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PHYSICS 331 MISSION 6: MAGNETOSTATICS IN MATTER AND ELECTRODYNAMICS (38+2)PTS)

Problems mentioned refer to David Griffith's *Introduction to Electrodynamics*, 5th edition. Skip two problems. Each problem worth 4pts.

Problem 54 Problem 6.1 (torque on loop)

Problem 55 Problem 6.8 (magnetization in matter)

Problem 56 Problem 6.12 (magnetic analog to that electric displacement problem where we contrasted methods)

Problem 57 Problem 6.17 (magnetic field in wire with uniform $\mathcal{X}_m$ including analysis of bound currents)

Problem 58 Problem 7.14 (changing magnetic field generates voltage in fixed loop)

Problem 59 Problem 7.16 (solenoid, induced electric field in quasistatic approximation)

Problem 60 Problem 7.30 (energy stored in magnetic field of toroidal coil)

Problem 61 Problem 7.39 (show given pair of electric and magnetic fields solve Maxwell's Equations and find the current and charge density)

Problem 62 Problem 8.3 (energy in charging capacitor)

Problem 63 Problem 8.9 (stress energy tensor problem)

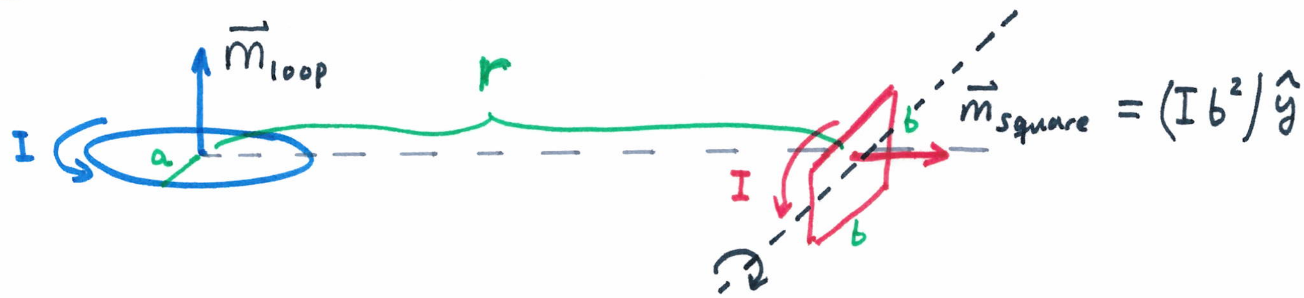
Problem 64 Problem 8.11 (angular momentum in electromagnetic fields and torque on sphere)

Problem 65 Problem 8.16 (cylindrical moving tube with charges, energy, momentum, power)

MISSION 6 SOLUTION

P54 GRIFFITHS 6.1

Calculate the torque exerted on the square loop due to the circular loop (assume $r >> a, b$) If the square loop is free to rotate, what will its equilibrium be?



$$\vec{N}_{square} = \vec{m}_{square} \times \vec{B}_{loop} = -\vec{B}_{loop} \times \vec{m}_{square}$$ (oops I did this)

$$\vec{B}_{loop} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}_{loop}\cdot\hat{r})\hat{r} - \vec{m}_{loop}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3((I\pi a^2\hat{z})\cdot\hat{y})\hat{y} - \pi a^2 I\hat{z}\right]$$

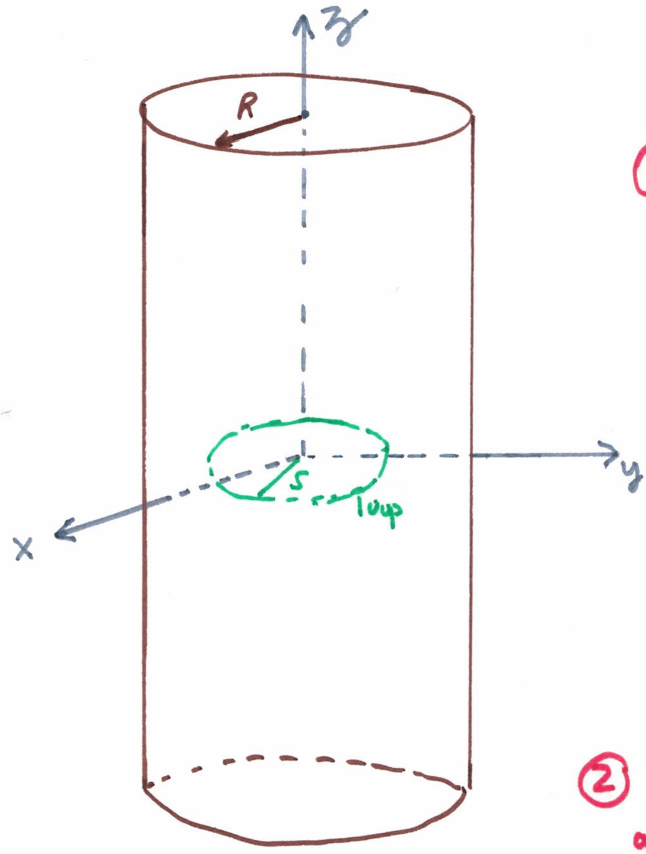
$$= \left(\frac{-\mu_0 a^2 I}{4r^3}\right)\hat{z}$$

$$\therefore \vec{N}_{square} = -\left(\frac{-\mu_0 I a^2}{4r^3}\hat{z}\right)\times\left(I b^2\hat{y}\right) = \boxed{\frac{-\mu_0 I^2 a^2 b^2}{4r^3}\hat{x}}$$

When $\vec{m}_{square} \parallel \vec{B}_{loop}$ we reach zero torque thus the equilbrium should be when $\underbrace{\vec{m}_{square} = -Ib^2\hat{z}}_{\text{"downward"}}$

PSS Problem 6.8

A very long circular cylinder of radius R carries a magnetization $\vec{M} = ks^2\hat{\phi}$ where k is constant. Find the $\vec{B}$ due to $\vec{M}$ for points inside and outside $s = R$ and assume you're considering a point far from endcaps.



Remark: there are at least two ways to solve this.

① use the auxillary field approach,

free current zero everywhere

$$\Rightarrow \vec{H} = 0$$

Then, by definition

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\therefore \boxed{\vec{B} = \mu_0 \vec{M} = \mu_0 k s^2 \hat{\phi}}$$

for $0 \le s \le R$

② CALCULATE Bound surface and volume currents and use Ampere's Law,

$$\vec{K}_b = \vec{M}\times\hat{s} = kR^2\hat{\phi}\times\hat{s} = -kR^2\hat{z} \quad \left(\begin{array}{c}\text{boundary at } s=R \\ \text{where } \hat{n}=\hat{s}\end{array}\right)$$

$$\vec{J}_b = \nabla\times\vec{M} = \frac{1}{s}\frac{\partial}{\partial s}\left[sM_\phi\right]\hat{z} = \frac{1}{s}\frac{\partial}{\partial s}\left[ks^3\right]\hat{z} = 3ks\,\hat{z}$$

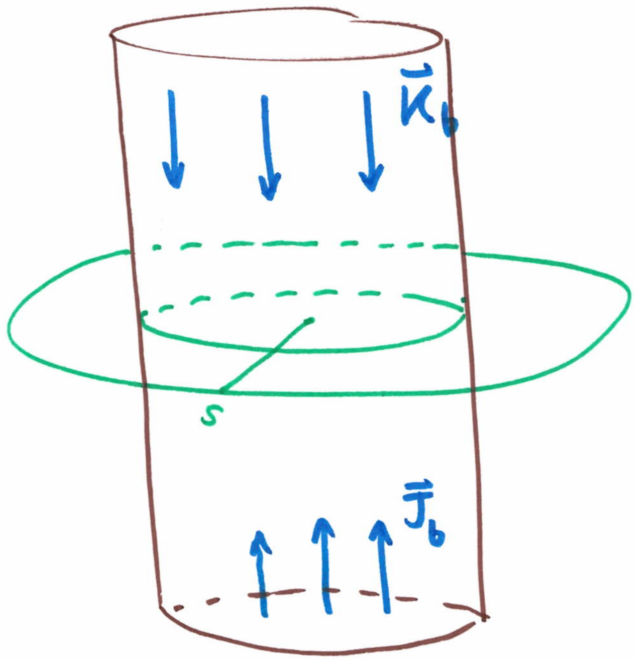
$$I_{enc} = \int_{loop} \vec{J}_b \cdot d\vec{a} = \int_0^s 3k\bar{s}\,(2\pi\bar{s}\,ds) = \frac{6k\pi\bar{s}^3}{3}\Bigg|_0^s = 2\pi k s^3$$

$$\mu_0 I_{enc} = \oint_{loop} \vec{B}\cdot d\vec{l} \Rightarrow \mu_0 2\pi k s^3 = 2\pi s B$$

$$\therefore \boxed{\vec{B} = \mu_0 k s^2 \hat{\phi}} \text{ for } s \le R$$

Then for $s > R$, $I_{enc} = 0$ hence $\boxed{\vec{B} = 0 \text{ for } s > R}$

P55 continued



$$\vec{I}_{enc} = \vec{I}_{\substack{bound\\interior}} + \vec{I}_{\substack{bound\\edge}}$$

$$= 2\pi k R^3 \hat{z} - (2\pi R)(kR^2)\hat{z}$$

$$= 0.$$

Of course, $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ makes it easier to see why where $\vec{H} = 0$ and $\vec{M} = 0$ $(s > R)$ we surely obtain $\vec{0} = \frac{1}{\mu_0}\vec{B} - \vec{0}$ $\therefore$ $\underline{\vec{B} = 0}$

(for $s > R$)

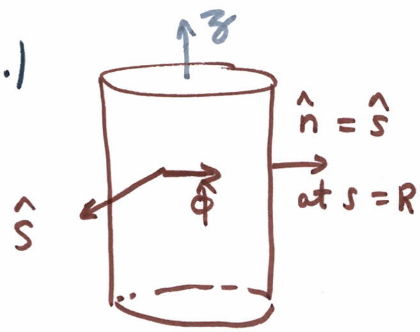
PS6 Problem 6.12

An infinitely long cylinder of radius R carries frozen-in magnetization $\vec{M} = ks\,\hat{z}$. There is no free current anywhere.

(a.) locate bound currents and find $\vec{B}$ created

(b.) find $\vec{H}$ and obtain $\vec{B}$

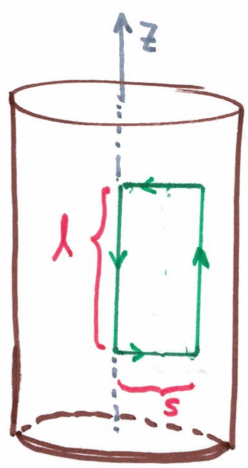
(a.)



$$\vec{K}_b = \vec{M}\times\hat{n} = kR\,\hat{z}\times\hat{s} = kR\,\hat{\phi}$$

$$\vec{J}_b = \nabla\times\vec{M} = -\frac{\partial M_z}{\partial s}\hat{\phi} \quad (\text{other terms zero})$$

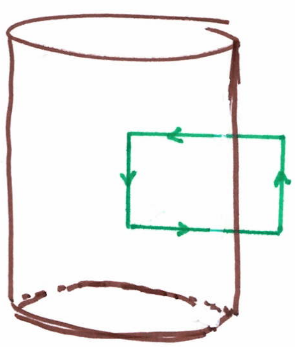
$$\vec{J}_b = -k\hat{\phi}$$



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \quad (d\vec{a} = (da)(-\hat{\phi}))$$

$$B_z(s)\,l = \mu_0(-lsk)$$

$$\Rightarrow \boxed{\vec{B} = \mu_0 sk\,\hat{z} \quad \text{for } 0 \le s < R}$$



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \quad (d\vec{a} = (da)(-\hat{\phi}))$$

$$B_z(s)\,l = \mu_0(lRk - kRl) = 0$$

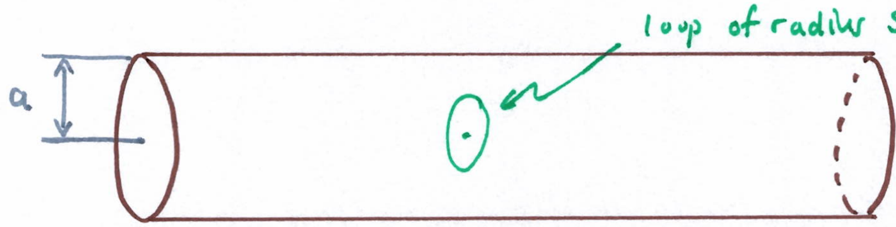
$$\Rightarrow \boxed{\vec{B} = 0 \quad \text{for } s > R}$$

(b.) since free current is everywhere zero $\oint_{loop} \vec{H}\cdot d\vec{l} = 0$ for all loops $\therefore$ $\vec{H} = 0$ everywhere. But, $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

$$\vec{M} = \begin{cases} ks\,\hat{z} & \text{for } s \le R \\ 0 & \text{for } s > R \end{cases} \Rightarrow \boxed{\begin{array}{ll} \vec{B} = \mu_0 ks\,\hat{z} & \text{for } s \le R \\ \vec{B} = 0 & \text{for } s > R \end{array}}$$

PS7 Problem 6.17

A (free) current I flows down a long straight wire of radius a. If the wire is made of linear material (copper or aluminum etc.) with susceptibility χ_m, and the current is distributed uniformly, what is the magnetic field a distance s from the axis? Find all bound currents. What is the net bound current flowing down the wire?



$$J_f = \frac{I}{\pi a^2}$$

For $s<a$, $\oint \vec{H}\cdot d\vec{l} = \pi s^2 J_f = \frac{Is^2}{a^2} = 2\pi s H \therefore H = \frac{Is}{2\pi a^2}$

For $s>a$ $\oint \vec{H}\cdot d\vec{l} = I = 2\pi s H \therefore H = \frac{I}{2\pi s}$

For linear media $\vec{B} = \mu \vec{H}$ where $\mu = \mu_0(1+\chi_m)$ thus, $\underbrace{\vec{H} = H\hat{\phi}}_{\text{from above work}}$

$$\boxed{\vec{B} = \frac{I\mu_0(1+\chi_m)s}{2\pi a^2}\hat{\phi} \text{ for } s \leq a}$$

When $s>a$ then $\mu = \mu_0$ and then $\vec{B} = \mu_0 \vec{H}$ hence

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \text{ for } s>a}$$

Bound currents? Recall $\vec{M} = \chi_m \vec{H}$ for linear media,

$$\vec{J}_b = \nabla\times\vec{M} = \nabla\times(\chi_m\vec{H}) = \chi_m \nabla\times\vec{H} = \chi_m \vec{J}_f$$

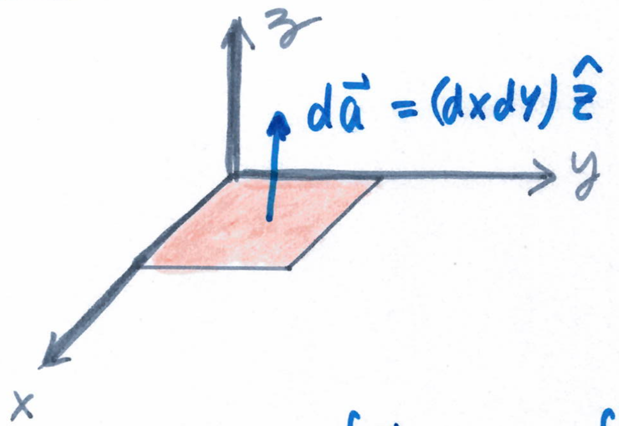
$$\boxed{\vec{J}_b = \frac{\chi_m I}{\pi a^2}\hat{z}}$$ (assuming I flows in $\hat{z}$-direction)

$$\vec{K}_b = \vec{M}\times\hat{n} = \chi_m \vec{H}\times\hat{s} = \frac{\chi_m I a}{2\pi a^2}\hat{\phi}\times\hat{s} = \boxed{\frac{-\chi_m I}{2\pi a}\hat{z} = \vec{K}_b}$$

$$\vec{I}_{enc\ total} = \pi a^2 \vec{J}_b + 2\pi a \vec{K}_b = (\chi_m I - \chi_m I)\hat{z} \therefore \boxed{I_{enc} = 0}$$

PS8 | Problem 7.14

A square loop of wire, side length a, lies in the 1^{st} quadrant of the xy-plane, with one corner at (0,0). In this region $\vec{B}(y,t) = k y^3 t^2 \hat{z}$ where k is constant. Find the emf induced in the loop



$$\Phi_B = \int \vec{B}\cdot d\vec{a} = \int_0^a \int_0^a k y^3 t^2\, dx\, dy = a k t^2 \int_0^a y^3\, dy$$

$$\Phi_B = \frac{a k t^2 a^4}{4} = \frac{1}{4} k t^2 a^5$$

$$\text{emf} = -\frac{d\Phi_B}{dt} = \boxed{-\frac{1}{2} k t a^5}$$

PS9 Problem 7.16

A long solenoid with radius a and n-turns per length carries time dependent current $I(t)$ in the $\hat{\phi}$-direction. Find the electric field (magnitude and direction) at distance s from the axis (both inside and out of the solenoid) in the quasi-static approximation.

- If I was steady current then the usual Amperian loop arguments show $B = 0$ outside the solenoid whereas $\vec{B} = \mu_0 n I \hat{z}$ for $s < a$. The quasistatic approximation posits we can still assume $\vec{B} = \mu_0 n I \hat{z}$ when $I = I(t)$.

Faraday's Law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ hence let $\underbrace{S_0 \text{ be } 0 \leq s \leq s_0 < a}_{d\vec{a} = (da)\,\hat{z}}$

$$\int_{S_0} (\nabla \times \vec{E}) \cdot d\vec{a} = -\int_{S_0} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

$$\int_{\partial S_0} \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_{S_0} \vec{B} \cdot d\vec{a}$$

$$E(2\pi s_0) = -\frac{\partial}{\partial t}\left(\mu_0 n I \pi s_0^2\right) = -\mu_0 n \pi s_0^2 \frac{dI}{dt}$$

$$\Rightarrow \boxed{\vec{E} = \frac{-\mu_0 n s}{2} \frac{dI}{dt} \hat{\phi}} \quad (s < a)$$

For disk of radius $s > a$,

$$E(2\pi s) = -\frac{\partial}{\partial t}\left(\pi a^2 \mu_0 n I\right) = -\pi a^2 \mu_0 n \frac{dI}{dt}$$

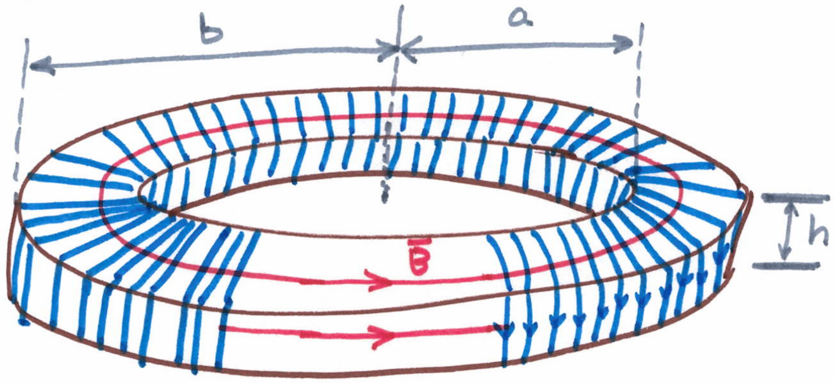
$$\vec{E} = -\frac{\mu_0 \pi a^2 n}{2\pi s} \frac{dI}{dt} \hat{\phi}$$

$$\boxed{\vec{E} = -\frac{\mu_0 n a^2}{2s} \frac{dI}{dt} \hat{\phi}} \quad (s > a)$$

P60 Problem 7.30

Calculate energy stored in toroidal coil of Ex. 7.12 by applying Eq. 7.44. Use the answer to check Eq. 7.37

$$W = \frac{1}{2\mu_0}\int B^2 d\tau \qquad\qquad L = \frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)$$



$$B = \frac{\mu_0 N I}{2\pi s} \quad \text{for} \quad 0 \le \phi \le 2\pi, \quad a \le s \le b, \quad 0 \le z \le h$$

$$W = \frac{1}{2\mu_0}\int_T B^2 d\tau = \frac{1}{2\mu_0}\left(\frac{\mu_0 N I}{2\pi}\right)^2 \int_0^h \int_0^{2\pi} \int_a^b \frac{1}{s^2}\cdot s\,ds\,d\phi\,dz$$

$$= \frac{\mu_0 N^2 I^2}{2(2\pi)^2}\cdot\left(\int_0^h dz\right)\left(\int_0^{2\pi} d\phi\right)\left(\int_a^b \frac{ds}{s}\right)$$

$$= \frac{\mu_0 N^2 h}{2\pi}\left(\ln(b) - \ln(a)\right)\frac{I^2}{2}$$

$$= \frac{1}{2}\left(\frac{\mu_0 N^2 h}{2\pi}\ln(b/a)\right) I^2 \underset{\uparrow}{=} \frac{1}{2}LI^2 \checkmark$$

(using Eq. 7.37)

P61 Problem 7.39

Given $\vec{E}(r,t) = \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{r^2}\,\theta(vt-r)\,\hat{r}$ and $\vec{B}(r,t) = 0$

Show $\vec{E}$ and $\vec{B}$ solve Maxwell's Eq$^{\underline{ns}}$. Determine ρ and $\vec{J}$ and describe the physical situation which gives rise to these fields

Notice $\nabla \cdot \vec{B} = 0$ holds for $\vec{B} = 0$. Furthermore, since $\vec{E} = E(r)\,\hat{r}$ we find $\nabla \times \vec{E} = 0 = -\dfrac{\partial \vec{B}}{\partial t}$ hence Faraday's Law holds. Gauss' and Ampere's Law remain,

$$\nabla \cdot \vec{E} = \frac{q}{4\pi\epsilon_0}\left(\nabla \cdot \frac{\hat{r}}{r^2}\right)\theta(vt-r) + \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r}\cdot\nabla\theta(vt-r)$$

$$= \frac{q}{4\pi\epsilon_0}\cdot 4\pi\delta^3(\vec{r})\,\theta(vt-r) + \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r}\cdot\hat{r}\,\frac{\partial}{\partial r}\left[\theta(vt-r)\right]$$

$$= \frac{q}{\epsilon_0}\delta^3(\vec{r})\,\theta(vt-r) + \frac{q}{4\pi\epsilon_0 r^2}\left(-\delta(vt-r)\right)$$

$$= \underbrace{\frac{q}{\epsilon_0}\delta^3(\vec{r})\,\theta(t)} - \frac{q}{4\pi\epsilon_0 r^2}\delta(vt-r)$$

notice $\delta^3(\vec{r}) = \begin{cases} \infty & \text{if } \vec{r} = 0 \\ 0 & \vec{r} \neq 0 \end{cases}$ then $v > 0$ is known so the simplification follows from DIRAC DELTA arithmetic.

Then since $\rho = \epsilon_0 \nabla \cdot \vec{E}$ we find

$$\boxed{\rho = q\,\delta^3(\vec{r})\,\theta(t) - \frac{q}{4\pi r^2}\delta(vt-r)}$$

Last, since $\nabla \times \vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\dfrac{\partial \vec{E}}{\partial t}$ we require $\vec{J} = -\epsilon_0\dfrac{\partial \vec{E}}{\partial t}$ since $\vec{B} = 0$ was given,

$$\vec{J} = -\epsilon_0\frac{q}{4\pi\epsilon_0 r^2}\frac{\partial}{\partial t}\left[\theta(vt-r)\right]\hat{r} \Rightarrow \boxed{\vec{J} = \frac{-q}{4\pi r^2}\,v\,\delta(vt-r)\,\hat{r}}$$

<u>Remark</u>: at $t=0$ a charge q appears at the origin and a charge $-q$ spreads out in spherical shell travelling at speed v for $r = vt$

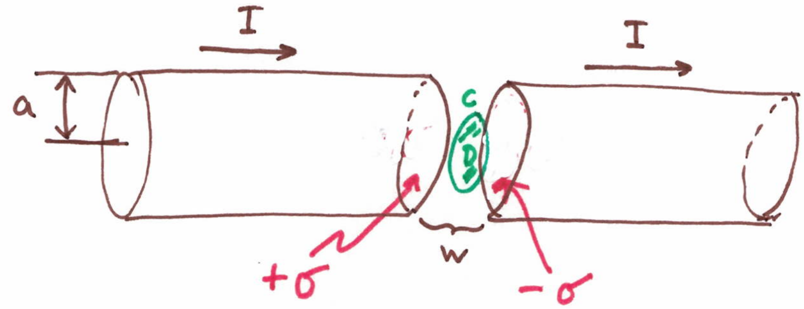
P62 Problem 8.3

Consider the charging capacitor of Problem 7.35

(a.) find $\vec{E}$ and $\vec{B}$ in the gap as functions of distance s from the axis at time t

(b.) find the energy density u_{em} and $\vec{S}$ in the gap. Note the direction of $\vec{S}$. Check Eq. 8.12 $\left(\frac{\partial u}{\partial t} = -\nabla \cdot \vec{S}\right)$

(c.) determine the total energy in the gap as fnct. of time. Calculate the total power flowing into the gap by integrating the Poynting vector over the appropriate surface. Check that the power input = rate of energy in gap increase ($W = 0$ since no charge in gap, assume fringing fields neglible)



(a.) $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z}$ where $\sigma = \frac{Q}{\pi a^2}$, assuming $Q = 0$ for $t = 0$

we find $Q = It$ thus $\boxed{\vec{E} = \left(\frac{It}{\pi a^2 \epsilon_0}\right)\hat{z}}$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \underbrace{\int_D (\nabla \times \vec{B}) \cdot d\vec{a}}_{\oint_C \vec{B} \cdot d\vec{l} = 2\pi s B} = \mu_0 \epsilon_0 \left(\frac{I}{\pi a^2 \epsilon_0}\right)\hat{z} \cdot \int_D d\vec{a}$$

$$= \frac{\mu_0 \epsilon_0 I}{\epsilon_0} \frac{\pi s^2}{\pi a^2}$$

$$\boxed{\vec{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}}$$

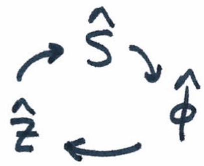
P 62 continued

(b.) $$u_{em} = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$$

$$= \frac{\epsilon_0}{2}\left(\frac{It}{\pi a^2 \epsilon_0}\right)^2 + \frac{1}{2\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2$$

$$= \underline{\frac{I^2}{2\pi^2 a^4}\left[\frac{t^2}{\epsilon_0} + \mu_0 s^2\right]}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$



$$= \frac{1}{\mu_0}\left(\frac{It}{\pi a^2 \epsilon_0}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right) \hat{z} \times \hat{\phi}$$

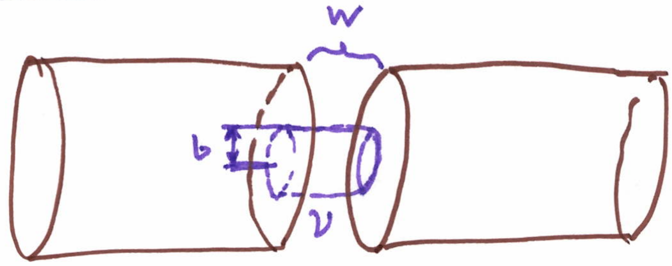
$$= \underline{\frac{-I^2 t s}{2\pi^2 a^4 \epsilon_0}} \hat{s}$$

$$\nabla \cdot \vec{S} = \frac{1}{s}\frac{\partial}{\partial s}\left[s\left(\frac{-I^2 t s}{2\pi^2 a^4 \epsilon_0}\right)\right] = \frac{-I^2 t}{2\pi^2 a^4 \epsilon_0}\frac{1}{s}(2s) = -\frac{I^2 t}{\pi^2 a^4 \epsilon_0}$$

$$\frac{\partial u}{\partial t} = \frac{I^2}{2\pi^2 a^4}\frac{\partial}{\partial t}\left[\frac{t^2}{\epsilon_0} + \mu_0 s^2\right] = \frac{I^2 t}{\pi^2 a^4 \epsilon_0} \quad \therefore \boxed{\frac{\partial u}{\partial t} = -\nabla \cdot \vec{S}} \checkmark$$

P62 continued

(c.)



$$\mathcal{V}: \quad 0 \le \phi \le 2\pi, \quad 0 \le s \le b << a, \quad 0 \le z \le w$$

$$U_{em} = \int_{\mathcal{V}} u_{em}\, d\tau$$

$$= \frac{I^2}{2\pi^2 a^4} \int_0^{2\pi} \int_0^{w} \int_0^{b} \left[\frac{t^2}{\epsilon_0} + \mu_0 s^2 \right] s\, ds\, dz\, d\phi$$

$$= \frac{I^2}{2\pi^2 a^4} (2\pi)(w) \left[\frac{t^2}{2\epsilon_0} s^2 + \frac{\mu_0}{4} s^4 \right] \Bigg|_0^b$$

$$= \underline{\frac{I^2 w}{\pi a^4} \left[\frac{t^2 b^2}{2\epsilon_0} + \frac{\mu_0 b^4}{4} \right]}.$$

$$P_{in} = -\int_{\partial \mathcal{V}} \vec{S} \cdot d\vec{a} = \int_0^{w} \frac{I^2 t b}{2\pi^2 a^4 \epsilon_0} \hat{s} \cdot (2\pi b\, dz\, \hat{s})$$

since $\vec{S} \cdot d\vec{a} = 0$ on caps just need consider $s = b$ part of $\partial \mathcal{V}$.

$$\Rightarrow \underline{P_{in} = \left(\frac{I^2 b^2 w}{\pi a^4 \epsilon_0} \right) t}$$

$$\underline{\frac{dU_{em}}{dt} = \left(\frac{I^2 w}{\pi a^4} \right) \left(\frac{2 t b^2}{2\epsilon_0} \right) = \left(\frac{I^2 b^2 w}{\pi a^4 \epsilon_0} \right) t = P_{in}}.$$

Remark: $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \Rightarrow c^2 = \frac{1}{\mu_0 \epsilon_0}$ can be used to reformulate my answers.

P63 Problem 8.9

Consider an infinite parallel plate capacitor, with lower plate at $z = -d/2$, carrying surface charge $-\sigma$ and the upper plate at $z = d/2$ carrying charge density σ

(a.) determine T_{ij} in region between plates

(b.) Use Eq$^{\underline{n}}$ 8.21 to find electromagnetic force per unit area on the top plate, compare Eq$^{\underline{n}}$ 2.51

(c.) What is electromagnetic momentum per unit area, per unit time crossing the xy-plane (or any other ∥ plane between plates)?

(d.) there must be mechanical forces holding plates apart, suppose we suddenly remove the insulator between the plates, the mom. flux (c.) is now absorbed by plates and they begin to move. Find momentum per unit time delivered to the top plate (that is the force acting on it) and compare answer to (b.) [this is two ways of looking at same force]

(a.) $T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \cancel{\frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)}^{\,0}$ $\quad \underline{\vec{B} = 0}.$

Now $\vec{E} = \frac{-\sigma}{\epsilon_0}\hat{z}$ for $-\frac{d}{2} < z < \frac{d}{2}$ thus $E_3 = \frac{-\sigma}{\epsilon_0}$

and $E = \frac{\sigma}{\epsilon_0}$ whereas $\vec{B} = 0$ and $E_2 = E_1 = 0$

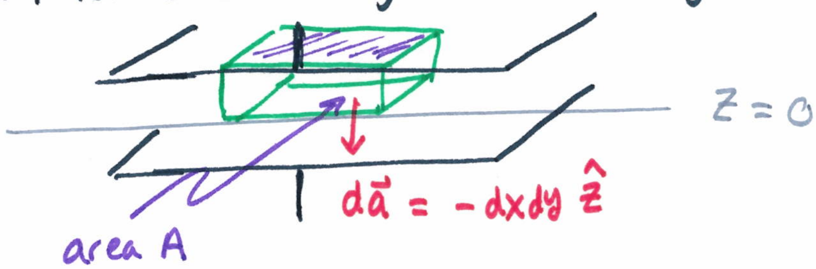
Consequently $T_{33} = \epsilon_0\left(E_3E_3 - \frac{1}{2}E^2\right) = \frac{\epsilon_0}{2}E_3^2 = \frac{\epsilon_0}{2}\left(\frac{\sigma}{\epsilon_0}\right)^2$

that is, $T_{33} = \frac{1}{2\epsilon_0}\sigma^2$ whereas $T_{11} = T_{22} = \epsilon_0\left(-\frac{1}{2}E^2\right) = \frac{-\sigma^2}{2\epsilon_0}$

$$\boxed{[T_{ij}] = \begin{bmatrix} -\sigma^2/2\epsilon_0 & 0 & 0 \\ 0 & -\sigma^2/2\epsilon_0 & 0 \\ 0 & 0 & \sigma^2/2\epsilon_0 \end{bmatrix}}$$

P63 continued

(b.) Eq. 8.21 for static configurations "$\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a}$"



$$F_z = \int T_{zz}\,da_z = \frac{-\sigma^2}{2\varepsilon_0} A \Rightarrow \boxed{\frac{\vec{F}}{A} = \frac{-\sigma^2}{2\varepsilon_0}\hat{z}}$$

(c.) "$-\overleftrightarrow{T} \cdot d\vec{a}$ is the electromagnetic momentum passing through area $d\vec{a}$"

$$-T_{zz} = \boxed{-\sigma^2/2\varepsilon_0}$$

momentum in z-direction crossing surface ⊥ to z-direction per unit area, per unit time

(d.) momentum per unit time, per unit area,

$$\boxed{\vec{f} = \frac{-\sigma^2}{2\varepsilon_0}\hat{z}}$$ (matches (b.))

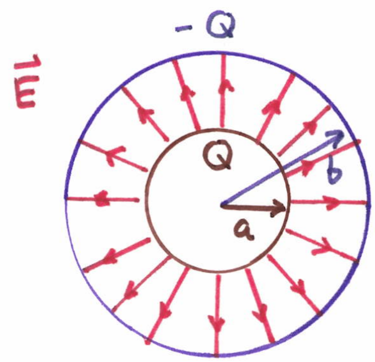
P64 Problem 8.11

Two concentric spherical shells carry uniformly distributed charges Q (at $r = a$) and $-Q$ (at $r = b > a$). They are immersed in a uniform magnetic field $\vec{B} = B_0 \hat{z}$

(a.) find the angular momentum of the fields with respect to the center

(b.) now $\vec{B}$ is gradually reduced to zero, find the torque on each sphere, and the resulting angular momentum of the system.

(a.)



$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \qquad \hat{r} = \langle \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \rangle$$

$$\vec{g} = \epsilon_0 \vec{E} \times \vec{B} = \frac{QB_0}{4\pi r^2} \hat{r} \times \hat{z}$$

$$\vec{l} = \vec{r} \times \vec{g} = \frac{QB_0}{4\pi r} \hat{r} \times (\hat{r} \times \hat{z}) \qquad \alpha = \frac{QB_0}{4\pi}$$

Notice $\hat{r} \times (\hat{r} \times \hat{z}) = (\hat{r} \cdot \hat{z})\hat{r} - (\hat{r} \cdot \hat{r})\hat{z}$

$$= \cos\theta \, \hat{r} - \hat{z}$$

$$= \langle \cos\phi \sin\theta \cos\theta, \sin\phi \sin\theta \cos\theta, \cos^2\theta - 1 \rangle$$

$$= \langle \cos\phi \sin\theta \cos\theta, \sin\phi \sin\theta \cos\theta, -\sin^2\theta \rangle$$

Thus,

$$\vec{L} = \int \vec{l} \, d\tau = \int_0^{2\pi} \int_0^{\pi} \int_a^b \frac{\alpha}{r} \Big(\sin\theta \cos\theta \underbrace{\langle \cos\phi, \sin\phi, 0 \rangle}_{\text{integrate to } 0} - \langle 0, 0, \sin^2\theta \rangle \Big) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\vec{L} = \Big\langle 0, 0, -\pi (b^2 - a^2) \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3} \Big\rangle \alpha$$

$$\vec{L} = \Big\langle 0, 0, \frac{-QB_0}{3} (b^2 - a^2) \Big\rangle$$

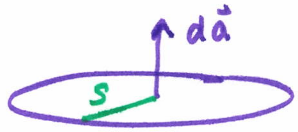
$$\boxed{\vec{L} = -\frac{1}{3} QB_0 (b^2 - a^2) \hat{z}}$$

P64 continued

(b.)

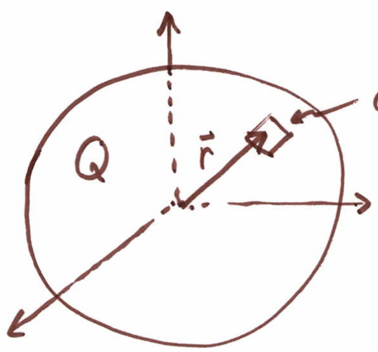
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial B}{\partial t}\hat{z}$$

$$\oint (\nabla \times \vec{E}) \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial B}{\partial t}\hat{z} \cdot d\vec{a}$$



$$E(2\pi s) = -\frac{\partial B}{\partial t}\pi s^2$$

$$\vec{E} = -\frac{1}{2}\left(\frac{\partial B}{\partial t}\right) s\hat{\phi}$$



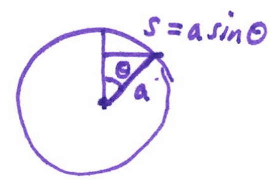
da has $dQ = \sigma\, da$, $da = a^2 \sin\theta\, d\theta\, d\phi$

$d\vec{F} = \vec{E}\sigma\, da$ gives force on da

$$d\vec{N} = \vec{r} \times d\vec{F}$$

$$d\vec{N} = (s\hat{s} + z\hat{z}) \times d\vec{F} = s\hat{s} \times d\vec{F} + \underbrace{z\hat{z} \times d\vec{F}}_{\text{integrates to zero}}$$

$$d\vec{N} = -s\, dF\, \hat{s} \times \hat{\phi} = -s\, dF\, \hat{z}$$



$$dN_z = -sE\sigma\, da = -s\left(\frac{1}{2}\frac{\partial B}{\partial t}s\right)\sigma\, da$$

$$dN_z = -(a\sin\theta)^2 \frac{1}{2}\frac{dB}{dt}\sigma \cdot a^2 \sin\theta\, d\phi$$

$$N_z = -\sigma\left(\frac{1}{2}\frac{dB}{dt}\right)a^4 \cdot \left(\int_0^{2\pi} d\phi\right)\left(\int_0^{\pi} \sin^3\theta\, d\theta\right)$$

$$N_z = -\left(\frac{Q}{4\pi a^2}\right)\left(\frac{1}{2}\frac{dB}{dt}\right) \cdot a^4 \cdot 2\pi \cdot \frac{4}{3} \Rightarrow \underline{\vec{N}_a = -\frac{Qa^2}{3}\frac{dB}{dt}\hat{z}}.$$

Likewise, $\underline{\vec{N}_b = \frac{Qb^2}{3}\frac{dB}{dt}\hat{z}}$. $\therefore$ $\underline{\vec{N} = \frac{Q}{3}\frac{dB}{dt}(b^2 - a^2)\hat{z}}$

total torque.

$$\vec{L} = \int \vec{N}\, dt = \frac{Q}{3}(b^2 - a^2)\hat{z}\int_0^{\infty} \frac{dB}{dt}\, dt = \boxed{-\frac{QB_0}{3}(b^2 - a^2)\hat{z}}$$

final mechanical ang. mom. matches initial electrom. ang. mom ✓

P65 **Problem 8.16**

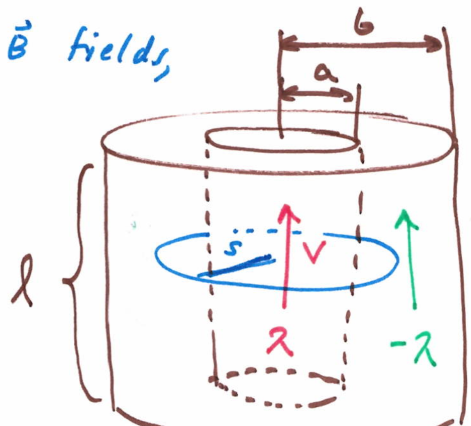
An infinitely long cylindrical tube of radius a moves at constant speed v along its axis. It carries a net charge per unit length λ, uniformly distributed over its surface. Surrounding it, at radius b, is another cylinder moving with same velocity but carrying $-\lambda$. Find:

(a.) the energy per unit length stored in the fields

(b.) the momentum per unit length in the fields

(c.) the energy per unit time transported by the fields across a plane $\perp$ to the cylinders.

(a.) Given $a < s < b$ we have nonzero $\vec{E}$ & $\vec{B}$ fields,



$$\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{s}\hat{s}$$

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{\lambda v}{s}\hat{\phi}$$

$(I_{enc} = \lambda v)$

Thus calculate

$$u = \frac{\varepsilon_0}{2}\left(\frac{\lambda}{2\pi\varepsilon_0 s}\right)^2 + \frac{1}{2\mu_0}\left(\frac{\mu_0\lambda v}{2\pi s}\right)^2$$

$$u = \frac{\lambda^2}{8\pi^2\varepsilon_0^2}\left[1 + \mu_0\varepsilon_0 v^2\right]\frac{1}{s^2} = \frac{\lambda^2}{8\pi^2\varepsilon_0^2}\left[1 + \frac{v^2}{c^2}\right]\frac{1}{s^2} \quad \left(c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right)$$

$$= \frac{\lambda^2}{8\pi^2\varepsilon_0^2}\left[1 + \frac{v^2}{c^2}\right]\int_0^l\int_0^{2\pi}\int_a^b \frac{s\,ds\,d\phi\,dz}{s^2}$$

$$W = \frac{\lambda^2}{8\pi^2\varepsilon_0^2}\left[1 + \frac{v^2}{c^2}\right] l(2\pi)\left[\ln(b) - \ln(a)\right]$$

$$\therefore \boxed{\frac{W}{l} = \frac{\lambda^2}{4\pi\varepsilon_0}\left(1 + \frac{v^2}{c^2}\right)\ln(b/a)}$$

P65 continued

(b.) Since $\vec{g} = \epsilon_0 \vec{E} \times \vec{B} = \epsilon_0 \left(\frac{\lambda}{2\pi \epsilon_0} \frac{\hat{s}}{s} \right) \times \left(\frac{\mu_0 \lambda v}{2\pi s} \hat{\phi} \right)$

we obtain $\vec{g} = \frac{\mu_0 \lambda^2 v}{4\pi^2 s^2} \hat{z}$ thus

$$\vec{P} = \int \vec{g}\, d\tau = \frac{\mu_0 \lambda^2 v}{4\pi^2} \int_0^l \int_0^{2\pi} \int_a^b \frac{s\, ds\, d\phi\, dz}{s^2} \hat{z}$$

$$\vec{P} = \left(\frac{\mu_0 \lambda^2 v}{4\pi^2} \right) (l) (2\pi) (\ln(b) - \ln(a)) \hat{z}$$

$$\boxed{\frac{\vec{P}}{l} = \frac{\mu_0 \lambda^2 v}{2\pi} \ln(b/a) \hat{z}}$$ ← momentum per length between cylinders

(c.) $\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{\epsilon_0 \mu_0} \vec{g} = c^2 \vec{g} = \frac{\mu_0 c^2 \lambda^2 v}{4\pi s^2} \hat{z}$

$$\frac{dW}{dt} = \int \vec{S} \cdot d\vec{a} = \frac{\mu_0 c^2 \lambda^2 v}{4\pi^2} \int_0^{2\pi} \int_a^b \frac{s\, ds\, d\phi}{s^2}$$

$$= \left(\frac{\mu_0 c^2 \lambda^2 v}{4\pi^2} \right) (2\pi) (\ln(b) - \ln(a))$$

$$= \boxed{\frac{\mu_0 c^2 \lambda^2 v}{2\pi} \ln(b/a)}$$