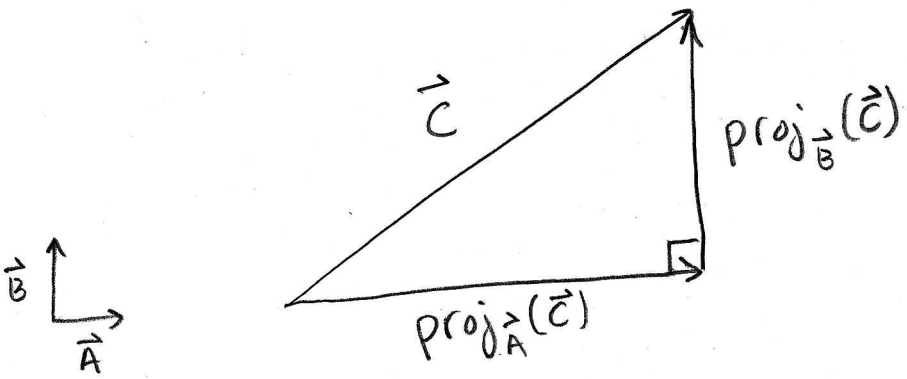
Math 231 Homework Project I: Vectors:

Follow instructions. Be careful to answer all the questions raised in each part. Please turn in neat work with problems clearly labeled and your name on each page. Thanks. Some of these problems are hard. You are not alone. If you get stuck, phone a friend. Or email me, or stop by office hours etc... Start early though, otherwise we may not be able to resolve your questions in time. Also, come to class. I may inadvertently work part of the homework. I plan for there to be 30 total problems. This assignment has 11 problems, this means it is worth 11pts of your final grade.

PROBLEM 1: Suppose that $\vec{A}, \vec{B}$ are non-zero two-dimensional vectors which are orthogonal. Show that if $\vec{C}$ is any two-dimensional vector then it can be written as a linear combination of the given vectors; that is show there exist $s, t \in \mathbb{R}$ such that $\vec{C} = s\vec{A} + t\vec{B}$. *Hint: if $\vec{u}$ is a unit vector then $\vec{u} \cdot \vec{C}$ gives the component of $\vec{C}$ in the $\vec{u}$ direction.*



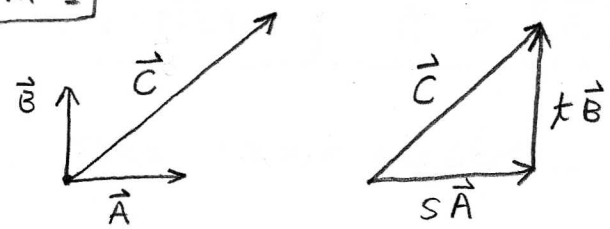
$$\vec{C} = \text{proj}_A(\vec{C}) + \text{proj}_{\vec{B}}(\vec{C})$$
$$= (\vec{C}\cdot\hat{A})\hat{A} + (\vec{C}\cdot\hat{B})\hat{B}$$
$$= \left(\frac{\vec{C}\cdot\vec{A}}{A^2}\right)\vec{A} + \left(\frac{\vec{C}\cdot\vec{B}}{B^2}\right)\vec{B} = s\vec{A} + t\vec{B}$$

$$\therefore \boxed{s = \frac{\vec{C}\cdot\vec{A}}{A^2} \quad \text{and} \quad t = \frac{\vec{C}\cdot\vec{B}}{B^2}}$$

comparing the LHS and RHS. Notice these must be seperately equal since $\vec{A}$ and $\vec{B}$ have no common vector component.

HELP FOR PROBLEMS 1 & 2 OF MATH 231 HOMEWORK PROJECT 1

PROBLEM 1



$s = ?$
$t = ?$

can use projections to find formulas for s & t.

PROBLEM 2 Projections not helpful since $\vec{A}$ and $\vec{B}$ are not "completely independent", I mean part of $\vec{A}$ can point in the $\vec{B}$-direction. The clean (non geometric) sol$^{\underline{n}}$ is best done with a little basic linear algebra. I'll supply those details here. Ultimately our goal is to find explicit formulas for s and t in terms of c_1, c_2, a, b, c, d where

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \vec{a} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} c \\ d \end{bmatrix}.$$

We want s, t such that

$$\vec{C} = s\vec{A} + t\vec{B}$$

we were given that $\vec{A} \nparallel \vec{B}$. This means there does not exist a k such that $\vec{A} = k\vec{B}$.

Lemma I. $\vec{A} \nparallel \vec{B} \Rightarrow \vec{A}$ and $\vec{B}$ are linearly independent

Proof: "linear independence" means $c_1\vec{A} + c_2\vec{B} = 0 \Rightarrow c_1 = c_2 = 0$. Suppose $\vec{A} \nparallel \vec{B}$ and $\vec{A}$ and $\vec{B}$ are not linearly independent then $c_1\vec{A} + c_2\vec{B} = 0$ and $c_1 \neq 0$ or $c_2 \neq 0$. Suppose $c_1 \neq 0$ then $\vec{A} + \frac{c_2}{c_1}\vec{B} = 0$ and $\vec{A} = \frac{-c_2}{c_1}\vec{B}$ so there exists $k = -c_2/c_1$ but this is a contradiction. Hence, by proof by contradiction the lemma holds true.

Lemma II: If $\vec{A}, \vec{B}$ are linearly independent then the matrix M made by concatenating the vectors is invertible; $M = [\vec{A} | \vec{B}]$ has $\det(M) \neq 0$.

Proof: take linear algebra.

The inverse matrix for $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is given by the formula $\boxed{M^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}}$

The inverse matrix M^{-1} has $M^{-1}M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{"}1\text{"}$.

- Lets return to the initial problem and see what linear algebra does for us,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = s\begin{bmatrix} a \\ b \end{bmatrix} + t\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} as + ct \\ bs + dt \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}$$

$$\vec{C} = M\begin{bmatrix} s \\ t \end{bmatrix} \Longrightarrow M^{-1}\vec{C} = M^{-1}M\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

Multiply the matrix and column vector and that gives us the values of s and t we were looking for. You are free to start with

$$\boxed{s = \frac{c_1 d - c_2 c}{ad - bc} \quad \Big| \quad t = \frac{-c_1 b + c_2 a}{ad - bc}}$$

plug them into $s\vec{A} + t\vec{B}$ and show you get $\vec{C} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

PROBLEM 2: Suppose that $\vec{A}, \vec{B}$ are non-zero two-dimensional vectors which are not parallel. Show that if $\vec{C}$ is any two-dimensional vector then it can be written as a linear combination of the given vectors; that is show there exist $s, t \in \mathbb{R}$ such that $\vec{C} = s\vec{A} + t\vec{B}$.

I showed $s = \dfrac{c_1 d - c_2 c}{ad - bc}$ and $t = \dfrac{-c_1 b + c_2 a}{ad - bc}$

should work given $\vec{A} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{B} = \begin{bmatrix} c \\ d \end{bmatrix}$ and $\vec{C} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

I asked for you to show these work.

$$s\vec{A} + t\vec{B} = \left(\frac{c_1 d - c_2 c}{ad - bc}\right)\begin{bmatrix} a \\ b \end{bmatrix} + \left(\frac{-c_1 b + c_2 a}{ad - bc}\right)\begin{bmatrix} c \\ d \end{bmatrix}$$

$$= \frac{1}{ad - bc}\left(\begin{bmatrix} c_1 da - c_2 ca \\ c_1 db - c_2 cb \end{bmatrix} + \begin{bmatrix} -c_1 bc + c_2 ac \\ -c_1 bd + c_2 ad \end{bmatrix}\right)$$

$$= \frac{1}{ad - bc}\begin{bmatrix} c_1 da - \cancel{c_2 ca} - c_1 bc + \cancel{c_2 ac} \\ \cancel{c_1 db} - c_2 cb - \cancel{c_1 bd} + c_2 ad \end{bmatrix}$$

$$= \frac{1}{ad - bc}\begin{bmatrix} (ad - bc)\, c_1 \\ (ad - bc)\, c_2 \end{bmatrix}$$

In the usual notation, $\alpha\langle v, w\rangle = \langle \alpha v, \alpha w\rangle$

$$= \frac{ad - bc}{ad - bc}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \vec{C} \checkmark$$

So my choice of s and t works.

PROBLEM 3: Inequalities of interest.

a. (Cauchy-Schwarz Inequality) Argue that $\vec{A} \cdot \vec{B} = AB\cos(\theta)$ implies $|\vec{A} \cdot \vec{B}| \leq |\vec{A}||\vec{B}|$.

b. (Triangle Inequality) Show that $|\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}|$. Hint: to accomplish this consider that $|\vec{A} + \vec{B}|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B})$ and use part a.

a.) $|\vec{A} \cdot \vec{B}| = |AB\cos\theta|$ $\quad \vec{A} = A\hat{A}$ where $|\vec{A}| = A$.

$= |A||B||\cos\theta|$

$\leq |A||B|$ this proves a.

b.) $|(\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B})| = |\vec{A} \cdot \vec{A} + 2\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{B}|$

$= |A^2 + 2\vec{A} \cdot \vec{B} + B^2|$

$\leq |A^2| + |2\vec{A} \cdot \vec{B}| + |B^2|$: using ordinary triangle ineq. for real #'s

$\leq A^2 + 2AB + B^2$: by a.)

$= (A+B)^2$

Notice $|\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}| = A + B$ is same as

$|\vec{A} + \vec{B}|^2 \leq (A+B)^2$

And $|\vec{A} + \vec{B}|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B})$ thus,

$$\boxed{|\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}|}$$

PROBLEM 4: Prove the following assertions:

a. $(\vec{v} \times \vec{w}) \cdot \vec{w} = 0.$

$$\langle V_2W_3 - V_3W_2,\ V_3W_1 - V_1W_3,\ V_1W_2 - V_2W_1 \rangle \cdot \langle W_1, W_2, W_3 \rangle$$

$$= W_1(\cancel{V_2W_3} - \cancel{V_3W_2}) + W_2(\cancel{V_3W_1} - \cancel{V_1W_3}) + W_3(\cancel{V_1W_2} - \cancel{V_2W_1})$$

1 2 2 3 3 1

$$= 0.$$

b. $(\vec{v} - \vec{w}) \times (\vec{v} + \vec{w}) = 2(\vec{v} \times \vec{w}).$

$$(\vec{V} - \vec{W}) \times (\vec{V} + \vec{W}) = \vec{V} \times (\vec{V} + \vec{W}) - \vec{W} \times (\vec{V} + \vec{W})$$

$$= \cancel{\vec{V} \times \vec{V}}^{\,0} + \vec{V} \times \vec{W} - \vec{W} \times \vec{V} - \cancel{\vec{W} \times \vec{W}}^{\,0}$$

$$= \vec{V} \times \vec{W} - (-\vec{V} \times \vec{W})$$

$$= 2\vec{V} \times \vec{W}$$

c. $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$.

See homework sol^{n} page (H11).

d. $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$.

See homework sol^{n} page (H11)

PROBLEM 5: Cancellation Properties: for ordinary real (or complex) numbers if $a \neq 0$ then $ab = ac$ implies that $b = c$. This problem investigates if that still is the case for the vectors' dot and cross products. Suppose that $\vec{A} \neq \vec{0}$ for the following questions.

a. If $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$ does it follow that $\vec{B} = \vec{C}$? If not give an argument why not. If true explain why it is true in general.

No consider $\vec{A} = \hat{i}$ then $\hat{i} \cdot \hat{j} = 0 = \hat{i} \cdot \hat{k}$ yet $\hat{j} \neq \hat{k}$ so this is a counter-example.

b. If $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$ does it follow that $\vec{B} = \vec{C}$? If not give an argument why not. If true explain why it is true in general.

No consider any $\vec{A}$ and take $\vec{B} = \vec{A}$ and $\vec{C} = -\vec{A}$

$\vec{A} \times \vec{A} = 0$ and $\vec{A} \times (-\vec{A}) = 0$ yet $\vec{A} \neq -\vec{A}$.

c. If $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$ and $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$ does it follow that $\vec{B} = \vec{C}$? If not give an argument why not. If true explain why it is true in general.

<u>Proof by contradiction.</u>

Suppose that $\vec{A} \neq 0$ and $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$ <u>AND</u> $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$ <u>BUT</u> $\vec{B} \neq \vec{C}$. Notice that $\vec{B} - \vec{C} \neq 0$ in this case.

Also notice,

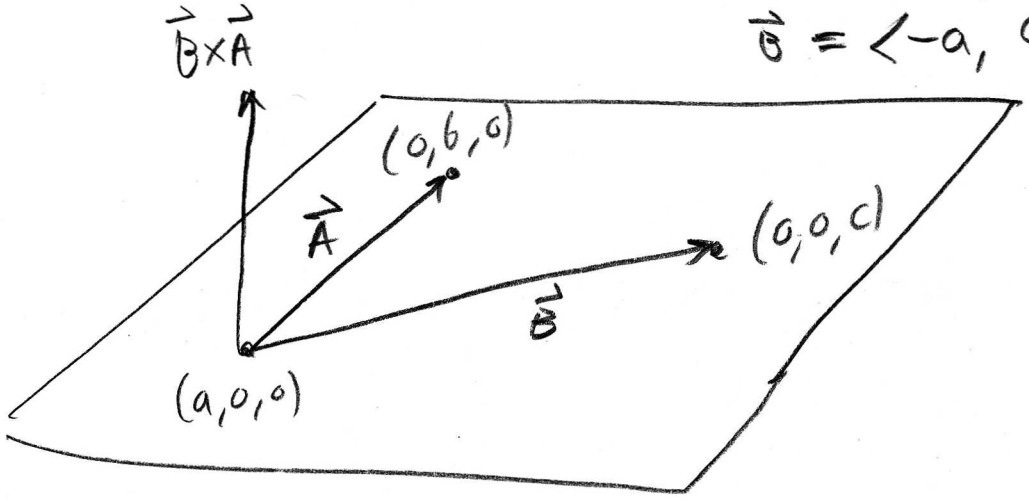
$$\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C} \iff \vec{A} \cdot (\vec{B} - \vec{C}) = 0$$
$$\vec{A} \times \vec{B} = \vec{A} \times \vec{C} \iff \vec{A} \times (\vec{B} - \vec{C}) = 0$$

Thus the vector $\vec{B} - \vec{C}$ is both parallel and perpendicular to $\vec{A} \neq \vec{0}$. It follows that $\vec{B} - \vec{C} = 0$ thus $\vec{B} = \vec{C}$ which is a contradiction. Therefore the proposition is true by proof via contradiction.

<u>Remark:</u> I will award bonus points for a direct proof. It escapes me at the moment.

PROBLEM 6: Find the equation of the plane that contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

$$\vec{A} = \langle -a, b, 0 \rangle$$
$$\vec{B} = \langle -a, 0, c \rangle$$



$$\vec{B} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$

$$= \hat{i}(bc) - \hat{j}(-ac) + \hat{k}(ab)$$

$$= \langle bc, ac, ab \rangle$$

Choose, $\vec{r}_0 = (a, 0, 0)$

$$\boxed{\begin{aligned} bc(x-a) + acy + abz &= 0 \\ bcx + acy + abz &= abc \end{aligned}}$$

PROBLEM 7: Use Mathematica (or another program you prefer) to graph the following equations. Name each surface.

a. $x^2 + 4y^2 + 9z^2 = 1$

b. $x = 2y^2 + 3z^2$,

c. $x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$

See course webpage. Will post good sol^{n} once permission is granted.

PROBLEM 8: Use Mathematica (or another program you prefer) to graph $z = f(x, y)$ for the following functions:

a. $f(x, y) = \sqrt{x^2 + y^2}$,

b. $f(x, y) = e^{\sqrt{x^2+y^2}}$,

c. $f(x, y) = \sin(\sqrt{x^2 + y^2})$,

d. $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$.

See course webpage. Will post good sol^{n} once permission is granted.

On the basis of those graphs, describe how the two-dimensional graph $y = g(x)$ and the three-dimensional graph $z = f(x, y) = g(\sqrt{x^2 + y^2})$ are related.

The graph $Z = f(x,y)$ is a surface of revolution. Its the rotation of the graph $Z = g(x)$ or $Z = g(y)$ about the z-axis.

PROBLEM 9: Calculus properties of vector-valued functions of a real variable. Prove each of the following assertions. Again the repeated index notation may be used if you find it helpful. Remember that to prove a vector equality we must prove equality for each component. We assume that f is a differentiable function of a real variable and $\vec{A}, \vec{B}, \vec{U}, \vec{r}$ are vector-valued functions of a real variable such that all the derivatives listed below exist.

a. $\frac{d}{dt}(f\vec{U}) = \frac{df}{dt}\vec{U} + f\frac{d\vec{U}}{dt}$

$$\frac{d}{dt}\left(f U_m e_m\right) = \frac{df}{dt} U_m e_m + f \frac{dU_m}{dt} e_m = f'\vec{U} + f\vec{U}'$$

$$\begin{aligned}
\frac{d}{dt}\left(f\vec{U}\right) &= \frac{d}{dt}\langle fU_1, fU_2, fU_3\rangle \\
&= \left\langle \frac{d}{dt}(fU_1), \frac{d}{dt}(fU_2), \frac{d}{dt}(fU_3)\right\rangle \\
&= \langle f'U_1 + fU_1', f'U_2 + fU_2', f'U_3 + fU_3'\rangle \\
&= \langle f'U_1, f'U_2, f'U_3\rangle + \langle fU_1', fU_2', fU_3'\rangle \\
&= f'\langle U_1, U_2, U_3\rangle + f\langle U_1', U_2', U_3'\rangle \\
&= \frac{df}{dt}\vec{U} + f\frac{d\vec{U}}{dt}.
\end{aligned}$$

b. $\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$

$$\begin{aligned}
\frac{d}{dt}\left(\vec{A}\cdot\vec{B}\right) = \frac{d}{dt}\left(A_m B_m\right) &= \frac{dA_m}{dt} B_m + A_m \frac{dB_m}{dt} \\
&= \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}.
\end{aligned}$$

c. $\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$

constant w.r.t. time.

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d}{dt}\left(\epsilon_{ijk} A_i B_j e_k\right) = \epsilon_{ijk}\frac{d}{dt}\left(A_i B_j\right) e_k$$

$$= \epsilon_{ijk}\left[\frac{dA_i}{dt}B_j + A_i\frac{dB_j}{dt}\right] e_k$$

$$= \epsilon_{ijk}\frac{dA_i}{dt}B_j e_k + \epsilon_{ijk}A_i\frac{dB_j}{dt}e_k$$

$$= \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}.$$

<u>Alternative Sol^{n}</u>: Use <u>def^{n}</u> of $\frac{d}{dt}(\vec{r}(t))$.

added zero.

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \lim_{h\to 0}\left[\frac{(\vec{A}\times\vec{B})(t+h) - (\vec{A}\times\vec{B})(t)}{h}\right]$$

$$= \lim_{h\to 0}\left[\left(\frac{\vec{A}(t+h)-\vec{A}(t)}{h}\right)\times\vec{B}(t+h) + \vec{A}(t)\times\left(\frac{\vec{B}(t+h)-\vec{B}(t)}{h}\right)\right]$$

$$= \lim_{h\to 0}\left[\frac{\vec{A}(t+h)-\vec{A}(t)}{h}\right]\times\lim_{h\to 0}\left(\vec{B}(t+h)\right) + \vec{A}(t)\times\lim_{h\to 0}\left[\frac{\vec{B}(t+h)-\vec{B}(t)}{h}\right]$$

$$= \left(\frac{d\vec{A}}{dt}\times\vec{B}\right)(t) + \left(\vec{A}\times\frac{d\vec{B}}{dt}\right)(t) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}.$$

This is the same as the calculus I argument for the ordinary product rule. One of your peers thought up this proof. I approve of this sort of independent thought.

d. $\frac{d}{dt}[\vec{r}(t)\times\vec{r}'(t)] = \vec{r}(t)\times\vec{r}''(t)$ (use part c. plus a well-known fact about the cross product)

$$\frac{d}{dt}(\vec{r}\times\vec{r}') = \frac{d\vec{r}}{dt}\times\vec{r}' + \vec{r}\times\frac{d}{dt}(\vec{r}') \quad \text{use (c.)}$$

$$= \cancel{\vec{r}'\times\vec{r}'}_{0} + \vec{r}\times\vec{r}''$$

$$= \vec{r}\times\vec{r}'.$$

$\left(\text{or } \frac{d}{dt}(\vec{r}\times\vec{v}) = \vec{r}\times\vec{a} \text{ when } \vec{r} \text{ is position}\right)$

PROBLEM 10: Newton's 2nd Law is an equation that uses the calculus of vector-valued functions of a real variable. We are told that $\vec{F} = m\vec{a}$ for a particle with mass m and position $\vec{r} =< x, y, z >$ under the influence of a force $\vec{F}$. Recall that acceleration was defined to be the second derivative of position; $\vec{a}(t) = \vec{r}''(t)$. Find the position at time t given that the particle is subject to $\vec{F} =< k, 0, -mg >$. State the answer in terms of the initial conditions $\vec{r}(0) =< x_o, y_o, z_o >$ and $\vec{v}(0) = \vec{r}'(0) =< v_{ox}, v_{oy}, v_{oz} >$.

$$m\vec{a} = \vec{F}$$

$$m\langle x'', y'', z'' \rangle = \langle k, 0, -mg \rangle$$

where k, m, g are constants, $m \neq 0$

We must solve by integrating twice while applying the initial conditions,

$$\frac{d\vec{V}}{dt} = \langle k/m, 0, -g \rangle$$

$$\vec{V} = \int \frac{d\vec{V}}{dt}\,dt = \int \langle k/m, 0, -g \rangle\,dt$$

$$= \langle (k/m)t, 0, -gt \rangle + \vec{C}_1$$

But $\vec{V}(0) = \langle V_{ox}, V_{oy}, V_{oz} \rangle = \langle 0, 0, 0 \rangle + \vec{C}$

Hence $\vec{V}(t) = \langle V_{ox} + \left(\frac{k}{m}\right)t, V_{oy}, V_{oz} - gt \rangle = \frac{d\vec{r}}{dt}$

$$\vec{r}(t) = \int \frac{d\vec{r}}{dt}\,dt = \int \langle V_{ox} + \left(\frac{k}{m}\right)t, V_{oy}, V_{oz} - gt \rangle\,dt$$

$$= \langle V_{ox}t + \frac{k}{2m}t^2, V_{oy}t, V_{oz}t - \frac{g}{2}t^2 \rangle + \vec{C}_2$$

But $\vec{r}(0) = \langle x_o, y_o, z_o \rangle = \vec{0} + \vec{C}_2$ thus,

$$\boxed{\vec{r}(t) = \langle x_o + V_{ox}t, y_o + V_{oy}t, z_o + V_{oz}t - \tfrac{1}{2}gt^2 \rangle}$$

PROBLEM 11: Kepler's Laws state that the planets orbit in elliptical paths. In other words, the motion of the planets lies in an orbital plane. Kepler made this assertion on the basis of experimental evidence gathered by Tycho Brahe. Later Isaac Newton postulated that gravity follows his universal law of gravitation. If we place the sun with mass M_s at the origin then the force on a planet of mass M_p at position $\vec{r}$ is given by the equation (G is the gravitation constant)

$$\vec{F}_{gravity} = \frac{-GM_sM_p\vec{r}}{r^3}$$

where $r = |\vec{r}|$ and the minus reflects the fact that gravity is an attractive force. For a particular planet we expect Newton's 2nd law should hold; $\vec{F}_{gravity} = M_p\vec{r}''$. Show that Newton's laws imply that the motion of the planet lies in some plane. A convenient characterization of a curve residing in a plane is that $\vec{r} \times \vec{r}' = \vec{c}$ for some constant vector $\vec{c}$.

(I would like to ask you to derive Kepler's Laws in their entirety but there is not quite enough time. It is not too hard, see my notes for a complete derivation, I doubt we will cover it in lecture this semester)

$$\vec{r} \times \vec{r}' = \vec{c} \iff \underline{\frac{d}{dt}(\vec{r} \times \vec{v}) = 0}$$

So if we can show $\frac{d}{dt}(\vec{r} \times \vec{v}) = 0$ then we're done. But, we know

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \vec{r} \times \vec{a}$$

So if we can show $\vec{r} \times \vec{a} = 0$ then we're done. Notice

$$\vec{F} = \frac{-GM_sM_p}{r^3}\vec{r} = M_p\vec{a}$$

$$\Rightarrow \vec{a} = \left(-\frac{GM_s}{r^3}\right)\vec{r}$$

(antiparallel.)

thus $\vec{a}$ and $\vec{r}$ are colinear

so $\underline{\vec{a} \times \vec{r} = 0}$.