POWER SERIES AND THE I.O.C. AND R.O.C.

A power series is a function defined by a rather special formula. In particular, $f(x)$ given below:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is a <u>power series</u> $\underbrace{\text{centered at}}$ a with coefficients $c_0, c_1, c_2, \ldots$

<u>Remark</u>: also use x_0 if $f(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots$

$\boxed{E1}$ $f(x) = 1 + x + x^2 + \cdots$ is power series centered at $a = 0$.

Notice $1 + x + x^2 + \cdots = \dfrac{1}{1-x}$ provided $|x| < 1$ by the geometric series result. In this case, we find <u>$\text{dom}(f(x)) = (-1, 1)$</u>. In fact, $f(x) = 1 + x + \cdots = \sum_{n=0}^{\infty} x^n$ is a

$\boxed{\text{Def}^{\underline{n}} \text{ a power series centered at zero is a } \underline{\text{MacClaurin}}\ \underline{\text{Series}}.}$

$\boxed{E2}$ If we begin with $g(x) = \dfrac{1}{1-x}$ and notice that,

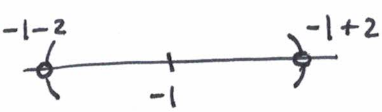
$$g(x) = \frac{1}{2-(x+1)} = \frac{1}{2[1 - \frac{1}{2}(x+1)]}$$ then we can

use geom. series result with $a = 1/2$ and $r = \frac{1}{2}(x+1)$ to expand $g(x)$ as a power series centered at $x_0 = -1$

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}(x+1)\right)^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(x-(-1))^n$$

only for $|r| < 1$

$$\left|\tfrac{1}{2}(x+1)\right| < 1 \implies |x+1| < 2$$

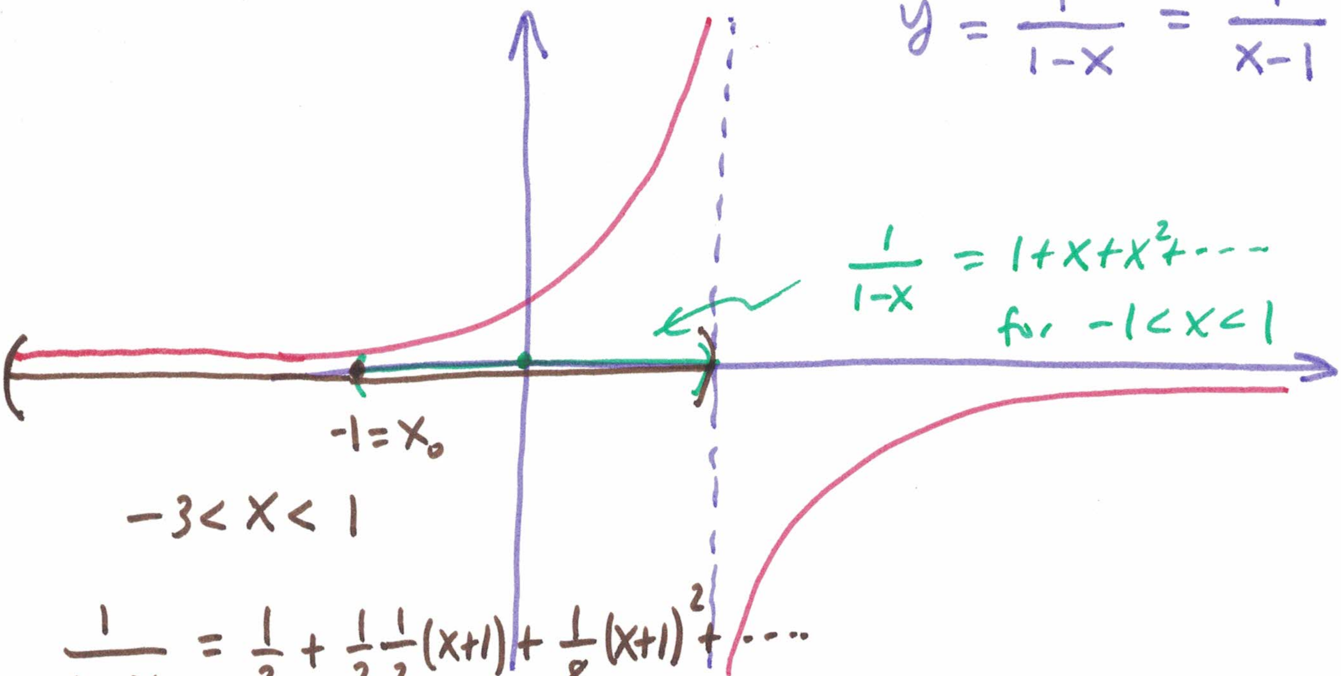


Thus define a new function (this $h(x)$ is a power series)

$$\boxed{h(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(x+1)^n \quad \text{with } \text{dom}(h(x)) = (-3, 1)}$$

centered at $x_0 = -1$, Coefficient $c_n = \dfrac{1}{2^{n+1}}$

$$y = \frac{1}{1-x} = \frac{-1}{x-1}$$



$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

for $-1 < x < 1$

$-3 < x < 1$

$$\frac{1}{1-x} = \frac{1}{2} + \frac{1}{2}\frac{1}{2}(x+1) + \frac{1}{8}(x+1)^2 + \cdots$$

Remark: Both $\boxed{E1}$ and $\boxed{E2}$ have domains given by an open interval of $\mathbb{R}$. An <u>interval</u> is a <u>connected</u> subset of $\mathbb{R}$ for this course. There are the following cases:

1.) $\mathbb{R} = (-\infty, \infty)$ $\quad (R = \infty)$

2,3.) $(-\infty, 0]$, $(-\infty, 0)$

4,5) $[0, \infty)$, $(0, \infty)$

THESE INTERVALS 2–5 ARE NOT POSSIBLE DOMAINS FOR POWER SERIES.

6.) $[a, b] = [x_0 - R, x_0 + R]$

7.) $[a, b) = [x_0 - R, x_0 + R)$

8.) $(a, b] = (x_0 - R, x_0 + R]$

9.) $(a, b) = (x_0 - R, x_0 + R)$

10.) $\{x_0\}$ $\quad (R = 0)$

In summary, cases 1, 6, 7, 8, 9, 10 are the only possible domains for a power series. These form the possible Interval Of Convergence (I.O.C.). Each of these has a characteristic Radius Of Convergence (R.O.C.)

1.) $(-\infty, \infty)$ has $R = \infty$

6, 7, 8, 9.) have $R = \# < \infty$ (finite radius case)

The "open I.O.C." is $(x_0 - R, x_0 + R)$ for all of these cases. Sometimes your instructor might ask for the "open IOC"

10.) $\{x_0\}$ means the power series $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ converges <u>only</u> for $x = x_0$.

Thm/ $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ has domain as described above, either $R = 0$, $R = \infty$ or R is a finite value

FINDING THE INTERVAL OF CONVERGENCE

Step 1: Use the Ratio Test to find the interval for which the series converges (absolutely), usually $a - R < x < a + R$

Step 2: If $(a-R, a+R)$ is a finite interval then check the endpoints for convergence/divergence. This usually will entail a n^{th} term test or an alternating series test.

Step 3: Using the Th^{m} (on previous page) we're done. (If Step 2 has both endpt.'s divergent then I.O.C is $(a-R, a+R)$)

<u>Remark</u>: this not a comprehensive advice, sometimes it'll be easier & sometimes you'll need to think or do some algebra before applying the steps.

<u>Example 18.1.3b</u>

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E3 Find the I.O.C for $\sum_{n=0}^{\infty} \frac{\sqrt{n}\, x^n}{3^n} = \sum_{n=0}^{\infty} a_n = S(x)$. STEP 1: RATIO TEST

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\, x^{n+1}}{3^{n+1}}\frac{3^n}{\sqrt{n}\, x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{3\sqrt{n}}x\right|$$

$$= \frac{1}{3}|x| \underbrace{\lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{\sqrt{n}}\right|}_{\to 1} \quad \text{since, } \frac{\sqrt{n+1}}{\sqrt{n}} = \frac{n+1}{\sqrt{n^2+n}} = \frac{1+1/n}{\sqrt{1+1/n}} \longrightarrow 1 \text{ as } n\to\infty.$$

Thus $L = \frac{1}{3}|x|$ so $L < 1$ if $\frac{1}{3}|x| < 1$ aka $|x| < 3$ and $L > 1$ if $|x| > 3$. Thus $S(x)$ converges on $(-3, 3)$.

STEP 2: Endpoints, test for conv./div at $x = \pm 3$

Consider thus

$$S(3) = \sum_{n=0}^{\infty}\frac{\sqrt{n}}{3^n}3^n = \sum_{n=0}^{\infty}\sqrt{n}$$: diverges since $\sqrt{n} \to \infty$ as $n \to \infty$ (using n^{th} term test)

$$S(-3) = \sum_{n=0}^{\infty}\frac{\sqrt{n}}{3^n}(-3)^n = \sum_{n=0}^{\infty}(-1)^n\sqrt{n}$$: diverges by n^{th} term test again $(-1)^n\sqrt{n} \not\to 0$ as $n\to\infty$.

$$\sum_{n=0}^{\infty} 2^n (x+1)^n$$

Example 18.1.4(sometimes we get the endpoints for free)

(a.) $\sum 2^n(x+1)^n$ this is a geometric series with $a = 1$ & $r = 2(x+1)$

it converges iff $|r| < 1 \implies |2(x+1)| < 1$

$$|x+1| < \frac{1}{2}$$

Thus the $R = \frac{1}{2}$ and $\frac{-3}{2} < x < \frac{-1}{2}$ is the I.O.C. $= \left(\frac{-3}{2}, \frac{-1}{2}\right)$

Example 18.1.5(this was a test question from a previous course)

Find the IOC and ROC for the power series defined below.

PROBLEM FIVE Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{3}\right)^{n-1}$.

$$L = \lim_{n\to\infty} \left| \frac{(x/3)^n}{n+1} \frac{n}{(x/3)^{n-1}} \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{x}{3}\right)\left(\frac{n}{n+1}\right) \right|$$

$$= \left|\frac{x}{3}\right| \underbrace{\lim_{n\to\infty}\left(\frac{n}{n+1}\right)}_{\to 1}$$

$$= \left|\frac{x}{3}\right| < 1 \implies -1 < \frac{x}{3} < 1 \implies -3 < x < 3$$

will converge

$x = \pm 3$ don't know yet.

$f(3) = \sum_{n=1}^{\infty} \frac{1}{n}$, p-series $p=1 \implies$ diverges.

$f(-3) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$, $b_n = \frac{1}{n} > 0$, $b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$

and $\frac{1}{n} \to 0$ as $n \to \infty$ $\therefore$ converges by A.S.T.

$$\boxed{\text{So we find I.O.C} = [-3, 3) \text{ and } R = 3}$$

HOMEWORK : §12.8 # 4, 6, 12, 25, 30, 32, 38, 41 : CALC. II : STEWART 6th Ed.

§12.8 #4 Find the I.O.C & R.O.C. for $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

Apply ratio test,

$$L = \lim_{n\to\infty} \left| \left(\frac{(-1)^{n+1} x^{n+1}}{n+2}\right)\left(\frac{n+1}{(-1)^n x^n}\right)\right| = \lim_{n\to\infty} \left| x \left(\frac{n+1}{n+2}\right)\right| = |x| \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) = |x|$$

Thus $L < 1$ iff $|x| < 1$ $\therefore$ $(-1,1) \subset$ I.O.C. Now check endpts.

$x = 1$) $f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n 1^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges by A.S.T.

$x = -1$) $f(-1) = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges by $p=1$ series test (harmonic series)

Therefore, $\boxed{\text{I.O.C.} = (-1, 1] \;\; \& \;\; \text{R.O.C.} = 1}$

§12.8 #6 Find I.O.C & R.O.C. for $\sum_{n=1}^{\infty} \sqrt{n}\, x^n$

Apply ratio test,

$$L = \lim_{n\to\infty} \left| \frac{\sqrt{n+1}\, x^{n+1}}{\sqrt{n}\, x^n}\right| = \lim_{n\to\infty} \left[\left(\sqrt{\frac{n+1}{n}}\right)|x|\right] = |x| \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = |x|.$$

Hence $L < 1$ iff $|x| < 1$ $\therefore$ $(-1,1) \subset$ I.O.C. & R.O.C. = 1.

Check end pts.

$x = 1$) $\sum_{n=1}^{\infty} \sqrt{n}$ diverges by n^{th} term test $\lim_{n\to\infty} \sqrt{n} \neq 0$

$x = -1$) $\sum_{n=1}^{\infty} \sqrt{n}(-1)^n$ diverges by n^{th} term test $\lim_{n\to\infty} (\sqrt{n})(-1)^n \neq 0$

Hence, $\boxed{\text{I.O.C.} = (-1, 1) \;\; \& \;\; \text{R.O.C.} = 1}$

§12.8 #12 $\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$ find the I.O.C. & R.O.C.

As usual, $L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{5^{n+1}(n+1)^5} \frac{5^n n^5}{x^n}\right| = \lim_{n\to\infty} \left[|x| \cdot \frac{n^5}{(n+1)^5} \cdot \frac{1}{5}\right] = \frac{|x|}{5}$.

Thus $L < 1$ iff $|x|/5 < 1 \Leftrightarrow |x| < 5$

Hence $(-5, 5) \subset$ I.O.C. & R.O.C. = 5. Check Endpts. $x = \pm 5$.

$x = 5$) $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges by $p=5$ test. $x = -5$) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ converges by A.S.T. test

Therefore, $\boxed{\text{I.O.C.} = [-5, 5] \;\; \& \;\; \text{R.O.C.} = 5}$

$\boxed{\S 12.8 \# 25}$ Find I.O.C & R.O.C. for $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{4^n}{n^2}\left(x+\frac{1}{4}\right)^n$

$x_0 = \frac{-1}{4}$

Notice $\displaystyle L = \lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}}{(n+1)^2} \frac{n^2}{(4x+1)^n} \right|$

$\displaystyle = \lim_{n\to\infty} \left[|4x+1| \left(\frac{n}{n+1}\right)^2 \right]$

$|4x+1| = |4(x+\frac{1}{4})| = 4|x+\frac{1}{4}| < 1$

$\underline{|x+\frac{1}{4}| < \frac{1}{4}}$

$= |4x+1| < 1 \Rightarrow -1 < 4x+1 < 1$

$\Rightarrow -2 < 4x < 0$

$\Rightarrow -\frac{1}{2} < x < 0 \quad \therefore \underline{(-1/2, 0) \subset \text{I.O.C.}}$

Check Endpts,

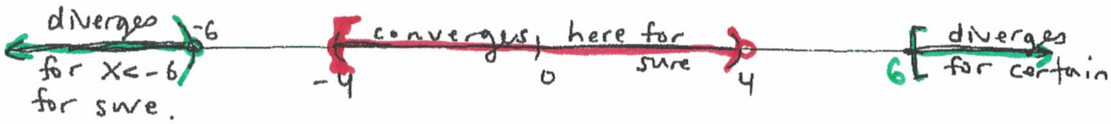
$\underline{x=0)}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by $p=2$

$\underline{x=\frac{-1}{2})}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ conv. by A.S.T.

Thus $\boxed{\begin{array}{l}\text{I.O.C.} = [-1/2, 0] \\ \text{R.O.C.} = 1/4\end{array}}$

$\boxed{\S 12.8 \# 30}$ Suppose $\displaystyle\underbrace{\sum_{n=0}^{\infty} C_n x^n}_{f(x)}$ converges when $x=-4$ and diverges for $x=6$. What can we say about (a.) $\displaystyle\sum_{n=0}^{\infty} C_n$, (b.) $\displaystyle\sum_{n=0}^{\infty} C_n 8^n$, (c.) $\displaystyle\sum_{n=0}^{\infty} C_n (-3)^n$ (d.) $\displaystyle\sum_{n=0}^{\infty} (-1)^n C_n 9^n$

To answer these we note that $\displaystyle\sum_{n=0}^{\infty} C_n x^n$ is centered at $a=0$ thus convergence at $x=-4 \Rightarrow$ I.O.C. $\underline{\text{includes}}$ $(-4, 4)$



Between $-6 \le x \le -4$ and $4 \le x < 6$ we're ignorant. Hence,

(a.) $\displaystyle\sum_{n=0}^{\infty} C_n$ converges.

(b.) $\displaystyle\sum_{n=0}^{\infty} C_n 8^n$ diverges

(c.) $\displaystyle\sum_{n=0}^{\infty} C_n (-3)^n$ converges

(d.) $\displaystyle\sum_{n=0}^{\infty} C_n (-9)^n$ diverges.

All of these based on the I.O.C. $\text{Th}^{\underline{m}}$ which says the I.O.C. is either $\mathbb{R}$, a (half)(open)(closed) interval or just the center.

§12.8 #32 Let $p, q \in \mathbb{R}$ such that $p < q$. Find power series whose I.O.C. is
(a.) (p, q), (b.) $(p, q]$, (c.) $[p, q)$ (d.) $[p, q]$

To begin notice we want our center at the <u>midpoint</u> thus we want $a = \frac{1}{2}(p+q)$. Thus

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n \left(x - \frac{p+q}{2}\right)^n}$$

Then use, for example,

(a.) $c_n = 1$ (endpts fail due to n^{th} term test)

(b.) $c_n = \frac{(-1)^n}{n}$ (left endpt harmonic, right A.S.T.)

(c.) $c_n = \frac{(-1)^{n+1}}{n}$ (left endpt. alt. harmonic, right harmonic)

(d.) $c_n = \frac{1}{n^2}$ (A.S.T. on left endpt, $p=2$ on right end pt.)

§12.8 #38 If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_{n+4} = c_n$ for all $n \geq 0$ then find the I.O.C. and find a nice f-la for $f(x)$

Notice $c_0 = c_4 = c_8 = \cdots = c_{4n}$, $c_1 = c_5 = \cdots = c_{4n+1}$, $c_2 = c_6 = \cdots = c_{4n+2}$ and $c_3 = c_7 = c_{11} = \cdots = c_{4n+3}$ thus the series breaks into 4 parts,

$$f(x) = \sum_{n=0}^{\infty} c_{4n} x^{4n} + \sum_{n=0}^{\infty} c_{4n+1} x^{4n+1} + \sum_{n=0}^{\infty} c_{4n+2} x^{4n+2} + \sum_{n=0}^{\infty} c_{4n+3} x^{4n+3} \quad \bigstar$$

$$= c_0 \sum_{n=0}^{\infty} x^{4n} + x c_1 \sum_{n=0}^{\infty} x^{4n} + x^2 c_2 \sum_{n=0}^{\infty} x^{4n} + x^3 c_3 \sum_{n=0}^{\infty} x^{4n}$$

$$= \underbrace{\boxed{\frac{c_0 + c_1 x + c_2 x^2 + c_3 x^3}{1 - x^4}}}_{\text{"nice" formula.}}$$

identifying $a = 1$ and $r = x^4$ for each summation above

$\therefore$ $|x^4| < 1 \Rightarrow \boxed{(-1, 1) = \text{I.O.C}}$

Question: why is it ok to rearrange the power series as I did in step $\bigstar$?

§12.8 #41 Suppose $\sum c_n x^n$ has R.O.C $= 2$ & $\sum d_n x^n$ has R.O.C. $= 3$ what is the R.O.C. of $\sum (c_n + d_n) x^n$?

$$\text{dom}(f+g) = \text{dom}(f) \cap \text{dom}(g)$$

$$\Rightarrow (-2, 2) \cap (-3, 3) = (-2, 2) \cdots \boxed{\text{R.O.C.} = 2}$$