Please solve these on paper without fuzzies, one-side, show steps, you may use results which are shown in Lectures 36,37,38,39 and 40. In fact, I would strongly encourage you to not reinvent all the wheels I derived in those talks.

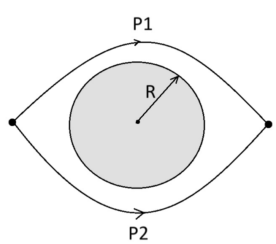
Problem 1 (20pts) A spherical rubber shell has polarization $\vec{P} = kr\,\hat{r}$ for $a \leq r \leq b$. There is no free charge in the system.

- **(a.)** Find the bound charge inside the rubber and find the bound surface charge on the inner and outside edge of the rubber shell,
- **(b.)** Calculate the potential inside the sphere given that $V(\infty) = 0$. Surely much partial credit can be earned by calculating $\vec{D}$ and hence $\vec{E}$ in each region.

Problem 2 (20pts) A charge Q is at the point $(0, L, 0)$ above the zx-plane where an hugely huge conductor extends to ludicrous distances. Find the charge density the conducting plane.

Problem 3 (30pts) In quantum mechanics the phase of a charge q moving along a path with vector potential $\vec{A}$ is given a phase shift $\varphi = \frac{q}{\hbar}\int_P \vec{A} \bullet d\vec{l}$.

- **(a.)** Consider a very long solenoid of radius R centered about the z-axis. A current I flows through the solenoid which has n turns per unit length. Find the vector potential for the magnetic field for $0 \leq s \leq R$.
- **(b.)** Find the vector potential outside the solenoid where $s > R$.
- **(c.)** Let P_1 and P_2 be two coterminal paths around the solenoid; that is suppose P_1, P_2 are paths for which $P_1 \cup (-P_2)$ then forms a loop enclosing around the solenoid. Let φ_j be the phase shift along P_j. Find explicitly how the difference in phase shift $\varphi_2 - \varphi_1$ depends on the current I.



Problem 4 (20pts) A very long circular cylinder of radius R carries a magnetization $\vec{M} = ks\widehat{\phi}$ where k is constant. Find the magnetic field for points inside and outside $s = R$ as follows:

- **(a.)** solve via the $\vec{H}$ arguments
- **(b.)** solve via analysis of Ampere's Law on the bound currents

Problem 5 (15pts) A square loop of side length a lies in the xy-plane in the first quadrant with one corner at $(0, 0)$. In this region $\vec{B} = 3ky^2t^2(\hat{y} + \hat{z})$ where k is a constant. Calculate the emf in the loop.

Problem 6 (15pts) Consider two inertial frames which are related by an x-velocity transformation. In particular, we suppose the primed frame is moving with velocity V in the x-direction. We have the following relations between frame (t, x, y, z) and (t', x', y', z'):

$$\begin{aligned} t' &= \gamma(t - Vx/c^2) \\ x' &= \gamma(x - Vt) \\ y' &= y, \\ z' &= z \end{aligned}$$

where $\gamma = \dfrac{1}{1 - V^2/c^2}$ and c is the speed of light. Notice these are linear relations between (t, x, y, z) and (t', x', y', z') hence we have the corresponding differential relations:

$$\begin{aligned} dt' &= \gamma(dt - V dx/c^2) \\ dx' &= \gamma(dx - V dt) \\ dy' &= dy, \\ dz' &= dz \end{aligned}$$

Let $\vec{u} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$ denote the velocity of a trajectory in the (t, x, y, z)-frame and let $\vec{u}' = \langle \frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'} \rangle$ denote the velocity of the same trajectory as measured in the (t', x', y', z') frame. Furthermore, let us denote $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{u}' = \langle u_1', u_2', u_3' \rangle$. Using formal calculus of differentials derive the following relations between the velocity measured in the unprimed and primed frame:

(a.) $u_1' = \dfrac{u_1 - V}{1 - Vu_1/c^2}$

(b.) $u_2' = \dfrac{u_2}{\gamma(1 - Vu_1/c^2)}$

(c.) $u_3' = \dfrac{u_3}{\gamma(1 - Vu_1/c^2)}$

Problem 7 (15pts) Another way to derive the velocity transformation rules is to use an appropriate 4-vector. We introduced the **proper velocity** as $U^\mu = \frac{dX^\mu}{d\tau}$ where τ is the proper time of the moving object. We saw the 4-velocity of an object w.r.t frame (ct, x, y, z) was given by:

$$(U^\mu) = (\gamma c, \gamma \vec{u}) = (U^0, U^1, U^2, U^3)$$

and $\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$ and $\vec{u} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$. If we consider the x-boosted frame (ct', x', y', z') as in the previous problem then we can transform the 4-velocity by:

$$U^{0'} = \gamma(V)(U^0 - VU^1/c) \qquad \& \qquad U^{1'} = \gamma(V)(U^1 - VU^0/c)$$

and $U^{2'} = U^2$ and $U^{3'} = U_3$ and yet

$$(U^{\mu'}) = (\gamma' c, \gamma' \vec{u}\,') = (\gamma' c, \gamma' u_1', \gamma' u_2', \gamma' u_3') = (U^{0'}, U^{1'}, U^{2'}, U^{3'})$$

where $\vec{u}' = \langle \frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'} \rangle$ and $u' = \|\vec{u}'\|$ and $\gamma' = \frac{1}{\sqrt{1 - u'^2/c^2}}$.

(a.) for $j = 1, 2, 3$, write a formula for u_j' in terms of $U^{j'}$ and $U^{0'}$ alone.

(b.) derive the velocity transformation rules using (a.).

Problem 8 (15pts) The speed of an object moving with x-velocity u_1' in a primed frame moving with velocity V in the x-direction is given by $u_1 = \dfrac{u_1' + V}{1 + Vu_1'/c^2}$ (this is the inverse of the relation in part (a.) of a previous problem). This is the colinear velocity addition rule in special relativity. As you can clearly see, *velocities do not add.* In contrast, if we define rapidity of the frame by

$$\tanh(\alpha) = V/c$$

and the rapidity of the particle in the primed frame by

$$\tanh(\beta') = u_1'/c$$

and the rapidity of the particle in the unprimed frame by

$$\tanh(\beta) = u_1/c$$

then show that $\beta = \alpha + \beta'$. That is, show that *rapidities add.* Let me give you a huge hint:

$$\tanh(a + b) = \frac{\tanh(a) + \tanh(b)}{1 + \tanh(a)\tanh(b)}.$$

Problem 9 (15pts) Show $\overline{U} \bullet \overline{A} = 0$ where $\overline{U}$ is the 4-velocity and $\overline{A}$ is the 4-acceleration (see Lecture 38).

Problem 10 (15pts) Show $\vec{\mathcal{A}} = \gamma^2\vec{a} + \gamma^4\left(\frac{\vec{u}\bullet\vec{a}}{c^2}\right)\vec{u}$ (see Lecture 38, page 3 where it is partly done).

Problem 11 (15pts) Consider two particles with 4-momenta P_1 and P_2 respectively. Suppose the particles collide **elastically**. Let P_1' and P_2' denote the 4-momenta of the particles after the collision. Conservation of momentum and energy are simultaneously combined in the conservation of 4-momentum. Assume 4-momentum of the system is conserved and prove $P_1 \bullet P_2 = P_1' \bullet P_2'$ where we intend the $\bullet$ to denote the Minkowski product:

$$V \bullet W = -V^1W^1 + V^1W^1 + V^2W^2 + V^3W^3$$

Problem 12 (15pts) A photon of energy E is asorbed by a stationary proton, resulting in a neutral pion and a recoiling proton. Show the threshold energy for a photon to cause this process is:

$$E_{min} = \frac{m(m + 2M)c^2}{2M}$$

where m is the pion mass and M is the proton mass. I'll walk you through how to show this result using a frames of reference technique:

(a.) In the Center of Momentum Frame (CM) we have total 3-momentum is zero by definition of the frame. The minimum energy occurs when the after collision particles have zero velocity which give γ-factors of 1 hence $E_{CM} = mc^2 + Mc^2 = (m + M)c^2$. Then the total 4-momentum in the CM-frame at the threshold energy is:

$$P_{CM} = \left\langle \frac{E_{CM}}{c}, \sum \vec{p} \right\rangle = \left\langle \frac{E_{CM}}{c}, 0 \right\rangle$$

Calculate $P_{CM} \bullet P_{CM}$.

(b.) In the Laboratory or target frame (LAB) the proton is at rest thus $P_{proton} = \langle Mc, 0 \rangle$ and the photon has $P_{photon} = (E/c, \vec{p})$ where $\|\vec{p}\| = E/c$. Let $P_{LAB} = P_{proton} + P_{photon}$ be the total 4-momentum in the LAB frame. Calculate P_{LAB} then compute $P_{LAB} \bullet P_{LAB}$.

(c.) Use $P_{CM} \bullet P_{CM} = P_{LAB} \bullet P_{LAB}$ to derived the threshold energy formula.

Problem 13 (10pts) Hidden momentum is a very interesting topic in relativistic electrodynamics which I failed to cover in lecture. Work through Example 12.13.

Problem 14 (15pts) Suppose $B_{\mu\nu}$ and $C^{\mu\nu}$ are tensors on spacetime. Show that $B_{\mu\nu}C^{\mu\nu}$ is an invariant. In particular, transform the expression to another frame of reference via a Lorentz transformation and show the value is unaltered. Recall,

$$B_{\mu'\nu'} = \Lambda^{\mu}_{\mu'}\Lambda^{\nu}_{\nu'}B_{\mu\nu} \qquad \& \qquad C^{\mu'\nu'} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}C^{\mu\nu}$$

and $\Lambda^{\sigma}_{\alpha'}\Lambda^{\alpha'}_{\beta} = \delta^{\sigma}_{\beta}$ and $\Lambda^{\beta}_{\alpha'}\Lambda^{\sigma'}_{\beta} = \delta^{\sigma'}_{\alpha'}$. Notice the placement of the indices defines the matrix. We distinguish between $\Lambda^{\beta}_{\alpha'}$ and $\Lambda^{\alpha'}_{\beta}$, these are inverse of one another.

Problem 15 (15pts) Lorentz transformations have matrices Λ which satisfy $\Lambda^T g \Lambda = g = diag(-1, 1, 1, 1)$. Show that:

(a.) $\Lambda = 1 \oplus R$ where $R^T R = I_3$ is a Lorentz transformation

(b.) Let $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. If $\Lambda = \begin{bmatrix} \cosh(\phi) & -\sinh(\phi) & 0 & 0 \\ -\sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ where $\tanh\phi = \beta$ then show Λ is a Lorentz transformation.

(c.) the Lorentz transformation of the previous part is an x-boost which corresponds to transforming to a moving frame with velocity v in the x-direction with respect to the initial frame of reference. Find the matrices for the y-boost and z-boost.

Problem 16 (15pts) Suppose $F^{\mu\nu}$ and $G^{\mu\nu}$ be the field tensors described in Lecture 39. Calculate:

$$F^{\mu\nu}F_{\mu\nu} \qquad \& \qquad F^{\mu\nu}G_{\mu\nu}$$

and express your answer nicely using electric and magnetic fields.

Problem 17 (10pts) Given $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, show $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$.

Problem 18 (10pts) We showed Gauss' Law and Ampere's Law with Maxwell's correction can be expressed as $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$. Show $\partial_\mu J^\mu = 0$ follows from the tensorial version of Maxwell's Equations. Explain the physical significance of this equation.

Problem 19 (15pts) Find how the electric and magnetic fields transform under a y-boost by velocity v.

Problem 20 (10pts) Suppose a long steady current flows a very long time hence Ampere's Law applies to give the usual purely magnetic field which encircles the current. Is it possible to find an inertial frame of reference in which the electromagnetic field is purely electric ?

Problem 21 (15pts) Suppose $\vec{E} = \langle E_0, 0, 0\rangle$ and $\vec{B} = \langle 0, 0, B_0\rangle$ where $E_0 < cB_0$. Find a new frame of reference where the electric field is zero if it is possible. If not explain why not.

Problem 22 (20pts) Consider a monochromatic plane wave which is traveling in the z-direction and polarized in the x-direction.

- **(a.)** Find the electric and magnetic fields of the wave in a frame which is z-boosted from the given frame,
- **(b.)** Find the electric and magnetic fields of the wave in a frame which is x-boosted from the given frame.

Problem 23 (15pts) Griffith's Problem 12.76 (generalize relativisitic electrodynamics to include magnetic charge).

Problem 24 (20pts) Suppose there exists a magnetic monopole at the origin. Explain why you cannot find a vector potential on $\mathbb{R}^3 - \{(0,0,0)\}$. What is the largest domain you can hope to find a vector potential in such a situation ? What makes electric charge different than magnetic charge ?

Problem 25 (20pts) Suppose the Lagrangian for electromagnetism given in Lecture is modified to include an extra term which couples A_μ to itself:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu + mA_\mu A^\mu$$

Find how Maxwell's equations are modified by the additional term proportional to m. (I have set $\mu_0 = 1$ in this, so don't sweat the units)

Problem 26 Explain the significance of 19 and 331.