = V(r)$ so $\frac{\partial V}{\partial r} = \frac{dV}{dr}$ and $\nabla^2 V = 0$ yields

$$\frac{d}{dr}\left[r^2 \frac{dV}{dr}\right] = 0$$

$$\Rightarrow r^2 \frac{dV}{dr} = C_1 \leftarrow \underline{\text{Constant}}.$$

$$\Rightarrow \frac{dV}{dr} = \frac{C_1}{r^2} \Rightarrow \boxed{V(r) = C_2 - \frac{C_1}{r}}$$

(b.) $$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left[s \frac{\partial V}{\partial s}\right] \quad \text{since } \frac{\partial V}{\partial \phi} = 0 \text{ and } \frac{\partial V}{\partial z} = 0$$

Then as $V = V(s)$ so $\frac{\partial V}{\partial s} = \frac{dV}{ds}$ and $\nabla^2 V = 0$ yields

$$\frac{d}{ds}\left[s \frac{dV}{ds}\right] = 0$$

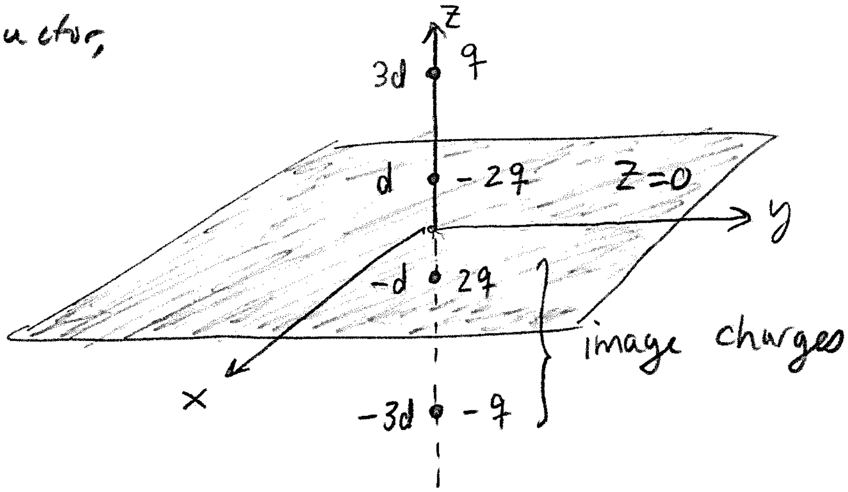
$$\Rightarrow s \frac{dV}{ds} = C_1$$

$$\Rightarrow \frac{dV}{ds} = \frac{C_1}{s}$$

$$\Rightarrow \boxed{V(s) = C_2 + \ln(s)}$$

P29 GRIFFITHS 3.7

Find force on charge q. Here xy-plane is grounded conductor,



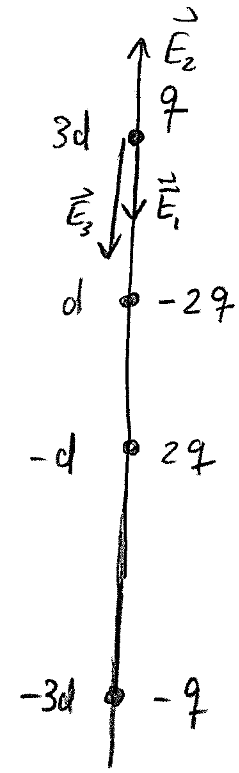
$$V_1(x,y,z) = \frac{1}{4\pi\varepsilon_0}\Bigg[\underbrace{\frac{-2q}{\sqrt{x^2+y^2+(z-d)^2}}}_{\substack{\text{potential}\\ \text{due to } -2q\\ \text{at } (0,0,d)}} + \underbrace{\frac{2q}{\sqrt{x^2+y^2+(z+d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+3d)^2}}}_{\substack{\text{image charge}\\ \text{portion of potential}}}\Bigg]$$

Now $V(x,y,z) = \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{\sqrt{x^2+y^2+(z-3d)^2}} + V_1(x,y,z)$ yields the potential due to both q and $-2q$ in the $z>0$ half-space (we can check $V(x,y,0)=0$ hence by uniqueness Thm the $V(x,y,z)$ formula solves Poisson's Eq^{n} in the $z>0$ space). To find $\vec{E}$-field due to $-2q$ at $(0,0,d)$ and the <u>induced charge</u> on the xy-plane is given by

$$\vec{E}_1 = -\nabla V_1$$ (this gives sum of Coulomb fields from $-2q$ at $(0,0,d)$
$2q$ at $(0,0,-d)$
$-q$ at $(0,0,-3d)$)

P29 continued

$$\vec{E}(0,0,3d) = \frac{q}{4\pi\varepsilon_0}\left[\overset{(\vec{E}_1)}{\frac{-2\hat{z}}{(2d)^2}} + \overset{(\vec{E}_2)}{\frac{2\hat{z}}{(4d)^2}} - \overset{(\vec{E}_3)}{\frac{\hat{z}}{(6d)^2}}\right]$$

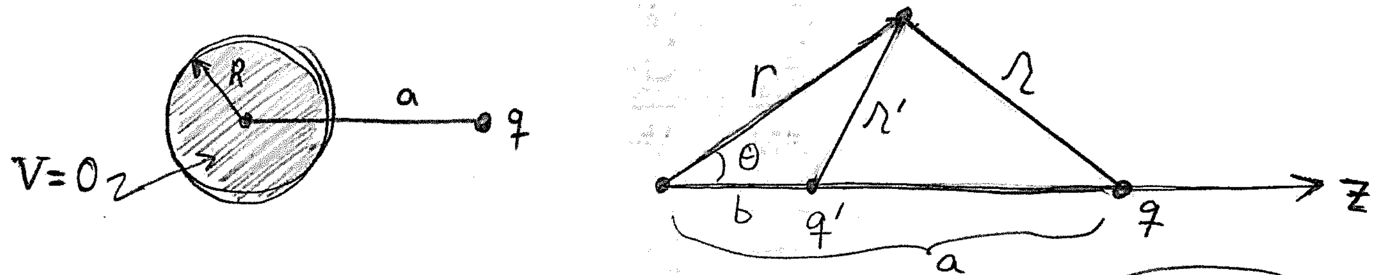


$$\vec{E}(0,0,3d) = \frac{q\hat{z}}{4\pi\varepsilon_0 d^2}\left[\frac{-2}{4} + \frac{2}{16} - \frac{1}{36}\right]$$

$$\underbrace{\vec{E}(0,0,3d) = \left(\frac{-29q}{288\pi\varepsilon_0 d^2}\right)\hat{z}}_{\text{electric field at } (0,0,3d)}$$

$$\boxed{\vec{F} = q\vec{E} = \frac{-29q^2\hat{z}}{288\pi\varepsilon_0 d^2}}$$

P30 GRIFFITHS 3.8



Example 3.2 tells us to use $q' = -\frac{R}{a} q$ at $b = \frac{R^2}{a}$ *

then $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{𝓇} + \frac{q'}{𝓇'}\right)$ gives potential due to q and the induced charge on the grounded sphere of radius R.

(a.) Apply the law of cosines to see that:

$$𝓇^2 = r^2 + a^2 - 2ra\cos\theta$$

$$(𝓇')^2 = r^2 + b^2 - 2rb\cos\theta$$

Thus,

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{R/a}{\sqrt{r^2+b^2-2rb\cos\theta}}\right]$$

by *

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\sqrt{\frac{a^2}{R^2}\left[r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta\right]}}\right]$$

$$\boxed{= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right]}$$

Notice $V = 0$ when $\|\vec{r}\| = r = R$ hence the image charge $q' = \frac{-R}{a}q$ at $b = R^2/a$ has succeeded in producing $V = 0$ along the sphere $r = R$.

P30 continued

(b.) find induced surface charge as function of θ

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R}$$

$$= \frac{-q}{4\pi} \frac{\partial}{\partial r}\Big|_{r=R}\left[\left(r^2+a^2-2ra\cos\theta\right)^{-1/2} - \left(R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta\right)^{-1/2}\right]$$

$$= \frac{-q}{4\pi}\left[\frac{\left(r^2+a^2-2ra\cos\theta\right)^{-3/2}}{-2}(2r-2a\cos\theta)\right.$$

$$\left. - \frac{\left(R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta\right)^{-3/2}}{-2}\left(2r\frac{a^2}{R^2}-2a\cos\theta\right)\right]\Big|_{r=R}$$

Now we're interested in the evaluation of the expression above at $r=R$,

$$\sigma = \frac{-q}{4\pi}\left[-\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}(R-a\cos\theta) + \left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R}-a\cos\theta\right)\right]$$

$$= \frac{q}{4\pi}\left[\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}\left[R-a\cos\theta-\frac{a^2}{R}+a\cos\theta\right]\right]$$

$$= \frac{q\,(R-a^2/R)}{4\pi\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}$$

$$= \boxed{\frac{q}{4\pi R}\left(\frac{R^2-a^2}{\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}\right) = \sigma_{induced}}$$

$$Q_{induced} = \int_{r=R}\sigma\, da = \int_0^{2\pi}\int_0^{\pi}\frac{q(R^2-a^2)}{4\pi R}\cdot\frac{R^2\sin\theta\, d\theta\, d\phi}{\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}$$

$$= \frac{2\pi q(R^2-a^2)R^2}{4\pi R}\;\frac{\left(R^2+a^2-2Ra\cos\theta\right)^{-1/2}}{-\frac{1}{2}(2Ra)}\Bigg|_0^{\pi}$$

$$= \frac{\pi q(R^2-a^2)R^2}{2\pi R^2 a}\left(\frac{-1}{\sqrt{R^2+a^2-2Ra}} + \frac{1}{\sqrt{R^2+a^2+2Ra}}\right)$$

$$= q\left(\frac{(R+a)(R-a)}{2a}\right)\left[\frac{-1}{R-a}+\frac{1}{R+a}\right] = \frac{-q}{2a}\left(R+a-(R-a)\right) = \boxed{\frac{-qR}{a}}$$

P30 continued

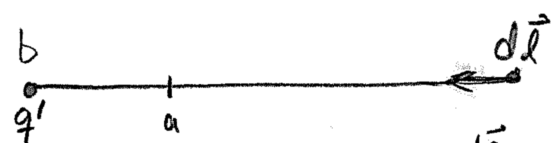
(b.) we found $\underline{Q_{induced} = -\frac{R}{a} q = q'}$.

(C.) work needed to construct system is given by

$$F = \frac{q q'}{4\pi\varepsilon_0 (r-b)^2}$$ acting on q as it moves

from $r=\infty$ to $r=a$. Notice moving the induced charge into place takes no work since we assume a perfect conducting sphere.

$$W = \int_{\infty}^{a} \vec{F}\cdot d\vec{l}$$



$d\vec{l} = \hat{r}\, dr$

$$= -\int_{a}^{\infty} \vec{F}\cdot d\vec{l}$$

$$\vec{F} = \frac{-(Rq^2/a)\hat{r}}{4\pi\varepsilon_0 (r-b)^2}$$

$$= -\int_{a}^{\infty} \frac{-Rq^2}{4\pi\varepsilon_0 a} \frac{dr}{(r-b)^2}$$

$$= \frac{Rq^2}{4\pi\varepsilon_0 a} \left. \frac{-1}{r-b} \right|_{a}^{\infty}$$

why is this wrong?

$$= \frac{Rq^2}{4\pi\varepsilon_0 a}\left(\frac{-1}{\infty} + \frac{1}{a-b}\right)$$

$$= \boxed{\frac{Rq^2}{4\pi\varepsilon_0 a(a-b)}} = \frac{Rq^2}{4\pi\varepsilon_0 a(a - R^2/a)}$$

$$= \boxed{\frac{Rq^2}{4\pi\varepsilon_0 (a^2-R^2)}}$$

P30 continued

(c.) the error was assuming the induced charge was the same when q is at $r=\infty$ as it was when q reached $r=a$. In fact, we must imagine

$$q' = \frac{-R}{r} q \quad \text{at} \quad b = \frac{R^2}{r}$$

and find dW from moving q' along $d\vec{l} = \hat{r}\,dr$ from $r=\infty$ all the way to $r=a$,

$$W = -\int_a^\infty \vec{F} \cdot d\vec{l}$$

$$= \int_a^\infty \frac{Rq^2}{4\pi\varepsilon_0} \cdot \frac{dr}{r(r - R^2/r)^2}$$

$$= \frac{q^2 R}{4\pi\varepsilon_0} \int_a^\infty \frac{r\,dr}{(r^2 - R^2)^2}$$

$u = r^2 - R^2$
$du = 2r\,dr$

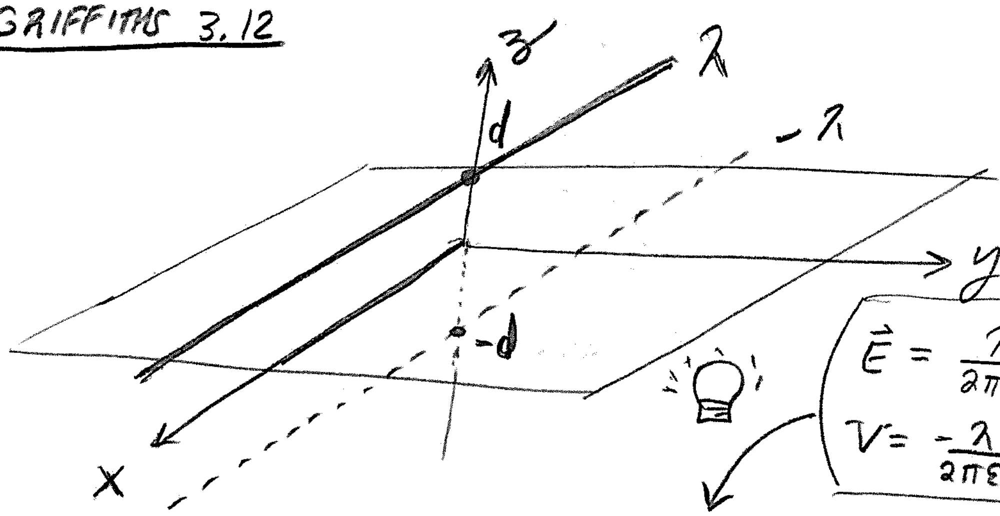
$$\int \frac{du/2}{u^2} = \frac{-1}{2u} + c$$

$$= \frac{q^2 R}{8\pi\varepsilon_0} \left(\frac{-1}{r^2 - R^2} \Big|_a^\infty \right)$$

$$= \boxed{\frac{q^2 R}{8\pi\varepsilon_0 (a^2 - R^2)}}$$

(funny, it's half my wrong answer, see why?)

P31 GRIFFITHS 3.12



$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s} \text{ for } \lambda \text{ on } z\text{-axis}$$

$$V = \frac{-\lambda}{2\pi\varepsilon_0}\ln(s) + C$$

(a.) $$V_\lambda(\vec{r}) = \frac{-\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{y^2+(z-d)^2}}{d}\right)$$

$$V_{-\lambda}(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{y^2+(z+d)}}{d}\right)$$

I've set these so both have $V = 0$ on x-axis.

Construct,

$$V(x,y,z) = V_\lambda(\vec{r}) + V_{-\lambda}(\vec{r})$$

d's cancel

$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{y^2+(z+d)^2}}{\sqrt{y^2+(z-d)^2}}\right)$$

$$= \boxed{\frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(y^2+(z+d)^2\right) - \ln\left(y^2+(z-d)^2\right)\right]}$$

(b.) $$\sigma = -\varepsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0}$$

$$= -\varepsilon_0\left(\frac{\lambda}{4\pi\varepsilon_0}\right)\left[\frac{2(z+d)}{y^2+(z+d)^2} - \frac{2(z-d)}{y^2+(z-d)^2}\right]\bigg|_{z=0}$$

$$= \frac{-\lambda}{2\pi}\left[\frac{d}{y^2+d^2} + \frac{d}{y^2+d^2}\right]$$

$$= \boxed{\frac{-\lambda d}{\pi(y^2+d^2)} = \sigma_{\text{induced}}}$$

P32 GRIFFITHS 3.15

Find potential of the infinite slot of Example 3.3 given the potential at $x=0$ is given by $V(0,y) = V_0$ for $0 < y < a/2$ and $V(0,y) = -V_0$ for $a/2 < y < a$.

We found $V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$ and argued

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

$$= \frac{2}{a}\left(\int_0^{a/2} V_0 \sin\left(\frac{n\pi y}{a}\right)dy - \int_{a/2}^{a} V_0 \sin\left(\frac{n\pi y}{a}\right)\right)$$

$$= \frac{2V_0}{a}\left[\frac{a}{n\pi}\cos\left(\frac{n\pi y}{a}\right)\Big|_0^{a/2} - \frac{a}{n\pi}\cos\left(\frac{n\pi y}{a}\right)\Big|_{a/2}^{a} \right]$$

$$= \frac{2V_0}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1 - \cos(n\pi) + \cos\left(\frac{n\pi}{2}\right)\right]$$

Thus,

$$C_{2j} = \frac{2V_0}{2j\pi}\left[2\cos\left(\frac{2j\pi}{2}\right) - 1 - \cos(2j\pi)\right] = \frac{V_0}{j\pi}\underbrace{\left[2(-1)^j - 2\right]}_{\substack{\text{zero if } j \text{ even} \\ -4 \text{ if } j = 2k-1}}$$

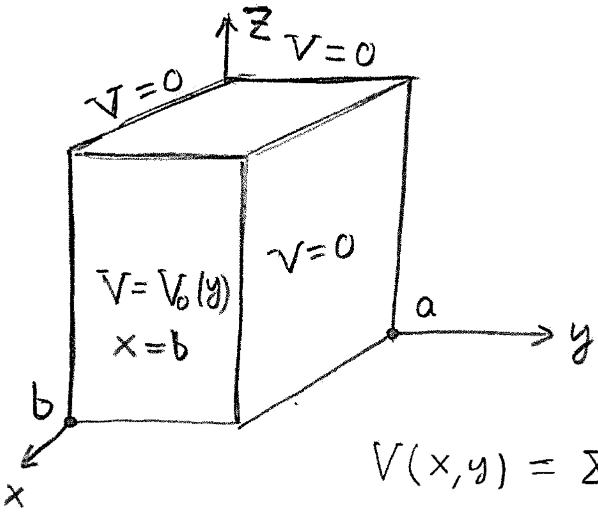
$$\underline{C_{2(2k-1)} = \frac{-4V_0}{\pi(2k-1)}}.$$

Likewise,

$$C_{2j-1} = \frac{2V_0}{(2j-1)\pi}\left[\underbrace{2\cos\left(\tfrac{\pi}{2}(2j-1)\right)}_{0} \overbrace{- 1 - \cos((2j+1)\pi)}^{-1-(-1)^{2j+1} = 0}\right] = 0$$

$$\boxed{V(x,y) = -\frac{4V_0}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1} e^{-\frac{(2k-1)\pi x}{a}} \sin\left(\frac{(2k-1)\pi y}{a}\right)}$$

P33 GRIFFITHS 3.17



Suppress z-dependence,

$$V(0,y) = 0 \quad (BC1)$$
$$V(x,0) = 0 \quad (BC2)$$
$$V(x,a) = 0 \quad (BC3)$$
$$V(b,y) = V_0(y) \quad (BC4)$$

$$V(x,y) = X(x)\, Y(y)$$

$$V_{xx} + V_{yy} = X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = C \Rightarrow \begin{pmatrix} X'' - CX = 0 \\ Y'' + CY = 0 \end{pmatrix}$$

Applying (BC2) and (BC3) $\Rightarrow$ $Y(0) = Y(a) = 0$

then $Y'' + cY = 0$ has no nontrivial solutions for $c \leq 0$. Hence $c = \beta^2$ and $Y'' + \beta^2 Y = 0$ has solution $Y(y) = c_1 \cos(\beta y) + c_2 \sin(\beta y)$.

$$Y(0) = c_1 = 0$$
$$Y(a) = c_2 \sin(\beta a) = 0 \Rightarrow \beta a = n\pi$$

where $n = 1, 2, 3, \dots$

$\therefore \beta = \frac{n\pi}{a}$

thus $\underline{Y_n(y) = \sin\left(\frac{n\pi y}{a}\right)}$.

Then $X'' - \beta^2 X = 0$ has solution

$$X_n = a_n \cosh(\beta x) + b_n \sinh(\beta x)$$

Notice (BC1) $\Rightarrow$ $X(0) = 0$ $\Rightarrow$ $\underline{a_n = 0}$. Thus,

(a.) $$\boxed{V(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)}$$

general formula, not implemented (BC4)

P33 continued

We found $V(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$

To find formula for b_n we need the Fourier trick,

$$\underline{\text{BC 4}} \quad V(b,y) = V_0(y) = \sum_{n=1}^{\infty} \underbrace{b_n \sinh(\beta b)}_{B_n} \sin(\beta y) \qquad \beta = \frac{n\pi}{a}$$

$$\int_0^a V_0(y)\sin\left(\frac{m\pi y}{a}\right)dy = \int_0^a \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dy$$

$$= \sum_{n=1}^{\infty} B_n \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dy$$

$$= \sum_{n=1}^{\infty} B_n \frac{2}{a}\delta_{n,m}$$

$$= \frac{2B_m}{a}$$

$$= \frac{2\, b_m \sinh\left(\frac{m\pi b}{a}\right)}{a}$$

$$\Rightarrow \boxed{b_n = \frac{a}{2\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy}$$

P33 continued

(b.) Suppose $V_0(y) = V_0$

$$b_n = \frac{a}{2\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a V_0 \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{a V_0}{2\sinh\left(\frac{n\pi b}{a}\right)} \cdot \frac{a}{n\pi}\left(-\cos(n\pi) + \cos(0)\right)$$

$$= \frac{V_0 a^2}{2n\pi \sinh\left(\frac{n\pi b}{a}\right)}\left(1 - (-1)^n\right)$$

Thus $b_{2j} = 0$ whereas $(-1)^{2j-1} = -1$ hence

$$b_{2j-1} = \frac{V_0 a^2}{\pi(2j-1)\sinh\left(\frac{(2j+1)\pi b}{a}\right)}.$$

Therefore,

$$\boxed{V(x,y,z) = \sum_{j=1}^{\infty} \frac{V_0 a^2 \sinh\left(\frac{(2j+1)\pi x}{a}\right)\sin\left(\frac{(2j+1)\pi y}{a}\right)}{\pi \sinh\left(\frac{(2j+1)\pi b}{a}\right)}}$$

P34 GRIFFITHS 3.21

The potential $V_0 = k\cos(3\theta)$ is given on surface of sphere of radius R. Here k is constant. Find potential inside & outside sphere if we assume $\rho = 0$ for $r \neq R$. Also, find surface charge density for the given sphere.

This problem has axial (no ϕ-dep.) hence the general solution has form

$$V(r,\theta) = \sum_{m=0}^{\infty}\left(A_m r^m + \frac{B_m}{r^{m+1}}\right)P_m(\cos\theta)$$

However, we can adjust A_m & B_m differently in separate regions. For this problem, for $0 \le r < R$, set $\frac{1}{r^{m+1}}$ terms to zero since they blow-up at $r=0$

$$V_{I}(r,\theta) = \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta)$$

whereas for $r > R$, since $r \to \infty$ makes A_m-term blow-up, $A_m = 0$

$$V_{II}(r,\theta) = \sum_{m=0}^{\infty} \frac{B_m}{r^{m+1}} P_m(\cos\theta)$$

Continuity at $r=R$, $V_I(R,\theta) = V_{II}(R,\theta)$,

$$\sum_{m=0}^{\infty} A_m R^m P_m(\cos\theta) = \sum_{m=0}^{\infty} \frac{B_m}{R^{m+1}} P_m(\cos\theta)$$

$$\Rightarrow A_m R^m = \frac{B_m}{R^{m+1}}$$

$$\therefore \underline{B_m = A_m R^{2m+1}}.$$

P34 continued

It remains to impose $V_0 = k\cos(3\theta)$ at $r = R$,
It will be helpful to rewrite $\cos(3\theta)$ in terms of $\cos\theta$

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$$

Now, recall

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta = x$$
$$P_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(\cos\theta) = \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right)$$

My goal is to express $\cos(3\theta)$ as sum of Legendre polynomials in $\cos\theta$. Let $x = \cos\theta$

$$\begin{aligned}\cos(3x) &= 4x^3 - 3x \\ &= 4\cdot\frac{2}{5}\cdot\frac{5}{2}\left(x^3 - \frac{3}{5}\right) + \frac{12}{5} - 3x \\ &= \frac{8}{5}P_3(x) - 3P_1(x) + \frac{12}{5}P_0(x)\end{aligned}$$

Hence, $k\cos(3\theta) = V(R,\theta)$ yields,

$$k\left(\frac{12}{5} - 3P_1(\cos\theta) + \frac{8}{5}P_3(\cos\theta)\right) = \sum_{m=0}^{\infty} A_m R^m P_m(\cos\theta)$$

Then by LI of Legendre polynomials, or the integration trick we've discussed, $A_m = 0$ for $m \neq 0, 1, 3$ whereas

$$A_0 = \frac{12k}{5}, \quad A_1 = \frac{-3k}{R}, \quad A_3 = \frac{(8/5)k}{R^3}$$

P34 continued: $B_m = A_m R^{2m+1}$

$$A_0 = \frac{12k}{5}, \quad A_1 = \frac{-3k}{R}, \quad A_3 = \frac{8k}{5R^3} \quad \text{for } V_I$$

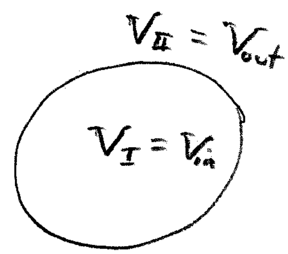
$$B_0 = \frac{12}{5}Rk, \quad B_1 = -3R^2k, \quad B_3 = \frac{8R^4k}{5} \quad \text{for } V_{II}$$

$$V(r,\theta) = \left(\frac{12}{5} - \frac{3r}{R}\cos\theta + \frac{8r^3}{5R^3}\cdot\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right)k$$

$$\text{for } 0 \le r < R$$

$$V(r,\theta) = \left(\frac{12R}{5r} - \frac{3R^2}{r^2}\cos\theta + \frac{8R^4}{5r^4}\cdot\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right)k$$

$$\text{for } r \ge R$$



Charge density:

$$\sigma_0(\theta) = -\epsilon_0\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\Bigg|_{r=R}$$

$$= -\epsilon_0\left(\left[\frac{-12R}{5R^2} + \frac{6R^2}{R^3}\cos\theta - \frac{32R^4}{5R^5}P_3(\cos\theta)\right] - \left[\frac{3}{R}\cos\theta + \frac{24R^2}{5R^3}P_3(\cos\theta)\right]\right)k$$

$$= -k\epsilon_0\left(\frac{-12}{5R} + \frac{3\cos\theta}{R} - \frac{8}{5R}\cdot\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right)$$

$$= \boxed{\frac{k\epsilon_0}{R}\left(\frac{12}{5} - \frac{3}{5}\cos\theta + 4\cos^3\theta\right)}$$