Printed Name: ______________________.

PHYSICS 331	MISSION 4: ELECTROSTATICS IN MATTER (37+3)PTS)

Problems mentioned refer to David Griffith's *Introduction to Electrodynamics*, 5th edition.

Problem 35 (5pt) Problem 4.4 (force of attraction nuetral polarized atom)

Problem 36 (6pt) Problem 4.6 (method of images, plane with ~~few charges~~) dipole)

Problem 37 (5pt) Problem 4.8 (energy of interaction for dipoles)

Problem 38 (6pt) Problem 4.14 (neutrality maintained via bound charge balanced)

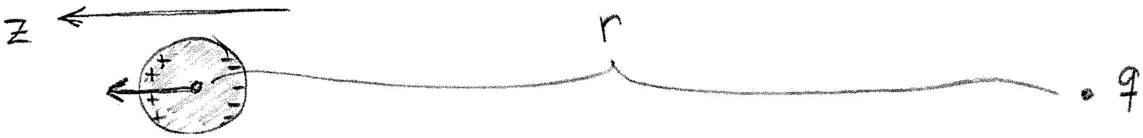
Problem 39 (6pt) Problem 4.15 (dielectric shell, contrasts methods)

Problem 40 (6pt) Problem 4.24 (conducting sphere, coated with insulator, find E-field)

Problem 41 (6pt) Problem 4.26 (conducting sphere, coated with insulator, find energy)

P35 GRIFFITHS PROBLEM 4.4

A point charge q is situated a large distance r from a neutral atom of polarization α. Find force of attraction between them.

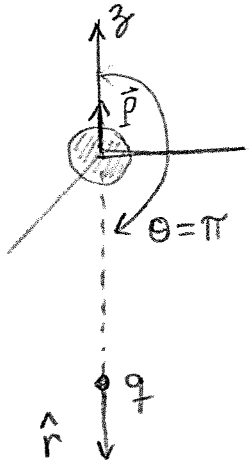


$\vec{E}_q = \frac{q}{4\pi\varepsilon_0} \frac{\hat{z}}{d^2}$ is electric field at origin due to q at $(0,0,-d)$

Then the polarization of the neutral atom is given by $\vec{P} = \alpha \vec{E}$

The neutral atom has dipole field,

$$\underbrace{\vec{E}_{dipole} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})}_{\text{assuming } \vec{p} = p\hat{z}}$$



$$\underbrace{\vec{E}_{dipole}}_{\text{at } (0,0,-d)} = \frac{\alpha E_q}{4\pi\varepsilon_0 d^3}(-2\hat{r}) = \frac{-2\alpha}{4\pi\varepsilon_0 d^3}\left(\frac{q}{4\pi\varepsilon_0 d^2}\right)\hat{z}$$

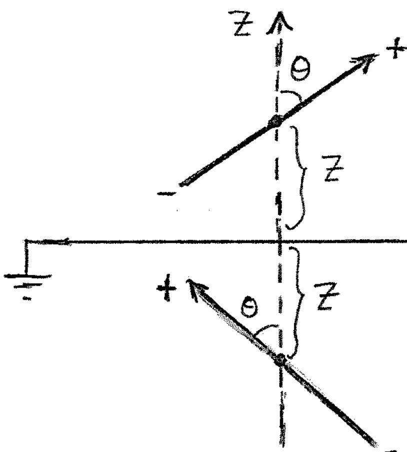
$$\vec{F} = q\vec{E}_{dipole}$$

$$|\vec{F}| = \frac{2\alpha q^2}{(4\pi\varepsilon_0)^2 d^5} = \boxed{\frac{\alpha q^2}{8\pi^2\varepsilon_0^2 d^5}}$$

(attractive force)

P36 GRIFFITHS PROBLEM 4.6

A perfect dipole $\vec{p}$ is situated distance z above an infinite grounded conducting plane. The dipole makes an angle θ with the normal to the plane. Find the <u>torque</u> on $\vec{p}$. If the dipole is free to rotate, in what orientation will it come to rest?



$$+\ \vec{P}_1 = P\cos\theta\,\hat{z} + P\sin\theta\,\hat{x} = \vec{P} = P\langle \sin\theta, 0, \cos\theta\rangle$$

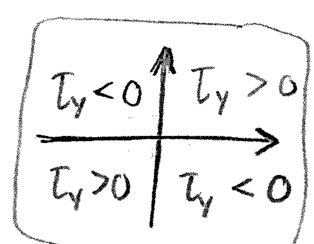
$$-\ \vec{P}_2 = P\cos\theta\,\hat{z} - P\sin\theta\,\hat{x} \quad (\text{image dipole})$$

Concept: we can replace σ induced at $z=0$ with $\vec{P}_2$ and the resulting E-field above conducting plane is same. We wish to find torque on $\vec{P}$, it is due to $\vec{E}$-field generated by induced charge on plane, so, use $\vec{P}_2$ to find E-field acting on $\vec{P}$ at $(0,0,z)$,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0(2z)^3}\left(3(\vec{P}_2\cdot\hat{z})\hat{z} - \vec{P}_2\right) \quad \left(\text{Using GRIFFITH'S PROBLEM 3.38}\right)$$

$$= \frac{P}{32\pi\varepsilon_0 z^3}\left((3\cos\theta)\hat{z} - (\cos\theta)\hat{z} + (\sin\theta)\hat{x}\right)$$

$$= \frac{P}{32\pi\varepsilon_0 z^3}\left((2\cos\theta)\hat{z} + (\sin\theta)\hat{x}\right)$$



The torque on a dipole is given by $\vec{\tau} = \vec{P}\times\vec{E}$

$$\vec{\tau} = \frac{p^2}{32\pi\varepsilon_0 z^3}\langle \sin\theta, 0, 2\cos\theta\rangle \times \langle \sin\theta, 0, \cos\theta\rangle$$

$$= \frac{p^2}{32\pi\varepsilon_0 z^3}\langle 0, \sin\theta\cos\theta, 0\rangle \Rightarrow \boxed{\vec{\tau} = \frac{p^2\sin\theta\cos\theta}{32\pi\varepsilon_0 z^3}\hat{y}}$$

$\boxed{\vec{P} \text{ rests } \perp \text{ to plane}} \longleftarrow \boxed{\theta = 0, \pi \text{ stable equilbria}}$

P37 GRIFFITH'S PROBLEM 4.8

Show that the interaction energy of two ideal dipoles seperated by a displacement $\vec{r}$ is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[\vec{P}_1\cdot\vec{P}_2 - 3(\vec{P}_1\cdot\hat{r})(\vec{P}_2\cdot\hat{r})\right]$$

Hint: we know the energy of permanant ideal dipole in electric field $\vec{E}$ is given by $U = -\vec{P}\cdot\vec{E}$. Furthermore

$$\vec{E}_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{P}\cdot\hat{r})\hat{r} - \vec{P}\right]$$

Place $\vec{P}_1$ at the origin then the electric field due to $\vec{P}_1$ at $\vec{r}$

$$\begin{aligned} U &= -\vec{P}_2\cdot\vec{E}_1 \\ &= -\vec{P}_2\cdot\left(\frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\vec{P}_1\cdot\hat{r})\hat{r} - \vec{P}_1\right]\right) \\ &= \underline{\frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[\vec{P}_2\cdot\vec{P}_1 - 3(\vec{P}_1\cdot\hat{r})(\vec{P}_2\cdot\hat{r})\right]}. \end{aligned}$$

Remark: we should have worked Problem 3.38

P38 PROBLEM 4.14

When you polarize a neutral dielectric, the charge moves a bit, but the total remains zero. Show that the total bound charge vanishes.

We have $\sigma_b = \vec{P} \cdot \hat{n}$ and $\rho_b = -\nabla \cdot \vec{P}$ where $\vec{P}$ is the polarization of an object with normal $\hat{n}$.

$$\int_{\mathcal{V}} \rho_b \, d\tau = -\int_{\mathcal{V}} (\nabla \cdot \vec{P}) \, d\tau$$

$$= -\int_{\partial \mathcal{V}} \vec{P} \cdot d\vec{a}$$

$$= -\int_{\partial \mathcal{V}} (\vec{P} \cdot \hat{n}) \, da$$

$$= -\int_{\partial \mathcal{V}} \sigma_b \, da$$

Therefore,

$$\underbrace{\int_{\mathcal{V}} \rho_b \, d\tau}_{\text{net bound charge within } \mathcal{V}} + \underbrace{\int_{\partial \mathcal{V}} \sigma_b \, da}_{\text{net bound charge on surface of } \mathcal{V}} = 0$$

P39 PROBLEM 4.15

A thick spherical shell $a<r<b$ is made of a dielectric material with a "frozen-in" polarization

$$\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$$

where k is a constant (there is no free charge in this problem)
Find electric field via

(a.) locate bound charge, use Gauss' law and calculate field from the bound charge.

(b.) use $\oint \vec{D}\cdot d\vec{a} = Q_{f\,enc}$ to find $\vec{D}$ and then get $\vec{E}$ from $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

(a.) $$\sigma_{b,a} = \vec{P}\cdot\hat{n}_a = \frac{k}{a}\hat{r}\cdot(-\hat{r}) = \frac{-k}{a} \quad (\text{at } r=a)$$

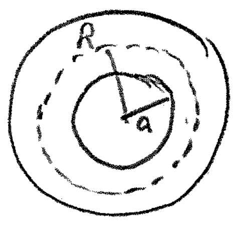
$$\sigma_{b,b} = \vec{P}\cdot\hat{n}_b = \frac{k}{b}\hat{r}\cdot(\hat{r}) = \frac{k}{b} \quad (\text{at } r=b)$$

$$\rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 P_r\right] \quad (P_\phi = P_\theta = 0)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{k}{r}\right]$$

$$= \frac{-k}{r^2}, \quad \rho_b\, d\tau = \frac{-k}{r^2} r^2\sin\theta\, d\theta\, d\phi\, dr = -k\, d\Omega\, dr$$

$$Q_{enc}(a<R<b) = \int_a^R \rho_b\, d\tau = -k\int d\Omega\int_a^R dr = -4\pi k(R-a)$$



$$\epsilon_0 \Phi_E = Q_{enc}(a) + Q_{enc}(a<R<b)$$

$$4\pi\epsilon_0 R^2 E = (4\pi a^2)\sigma_{b,a} + 4\pi k(a-R)$$

$$4\pi\epsilon_0 R^2 E = (4\pi a^2)\left(\frac{-k}{a}\right) + 4\pi k(a-R)$$

$$E = \frac{4\pi k}{4\pi\epsilon_0 R^2}\left[-a+a-R\right] = \frac{-4\pi k}{4\pi\epsilon_0 R}$$

P39 continued (a.)

We find for $a < r < b$ that

$$\boxed{\vec{E}(\vec{r}) = \left(\frac{-k}{\epsilon_0 r}\right)\hat{r} \quad \text{for } a<r<b}$$

Consider $r > b$ then the Gaussian sphere at $r = R$ includes $4\pi a^2 \sigma_{b,a} + 4\pi b^2 \sigma_{b,b} + Q_{enc}$ (total from $r=a$ to $r=b$)

$$Q_{enc} = 4\pi a^2\left(\frac{-k}{a}\right) + 4\pi b^2\left(\frac{k}{b}\right) + 4\pi k(a-b) = 0$$

Therefore Gauss' Law yields

$$4\pi R^2 E = \frac{Q_{enc}}{\varepsilon_0} = 0 \Rightarrow \boxed{\vec{E}(\vec{r}) = 0 \text{ For } r>b}$$

Likewise, $\boxed{\vec{E} = 0 \text{ for } 0 \leq r < a}$ since $Q_{enc} = 0$ within the inner sphere.

(b.) Since $Q_f = 0$ everywhere we have $\oint \vec{D}\cdot d\vec{a} = Q_f$ yields $\vec{D}(\vec{r}) = 0$ everywhere. Now $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

(i.) $0 \leq r < a$ have $\vec{P} = 0 \therefore \boxed{\vec{E} = \frac{1}{\epsilon_0}\vec{D} = \vec{0} \text{ for } 0\leq r<a}$

(ii.) $a<r<b$ have $\vec{P} = \frac{k}{r}\hat{r}$ thus

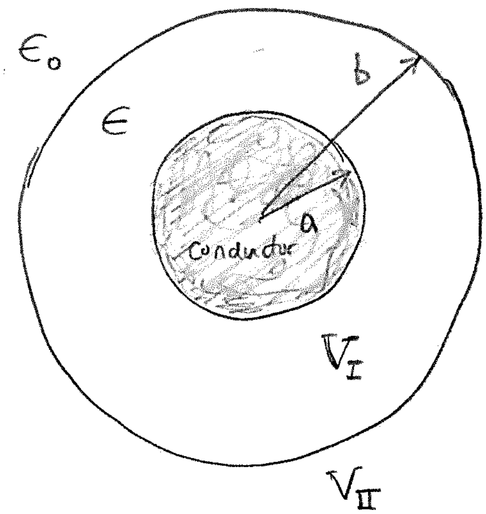
$$0 = \epsilon_0\vec{E} + \frac{k}{r}\hat{r} \Rightarrow \boxed{\vec{E} = \frac{-k}{\epsilon_0 r}\hat{r} \text{ for } a<r<b}$$

(iii) $r > b$ have $\vec{P} = 0 \therefore \boxed{\vec{E} = \frac{1}{\epsilon_0}\vec{D} = 0 \text{ for } r>b}$

<u>Remark</u>: if $\vec{D}$ is <u>defined</u> by $\varepsilon_0\vec{E} + \vec{P} = \vec{D}$ then we cannot really say $\vec{D} = 0$ at $r = a$ or $r = b$ where charge is localized

$\boxed{P40}$ PROBLEM 4.24 $\qquad \epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$

An uncharged conducting sphere of radius a is coated with a thick insulating shell (with dielectric constant ϵ_r) out to radius b. The object is now placed in otherwise uniform electric field $\vec{E}_0$. Find electric field in the insulator



Suppose $\vec{E}_0 = E_0 \hat{z}$ for $r \gg b$

then $\vec{E} = -\nabla V \Rightarrow V = -E_0 z$

or $\underline{V(\vec{r}) = -E_0 r \cos\theta \text{ for } r \gg b}$ $\boxed{BC1}$

$\underline{V_I = V_{II} \text{ for } r = b}$ $\boxed{BC2}$

We can set $V = 0$ within conductor, you might worry this is inconsistent with $V(\vec{r}) = -E_0 r\cos\theta$ for $r \gg b$ but, following path along xy-plane it is somewhat clear $\vec{E} \perp$ to xy-plane.

$\underline{V_I = 0 \text{ for } r = a}$ $\boxed{BC3}$

Finally, we have $\underline{\epsilon \frac{\partial V_I}{\partial r} = \epsilon_0 \frac{\partial V_{II}}{\partial r} \text{ at } r = b}$ $\boxed{BC4}$

Since V_I has $a < r < b$ we face all terms in the Legendre expansion (thanks to axial symmetry, no ϕ dep)

$$V_I(\vec{r}) = \sum_{m=0}^{\infty} \left(A_m r^m + \frac{B_m}{r^{m+1}} \right) P_m(\cos\theta)$$

Apply $\boxed{BC3}$,

$$0 = \sum_{m=0}^{\infty} \left(A_m a^m + \frac{B_m}{a^{m+1}} \right) P_m(\cos\theta)$$

$$\Rightarrow A_m a^m + \frac{B_m}{a^{m+1}} = 0 \Rightarrow \underline{B_m = -A_m a^{2m+1}} \quad \text{(III)}$$

P40 continued

For $r > b$, thanks to $\boxed{BC1}$ we find

$$V_{II}(\vec{r}) = -E_o r\cos\theta + \sum_{m=0}^{\infty} \frac{C_m}{r^{m+1}} P_m(\cos\theta)$$

Notice $B_m = -A_m a^{2m+1}$ yields

$$V_I(\vec{r}) = \sum_{m=0}^{\infty} A_m\left(r^m - \frac{a^{2m+1}}{r^{m+1}}\right) P_m(\cos\theta)$$

Apply $\boxed{BC2}$ and equate V_I and V_{II} at $r = b$,

$$\sum_{m=0}^{\infty} A_m\left(b^m - \frac{a^{2m+1}}{b^{m+1}}\right) P_m(\cos\theta) = -E_o b\cos\theta + \sum_{m=0}^{\infty} \frac{C_m}{b^{m+1}} P_m(\cos\theta)$$

We find conditions from LI of $P_m(\cos\theta)$

$\underline{m=0}$ $A_o(1 - a/b) = C_o/b$ ← (these must be zero since $V = 0$ where $z = 0$)

$\underline{m=1}$ $A_1(b - a^3/b^2) = -E_o b + C_1/b^2$

$\underline{m \geq 2}$ $A_m\left(b^m - \frac{a^{2m+1}}{b^{m+1}}\right) = \frac{C_m}{b^{m+1}}$

$$\underline{C_m = A_m(b^{2m+1} - a^{2m+1})}. \quad \text{IV}$$

It remains to apply $\boxed{BC4}$, applying IV to V_{II} formula,

$$V_{II}(\vec{r}) = \left(-E_o r + \frac{C_1}{r^2}\right)\cos\theta + \sum_{m=2}^{\infty} \frac{A_m(b^{2m+1} - a^{2m+1})}{r^{m+1}} P_m(\cos\theta)$$

P40 continued

$$\epsilon \frac{\partial V_I}{\partial r} = \epsilon_0 \frac{\partial V_{II}}{\partial r} \quad \text{at} \quad r = b$$

$$\epsilon \sum_{m=1}^{\infty} A_m \left(m b^{m-1} + \frac{(m+1) a^{2m+1}}{b^{m+2}} \right) P_m(\cos\theta) = * $$

$$= \epsilon_0 \left[\left(-E_0 - \frac{2 C_1}{b^3} \right) \cos\theta + \sum_{m=2}^{\infty} \frac{-(m+1) A_m \left(b^{2m+1} - a^{2m+1} \right)}{b^{m+2}} P_m(\cos\theta) \right]$$

Equating coefficients of $P_m(\cos\theta)$

$\cos\theta$ | $\epsilon A_1 \left(1 + \frac{2a^3}{b^3} \right) = \epsilon_0 \left(-E_0 - \frac{2 C_1}{b^3} \right)$

From $\boxed{BC2}$, $m=1$ we also have $\frac{C_1}{b^3} = E_0 + A_1 \left(1 - a^3/b^3 \right)$

Thus,

$$\epsilon A_1 \left(1 + \frac{2a^3}{b^3} \right) = -\epsilon_0 E_0 - 2\epsilon_0 \left(E_0 + A_1 \left(1 - a^3/b^3 \right) \right)$$

$$A_1 \left[\epsilon \left(1 + 2a^3/b^3 \right) + 2\epsilon_0 \left(1 - a^3/b^3 \right) \right] = -3\epsilon_0 E_0$$

$$A_1 \left[\epsilon \left(b^3 + 2a^3 \right) + 2\epsilon_0 \left(b^3 - a^3 \right) \right] = -3\epsilon_0 b^3 E_0$$

$$A_1 = \frac{-3 b^3 E_0}{\epsilon_r (2a^3 + b^3) + 2(b^3 - a^3)}$$

From * for $m \geq 2$ we find

$$\epsilon A_m \left(m b^{m-1} + \frac{(m+1) a^{2m+1}}{b^{m+2}} \right) = \epsilon_0 A_m \left(\frac{(m+1)(a^{2m+1} - b^{2m+1})}{b^{m+2}} \right)$$

$$\Rightarrow \underline{A_m = 0 \text{ for } m = 2, 3, \ldots}$$

P40 continued

The potential within the media is V_I in my notation, we've shown for $a<r<b$,

$$\vec{V}(\vec{r}) = \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta \quad : \text{ since } P_1(\cos\theta) = \cos\theta$$

$$= A_1\left(r - \frac{a^3}{r^2}\right)\cos\theta \quad : \quad B_1 = -A_1 a^3$$

$$= \frac{-3E_0\cos\theta}{\epsilon_r\left(2a^3/b^3+1\right)+2\left(1-a^3/b^3\right)}\left(r-\frac{a^3}{r^2}\right)$$

Finally, we can answer the question: what is the electric field in the insulator?

$$\vec{E} = -\nabla V$$

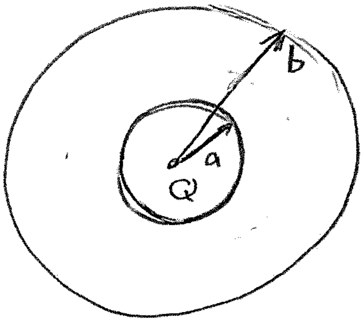
$$= -\hat{r}\left(\frac{\partial V}{\partial r}\right) - \hat{\theta}\left(\frac{1}{r}\frac{\partial V}{\partial\theta}\right)$$

$$= \frac{3E_0}{\epsilon_r\left(\frac{2a^3}{b^3}+1\right)+2\left(1-a^3/b^3\right)}\left[\cos\theta\left(1+\frac{2a^3}{r^3}\right)\hat{r}-\sin\theta\left(1-\frac{a^3}{r^3}\right)\hat{\theta}\right]$$

$$\boxed{\vec{E}(\vec{r}) = \frac{3E_0}{2\left[1-a^3/b^3\right]+\epsilon_r\left[1+2a^3/b^3\right]}\left[\cos\theta\left(1+\frac{2a^3}{r^3}\right)\hat{r}-\sin\theta\left(1-\frac{a^3}{r^3}\right)\hat{\theta}\right]}$$

P41 PROBLEM 4.26

A spherical conductor, of radius a, carries a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration



$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$\vec{D} = \epsilon_0 (1+\chi_e) \vec{E} = \epsilon \vec{E}$$

(linear dielectric)

If we consider any sphere of radius $r > a$ then $Q_{fenc} = Q$ and $\oint \vec{D} \cdot d\vec{a} = Q$ yields

$$\vec{D}(\vec{r}) = \frac{Q}{4\pi} \frac{\hat{r}}{r^2}$$

Now for $r > b$ we have $\vec{D} = \epsilon_0 \vec{E} \Rightarrow \underbrace{\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}}_{r>a}$.

On the other hand, for $a < r < b$,

$$\vec{D} = \epsilon \vec{E} \Rightarrow \underline{\vec{E} = \frac{Q}{4\pi\epsilon} \frac{\hat{r}}{r^2}} \quad (a < r < b)$$

The work to assemble this configuration, given by

$$W = \frac{1}{2} \int (\vec{D} \cdot \vec{E}) d\tau$$

$$= \frac{1}{2} \left(\iiint_{a<r<b} \left(\frac{Q}{4\pi} \frac{\hat{r}}{r^2} \right) \cdot \left(\frac{Q}{4\pi\epsilon} \frac{\hat{r}}{r^2} \right) d\tau + \iiint_{r>b} \left(\frac{Q}{4\pi} \frac{\hat{r}}{r^2} \right) \cdot \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \right) d\tau$$

$$= \frac{Q^2 \cdot 4\pi}{32\pi^2 \epsilon} \int_a^b \frac{dr}{r^2} + \frac{Q^2 \cdot 4\pi}{32\pi^2 \epsilon_0} \int_b^\infty \frac{dr}{r^2}$$

$$= \boxed{\frac{Q^2}{8\pi} \left[\frac{1}{\epsilon} \left(\frac{1}{a} - \frac{1}{b} \right) + \frac{1}{\epsilon_0} \left(\frac{1}{b} \right) \right]}$$