

Math 231 Homework Project II: Derivatives:SOLUTION

PROBLEM 12: Let $f(x, y) = ye^{-xy}$. Find the direction(s) in which the directional derivative of f at the point $(0, 2)$ has the value 1. Give the direction(or directions) in terms of unit vector(s).

$$\begin{aligned}\nabla f &= \langle f_x, f_y \rangle \\ &= \left\langle y \frac{\partial}{\partial x}(e^{-xy}), e^{-xy} + y \frac{\partial}{\partial y}(e^{-xy}) \right\rangle \\ &= \left\langle -y^2 e^{-xy}, e^{-xy} - xy e^{-xy} \right\rangle \\ &= e^{-xy} \langle -y^2, 1 - xy \rangle\end{aligned}$$

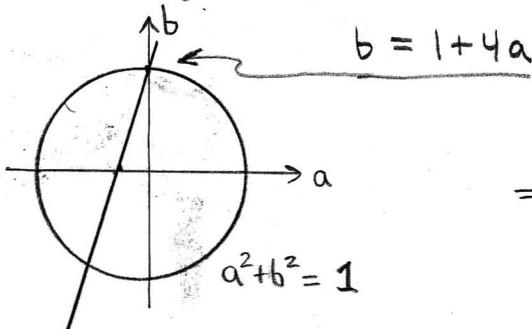
Evaluate the grad(f) = ∇f at $(0, 2)$,

$$(\nabla f)(0, 2) = \langle -4, 1 \rangle$$

We want to find $\hat{u} = \langle a, b \rangle$ with $|\hat{u}| = \sqrt{a^2 + b^2} = 1$
such that $D_{\hat{u}} f(0, 2) = 1$. This gives

$$\begin{aligned}(\nabla f)(0, 2) \cdot \hat{u} &= 1 && \text{need to solve simultaneously} \\ \langle -4, 1 \rangle \cdot \langle a, b \rangle &= -4a + b = 1 && \text{and } a^2 + b^2 = 1\end{aligned}$$

Now the way I picture the algebra is $a^2 + b^2 = 1$



$$\begin{aligned}b^2 &= (1+4a)^2 \\ \Rightarrow a^2 + (1+4a)^2 &= 1 \\ a^2 + 1 + 8a + 16a^2 &= 1 \\ 17a^2 + 8a &= 0 \\ a(17a + 8) &= 0 \\ a = 0 \text{ or } a = -\frac{8}{17} &\end{aligned}$$

If $a = 0$ then $b = 1$

If $a = -\frac{8}{17}$ then $b = \pm\sqrt{1 - \frac{64}{289}} = \pm\sqrt{\frac{225}{289}} = \pm\frac{15}{17}$ choose (-) from picture.

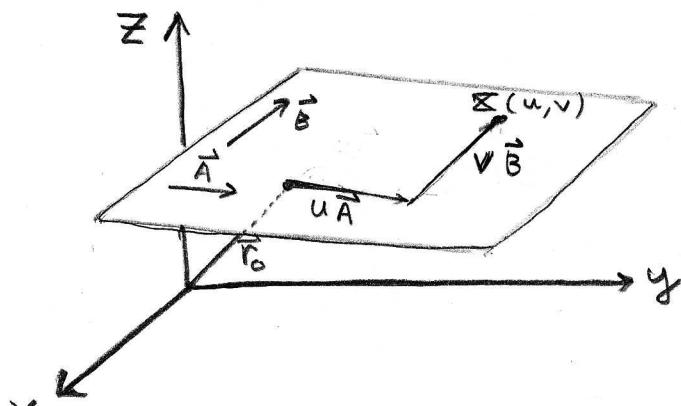
Thus we find $D_{\hat{u}} f(0, 2) = 1$ for $\hat{u} = \langle 0, 1 \rangle \text{ or } \langle -\frac{8}{17}, -\frac{15}{17} \rangle$

Reminder: the parametrization of a surface is a mapping X from $U \subseteq \mathbb{R}^2$ to \mathbb{R}^3 . For full credit you must supply both the equations of $X(u, v) = \langle X_1(u, v), X_2(u, v), X_3(u, v) \rangle$ and the domain U from which the point (u, v) are taken. Of course the parameters need not be labeled by "u" and "v". You can use any two appropriate chosen variables.

PROBLEM 13: Find parametrizations for the surfaces described in problems 19, 20 and 21 of section 17.6.
 Find the tangent plane at $(1, 2, -3)$ for #19. Find the tangent plane at $(0, 0, 1)$ for #20. Find the tangent plane at $(0, 1, 0)$ for #21.

$(1, 0, 0)$

§17.6 #19 We have a plane containing the vectors $\vec{A} = \langle 1, 1, -1 \rangle$ and $\vec{B} = \langle 1, -1, 1 \rangle$ and the point $(1, 2, -3)$.



Vector addition suggests we can write the parametrization

$$\Sigma(u, v) = \vec{r}_0 + u\vec{A} + v\vec{B}$$

Thus we find $\Sigma(u, v) = (1, 2, -3) + u\langle 1, 1, -1 \rangle + v\langle 1, -1, 1 \rangle$

$$\Sigma(u, v) = \langle 1+u+v, 2+u-v, -3-u+v \rangle$$

for $u \in \mathbb{R}$ and $v \in \mathbb{R}$ (no restrictions)

Alternatively: Find the plane's eqⁿ in x, y, z and use $\vec{r}(x, y)$,

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \langle 0, -2, -2 \rangle \text{ normal to plane, } \vec{r}_0 = (1, 2, -3)$$

$$\text{Thus } -2(y-2) - 2(z+3) = 0$$

$$y-2 + z+3 = 0 \Rightarrow z = -1-y, x \text{ is free}$$

Anytime we have a graph the parametrization follows easily,

$$\vec{r}(x, y) = \langle x, y, -1-y \rangle$$

Question: these are two versions of the same surface, I'm curious which x, y map to $\Sigma(0, 0) = \langle 1, 2, -3 \rangle$. Let's see
 $x=1, y=2 \Rightarrow \vec{r}(1, 2) = \langle 1, 2, -1-2 \rangle = \langle 1, 2, -3 \rangle$.

§ 17.6 #19 (Continued)

(3)

There are three ways to find the tangent plane at the point $(1, 2, -3)$.

1.) Graph View Point: we found it was $\langle 0, -2, -2 \rangle$ the plane has same normal vector everywhere thus at $(1, 2, -3)$ we have tangent plane

$$-2(y-2) - 2(3+3) = 0$$

$$y-2 + 3+3 = 0$$

$$\boxed{z = -1-y}$$

Tangent plane to a plane is the plane itself.

2.) Parametric:

$$\vec{r}_x = \langle 1, 0, 0 \rangle$$

$$\vec{r}_y = \langle 0, 1, -1 \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0, 1, 1 \rangle$$

\Rightarrow tangent plane at $(1, 2, -3)$

$$\text{which is } \vec{r}(1, 2) = \langle 1, 2, -3 \rangle$$

$$y-2 + 3+3 = 0 \rightarrow \boxed{z = -1-y}$$

3.) Level Surface: the plane is same as $F(x, y, z) = z+y = -1$

$$\text{note } \nabla F = \langle F_x, F_y, F_z \rangle = \langle 0, 1, 1 \rangle \text{ hence}$$

$$\text{the tangent plane is } \underline{y-2+3+3=0} \rightarrow \boxed{z = -1-y}$$

Remark: the contrast between these viewpoints is more pronounced on curvy surfaces.

§17.6 #20 Parametrize the lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1$. (4)
 We can use modified spherical coordinates, θ and ϕ still work we just need different radii in x, y, z .

$$\left. \begin{array}{l} x = \frac{1}{\sqrt{2}} \cos \theta \sin \varphi \\ y = \frac{1}{2} \sin \theta \sin \varphi \\ z = \cos \varphi \end{array} \right\} \text{with } \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \underline{\pi/2} \end{array} \leftarrow \begin{array}{l} \text{just want} \\ \text{top half} \end{array}$$

You might doubt me, let's check my answer,

$$2x^2 = 2 \cdot \frac{1}{2} \cos^2 \theta \sin^2 \varphi$$

$$4y^2 = 4 \cdot \frac{1}{4} \sin^2 \theta \sin^2 \varphi$$

$$z^2 = \cos^2 \varphi$$

$$\begin{aligned} \Rightarrow 2x^2 + 4y^2 + z^2 &= \cos^2 \theta \sin^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \varphi \\ &= \sin^2 \varphi (\cos^2 \theta + \sin^2 \theta) + \cos^2 \varphi \\ &= \sin^2 \varphi + \cos^2 \varphi \\ &= 1. \quad (\text{it works}) \end{aligned}$$

The parametrization is

$$\Sigma(\theta, \varphi) = \left\langle \frac{1}{\sqrt{2}} \cos \theta \sin \varphi, \frac{1}{2} \sin \theta \sin \varphi, \cos \varphi \right\rangle \quad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \underline{\pi/2} \end{array}$$

To find the tangent plane at $(1, 0, 0)$. We have several options,

1.) Could use level surface viewpoint

$$F(x, y, z) = 2x^2 + 4y^2 + z^2 - 1 = 0$$

$$\nabla F = \langle 4x, 8y, 2z \rangle$$

$$\nabla F(1, 0, 0) = \langle 4, 0, 0 \rangle$$

Thus the tangent plane is $4(x-1) = 0$

which is simply $\boxed{x=1}$

(this will look nice in Mathematica)

2.) Calculate the normal to tangent plane at $(1, 0, 0)$ via the parametric viewpoint.

$$\frac{\partial \Sigma}{\partial \theta} = \left\langle \frac{1}{\sqrt{2}} \sin \theta \sin \varphi, \frac{1}{2} \cos \theta \sin \varphi, 0 \right\rangle$$

$$\frac{\partial \Sigma}{\partial \varphi} = \left\langle \frac{1}{\sqrt{2}} \cos \theta \cos \varphi, \frac{1}{2} \sin \theta \cos \varphi, -\sin \varphi \right\rangle$$

The normal is $\Sigma_\theta \times \Sigma_\varphi$ at $(1, 0, 0)$ which corresponds to $\theta = 0$ and $\varphi = \pi/2$.

$$\Sigma_\theta(0, \pi/2) = \left\langle 0, \frac{1}{2}, 0 \right\rangle$$

$$\Sigma_\varphi(0, \pi/2) = \left\langle 0, 0, -1 \right\rangle$$

The normal is \perp to these tangents to the coordinate curves,

$$N(0, \pi/2) = \Sigma_\theta(0, \pi/2) \times \Sigma_\varphi(0, \pi/2)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= \underline{\left\langle -\frac{1}{2}, 0, 0 \right\rangle}$$

This gives tangent plane at $(1, 0, 0)$ of $-\frac{1}{2}(x-1) = 0$ which is simply $\boxed{x=1}$

Remark: the normal seems harder to calculate in the parametric view. However, it's easier to focus in on just part of a surface using parameters. For example if I want the ellipsoid above $\varphi = \pi/4$ I just say

$0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi/4$. This would be more difficult to phrase using the level surface idea.

§17.6 #21 | Find "right" part of hyperboloid $x^2 + y^2 - z^2 = 1$ that lies in front of the $x=0$ plane. (that is $x \geq 0$ for our surface) ⑥
 We can use hyperbolic angle γ and ordinary angle θ (polar) to parametrize this,

$$\left. \begin{array}{l} x = \cos \theta \cosh \gamma \\ y = \sin \theta \cosh \gamma \\ z = \sinh \gamma \end{array} \right\} \begin{array}{l} \text{you can check} \\ x^2 + y^2 - z^2 = \cosh^2 \gamma - \sinh^2 \gamma = 1. \end{array}$$

The graphs of $\cosh \gamma$ and $\sinh \gamma$ are

$$y = \cosh \gamma = \frac{1}{2}(e^\gamma + e^{-\gamma})$$

$$\frac{\cosh \gamma \geq 1}{\text{range}(\cosh \gamma) = [1, \infty)}$$

$$y = \sinh \gamma = \frac{1}{2}(e^\gamma - e^{-\gamma})$$

$$\text{range}(\sinh \gamma) = (-\infty, \infty)$$

$$y = \cos \theta$$

You'll find that θ works like a polar angle despite the presence of γ in the eq's.

$$y = \sin \theta$$

If we fix $z = z_0$ then $x^2 + y^2 = 1 + z_0^2$, thus the horizontal cross-section of this surface is a circle at each z . Whenever this happens, we can use θ to parametrize the circle. Then with that in mind if you fix $\theta = \theta_0$ then you'll find a hyperbola in that slice.

We can project each circle to xy -plane and measure θ as usual, we'd have $r_0^2 = 1 + z_0^2 \Rightarrow r_0 = \sqrt{1 + z_0^2}$. You can do this for each z thus we can look at all (x, y, z) of constant θ say $\theta = \theta_0$. For that slice $x = r \cos \theta_0$ and $y = r \sin \theta_0$ thus $\tan \theta_0 = y/x$ so $y = \tan \theta_0 x$

(7)

Continuing to try to show geometrically why we use $\cosh \theta$ and $\sinh \theta$ to parametrize.

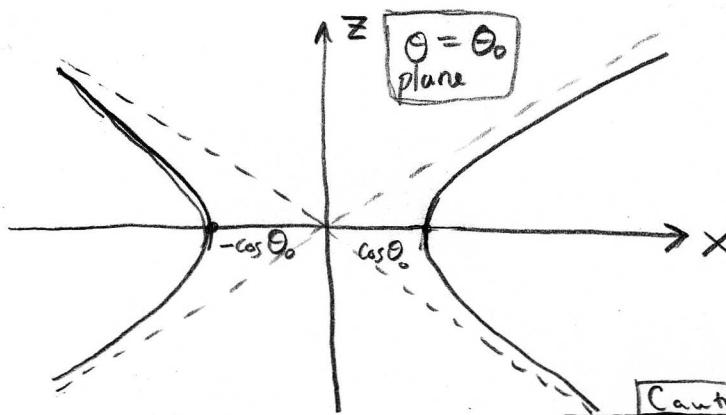
I had just discussed for constant z_0 we have a circle thus θ is a natural parameter for each such circle over the whole paraboloid. Thus we can ask what does the hyperboloid look like if we just consider the points with $\theta = \theta_0$. Geometrically this is an intersection of the hyperboloid and the plane $\theta = \theta_0$. I want to do this for an arbitrary angle because I want to eventually cover the whole hyperboloid. Let's return to our equations again, for $\theta = \theta_0 \Rightarrow y = \tan \theta_0 X$

$$X^2 + Y^2 - Z^2 = 1$$

$$\Rightarrow X^2 + X^2 \tan^2 \theta_0 - Z^2 = 1$$

$$X^2 \sec^2 \theta_0 - Z^2 = 1$$

(hyperbola)



$$Z=0 \text{ gives } X^2 \sec^2 \theta_0 = 1$$

$$X^2 = \frac{1}{\sec^2 \theta_0} = \cos^2 \theta_0$$

for $X \gg 1$

$$Z^2 = (\sec^2 \theta_0) X^2$$

$$\frac{X^2}{\cos^2 \theta_0} - Z^2 = 1$$

Caution:

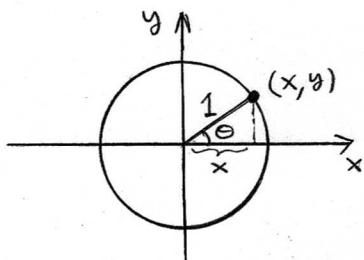
We only need to parametrize $\frac{1}{2}$ of this since θ will sweep over and pick up the other half. Note that the plane

$\theta = \theta_0 + \pi$
is the same plane!

Let's see now we need to develop some background about hyperbolic cosine and sine. In lecture we've not said enough yet.

Hyperbolic Trigonometric Functions and the Unit Hyperbola

Let me make an analogy to trig. functions and how to parametrize an ellipse. Recall we can define sine and cosine via the unit circle

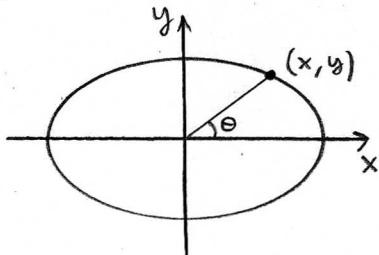


$$\cos \theta \equiv x \text{ and } \sin \theta \equiv y \\ \text{where } x^2 + y^2 = 1. \text{ Thus } \cos^2 \theta + \sin^2 \theta = 1$$

Then a circle of radius R has the natural parametrization by θ

$$x = R \cos \theta \quad \& \quad y = R \sin \theta \quad 0 \leq \theta \leq 2\pi$$

The parametrization of an ellipse is a little twist on these,



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

see that x/a is like "x" for the unit circle and y/b is like "y" for the unit circle. This suggests we choose $\cos \theta = x/a$ and $\sin \theta = y/b$

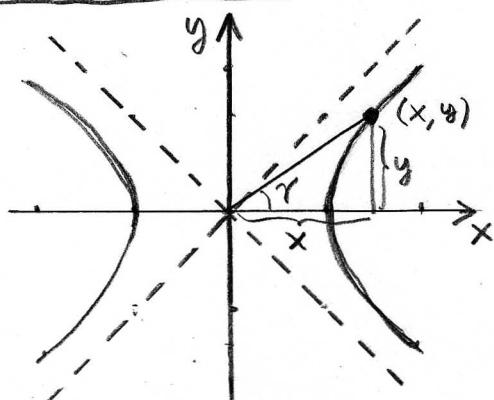
Hence the ellipse is given parametrically by

$$x = a \cos \theta \quad \& \quad y = b \sin \theta \quad 0 \leq \theta \leq 2\pi$$

The Unit Hyperbola: is defined to be $x^2 - y^2 = 1$. Its graph has

slant asymptotes of $y = \pm x$

Since $y = \pm \sqrt{x^2 - 1} \rightarrow \pm x$ as $x \gg 1$.



We can relate hyperbolic sine & cosine to the unit hyperbola

$$\cosh \gamma = x \quad \sinh \gamma = y \\ \text{where } x^2 - y^2 = 1. \text{ Note, } \cosh^2 \gamma - \sinh^2 \gamma = 1$$

Where γ is the hyperbolic angle and $\cosh \gamma = \frac{1}{2}(e^\gamma + e^{-\gamma})$ and $\sinh \gamma = \frac{1}{2}(e^\gamma - e^{-\gamma})$. Thus the right half of the unit hyperbola has,

$$x = \cosh \gamma \quad \& \quad y = \sinh \gamma \quad \text{for } -\infty < \gamma < \infty$$

Remark: I would like to define $\cosh(\gamma)$ and $\sinh(\gamma)$ via the unit hyperbola. I can't see how to get the formulas $\cosh \gamma = \frac{1}{2}(e^\gamma + e^{-\gamma})$ & $\sinh \gamma = \frac{1}{2}(e^\gamma - e^{-\gamma})$ from my geometric idea at the moment. In contrast, I defined $\sin \theta$ & $\cos \theta$ via the unit circle.

The intersection of $\theta = \theta_0$ & $x^2 + y^2 - z^2 = 1$ was found on ⑦,

⑨

$$\frac{x^2}{\cos^2 \theta_0} - z^2 = 1$$

this is a squished version of the unit hyperbola. You can see $x/\cos \theta_0$ is like "x" in $x^2 - y^2 = 1$ and z is like "y" in $x^2 - y^2 = 1$. Hence we propose $\cosh \gamma = x/\cos \theta_0$ & $\sinh \gamma = z$. This yields

$$x = \cos \theta_0 \cosh \gamma$$

$$z = \sinh \gamma$$

Returning to bottom of ⑥ we had $y = \tan \theta_0 x$ thus

$$y = \tan \theta_0 (\cos \theta_0 \cosh \gamma)$$

$$= \frac{\sin \theta_0}{\cos \theta_0} \cancel{\cos \theta_0} \cosh \gamma$$

$$= \sin \theta_0 \cosh \gamma$$

This is why I wrote ($0 \leq \theta \leq \pi$ to insure $x \geq 0$)

$$x = \cos \theta \cosh \gamma$$

$$y = \sin \theta \cosh \gamma$$

$$z = \sinh \gamma$$

$$\boxed{X(\theta, \gamma) = \langle \cos \theta \cosh \gamma, \sin \theta \cosh \gamma, \sinh \gamma \rangle}$$

$$0 \leq \theta \leq \pi, \gamma \in \mathbb{R}$$

to begin with. Perhaps my verbose solⁿ here has helped a few of you see why this parametrization is natural.

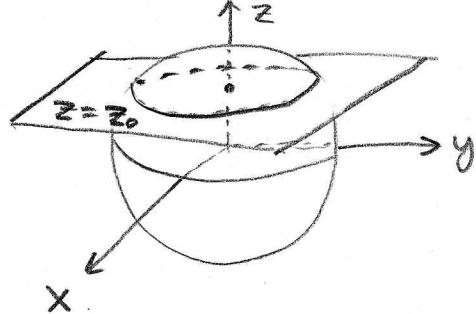
Remark: The bigger idea here is to parametrize a surface we try to find cross-sections which give curves on the surface which are something familiar like a circle, line, or hyperbola. Then once that is done fix the parameter already found and see what remains on the surface. Hopefully its not too ugly. Let me illustrate this method on a sphere,

2

How to parametrize a sphere using cross-section method

(10)

Consider $x^2 + y^2 + z^2 = R^2$. Nice cross-section is just horizontal slice $z = z_0$. This gives us



$$x^2 + y^2 + z_0^2 = R^2$$

$$x^2 + y^2 = R^2 - z_0^2$$

A circle of radius $\sqrt{R^2 - z_0^2}$. We can parametrize via θ ,

$$x = \sqrt{R^2 - z_0^2} \cos \theta$$

$$y = \sqrt{R^2 - z_0^2} \sin \theta$$

Clearly we can do this for each z in $-R \leq z \leq R$. So it makes sense to look at $\theta = \theta_0$ on the sphere.
-(I'm not insisting θ be the polar angle in this context although it is of course)-

$$\theta = \theta_0 \rightarrow \tan \theta_0 = y/x \rightarrow y = (\tan \theta_0) x$$

Substitute into sphere eqⁿ we get

$$x^2 + (\tan^2 \theta_0) x^2 + z^2 = R^2$$

$$(1 + \tan^2 \theta_0) x^2 + z^2 = R^2$$

$$\sec^2 \theta_0 x^2 + z^2 = R^2$$

$$\frac{x^2}{R^2 \cos^2 \theta_0} + \frac{z^2}{R^2} = 1 \quad \text{an ellipse with } \begin{cases} a = R \cos \theta_0 \\ b = R \end{cases}$$

We already used θ , let's use φ to parametrize this ellipse,
 $\sin \varphi = \frac{x}{R \cos \theta_0}$ & $\cos \varphi = z/R$. Thus

$$x = R \cos \theta_0 \sin \varphi \quad \text{and} \quad y = (\tan \theta_0) x \\ z = R \cos \varphi$$

$$= \frac{\tan \theta_0 R \cos \theta_0 \sin \varphi}{R} \\ = R \sin \theta_0 \sin \varphi$$

We recover the usual parametrization we've been using,

$$x = R \cos \theta \sin \varphi$$

$$0 \leq \theta \leq 2\pi$$

$$y = R \sin \theta \sin \varphi$$

$$0 \leq \varphi \leq \pi \quad (\text{to avoid double counting})$$

$$z = R \cos \varphi$$

§17.6 #21 Concluded

$$\Sigma(\theta, \gamma) = \langle \cos \theta \cosh \gamma, \sin \theta \cosh \gamma, \sinh \gamma \rangle \quad 0 \leq \theta < 2\pi, \gamma \in \mathbb{R}$$

$$\frac{\partial \Sigma}{\partial \theta} = \langle -\sin \theta \cosh \gamma, \cos \theta \cosh \gamma, 0 \rangle$$

$$\frac{\partial \Sigma}{\partial \gamma} = \langle \cos \theta \sinh \gamma, \sin \theta \sinh \gamma, \cosh \gamma \rangle$$

We wish to find tangent plane at $(0, 1, 0)$. Which pair of parameters (θ_0, γ_0) map to $(0, 1, 0)$? Need

$$\begin{aligned} \cos \theta_0 \cosh \gamma_0 &= 0 \rightarrow \underline{\theta_0 = \pi/2} \quad (\cosh(0) = 1 \text{ so } \cos \theta_0 \text{ must be zero}) \\ \sin \theta_0 \cosh \gamma_0 &= 1 \\ \sinh \gamma_0 &= 0 \rightarrow \underline{\gamma_0 = 0} \end{aligned}$$

We need the normal vector at $(0, 1, 0)$ which is

$$\begin{aligned} \vec{N}\left(\frac{\pi}{2}, 0\right) &= \Sigma_0\left(\frac{\pi}{2}, 0\right) \times \Sigma\left(\frac{\pi}{2}, 0\right) \\ &= \langle -\sin\left(\frac{\pi}{2}\right) \cosh(0), \cos\left(\frac{\pi}{2}\right) \cosh(0), 0 \rangle \times \langle \cos\frac{\pi}{2} \sinh 0, \sin\frac{\pi}{2} \sinh 0, 1 \rangle \\ &= -\hat{i} \times \hat{k} \\ &= \hat{j} = \langle 0, 1, 0 \rangle \quad (\text{that's funny}) \end{aligned}$$

Thus the tangent plane at $(0, 1, 0)$ is $y - 1 = 0$ or $\boxed{y = 1}$

Alternatively: Use level surface formalism,

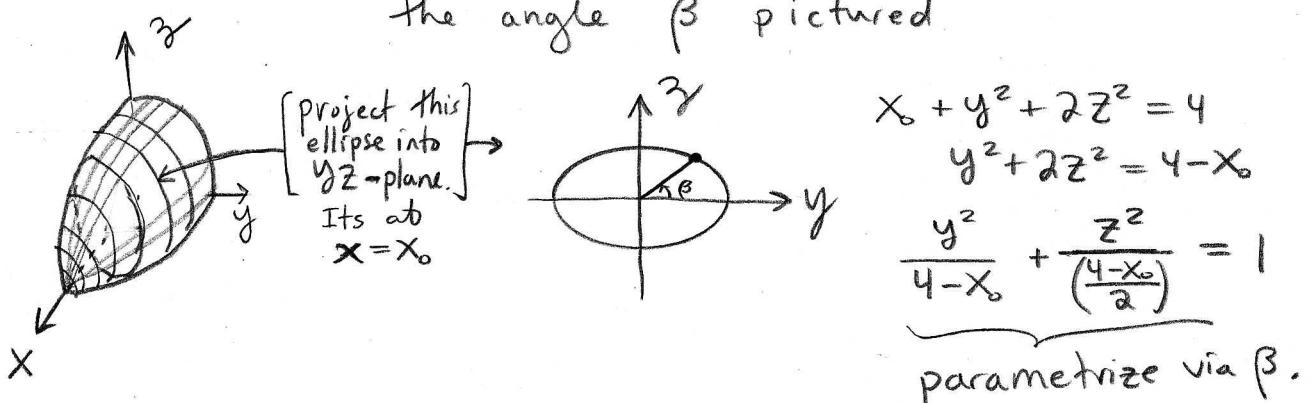
$$F(x, y, z) = x^2 + y^2 - z^2 = 1$$

$$\nabla F = \langle 2x, 2y, -2z \rangle$$

Thus $\nabla F(0, 1, 0) = \langle 0, 2, 0 \rangle$ is normal to surface at the point $(0, 1, 0)$ giving $2(y-1) = 0$ a.k.a. $\boxed{y = 1}$

PROBLEM 14: Find parametrizations for the surfaces described in problems 22, 23 and 24 of section 17.6.
 Find the tangent plane at $(0, \sqrt{2}, 1)$ for #22. Find the tangent plane at $(0, 0, 2)$ for #23. Find the tangent plane at $(4, 0, 0)$ for #24.

S17.6 #22 The part of the elliptic paraboloid $x + y^2 + 2z^2 = 4$ that lies in front of plane $x = 0$. I'd draw a picture to start. one natural parametrization here is to use x and the angle β pictured



We find " a " = $\sqrt{4-x_0}$ and " b " = $\sqrt{2-x_0/2}$ for the ellipse at $x=x_0$,
 $y = \sqrt{4-x_0} \cos \beta$ & $z = \sqrt{2-x_0/2} \sin \beta$ for $0 \leq \beta \leq 2\pi$

Hence,

$$\boxed{\vec{r}(x, \beta) = \langle x, \sqrt{4-x} \cos \beta, \sqrt{2-x/2} \sin \beta \rangle}$$

for $0 \leq x \leq 4$, $0 \leq \beta \leq 2\pi$

I'm guessing most of you used the more algebraic approach,

$$x = 4 - y^2 - 2z^2 \quad \text{and} \quad x \geq 0 \rightarrow y^2 + 2z^2 \leq 4$$

$$\boxed{\Sigma(y, z) = \langle 4 - y^2 - 2z^2, y, z \rangle}$$

for $(y, z) \in U = \{(y, z) \mid y^2 + 2z^2 \leq 4\}$

This approach is better than my $\vec{r}(x, \beta)$ in some ways. For example, the $\Sigma(0, 0) = \langle 4, 0, 0 \rangle$. My $\vec{r}(x, \beta)$ misses that point because β is undefined at that exceptional point.

8/7.6 # 23 Continued

(14)

The point $(0, 0, 2)$ is not covered by $\vec{r}(\theta, \varphi)$ because θ is not defined along the z -axis. Ooops. You probably used the level surface idea so it was not apparent anyway.

$$F(x, y, z) = x^2 + y^2 + z^2 - 4$$

$$\nabla F = \langle 2x, 2y, 2z \rangle = 2\langle x, y, z \rangle$$

$$\nabla F(0, 0, 2) = 2\langle 0, 0, 2 \rangle = \langle 0, 0, 4 \rangle$$

\Rightarrow tangent plane at $(0, 0, 2)$ is $4(z-2) = 0$
or simply $z=2$.

I chose that point because I thought the Mathematica would work out nice there. Geometrically the tangent plane to a sphere shares the same geometry as any other tangent plane to any other point on the sphere. (Spherical Symmetry!) Silly comments aside I should have given you the point $(2, 0, 0)$ where $\theta=0, \phi=\frac{\pi}{2}$

$$\vec{r}_\theta = \langle -2\sin\theta\sin\varphi, 2\cos\theta\sin\varphi, 0 \rangle$$

$$\vec{r}_\phi = \langle 2\cos\theta\cos\varphi, 2\sin\theta\cos\varphi, -2\sin\varphi \rangle$$

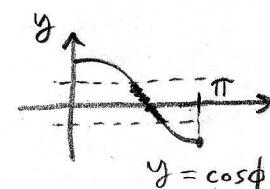
$$(\vec{r}_\theta \times \vec{r}_\phi)(0, \frac{\pi}{2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = \langle -4, 0, 0 \rangle$$

gives tangent plane $-4(x-2) = 0$ at $(2, 0, 0)$
which is simply $x=2$

§17.6 #24 Consider part of sphere $x^2 + y^2 + z^2 = 16$ that lies between $z = -2$ and $z = 2$. Find parametrization and then the tangent plane at $(4, 0, 0)$.

- Pictures are helpful, but I'm going to use algebraic arguments for a change.
- Observe that this is a sphere so θ & ϕ are the natural parameters, moreover $\rho = 4$.
- We have $-2 \leq z \leq 2$ this means

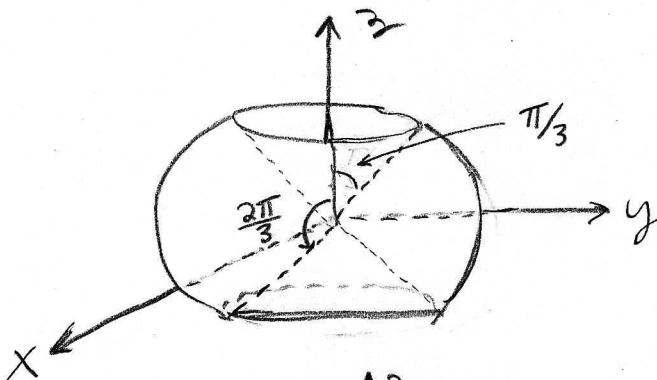
$$\begin{aligned} -2 &\leq 4\cos\phi \leq 2 \\ -\frac{1}{2} &\leq \cos\phi \leq \frac{1}{2} \\ \Rightarrow \frac{\pi}{3} &\leq \phi \leq \frac{2\pi}{3} \end{aligned}$$



Hence

$$\Sigma(\theta, \phi) = \langle 4\cos\theta\sin\phi, 4\sin\theta\sin\phi, 4\cos\phi \rangle$$

$$0 \leq \theta \leq 2\pi \text{ & } \frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$$

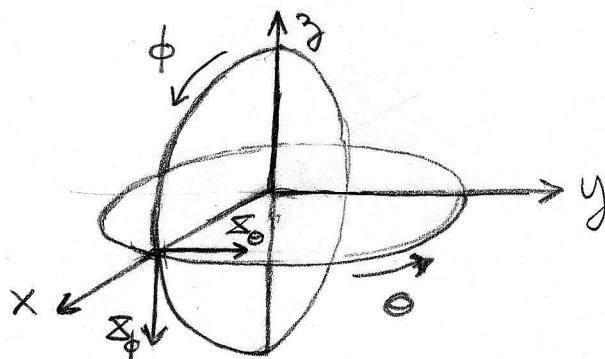


Notice $\Sigma(0, \pi/2) = (4, 0, 0)$ and

$$\Sigma_\theta(0, \pi/2) = \langle 0, 4, 0 \rangle = 4\hat{j}$$

$$\Sigma_\phi(0, \pi/2) = \langle 0, 0, -4 \rangle = -4\hat{k}$$

Thus $\vec{N}(0, \pi/2) = 4\hat{j} \times (-4\hat{k}) = -16\hat{i}$
and the tangent plane is $-16(x-4) = 0$
or simply $\boxed{x = 4}$



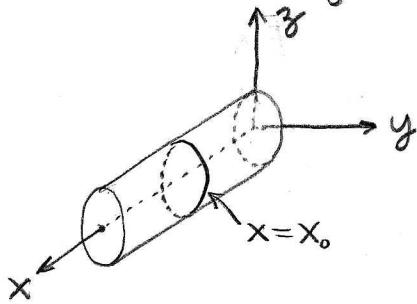
What happens is $\frac{\partial \Sigma}{\partial \theta}$ points in direction of increasing θ while $\frac{\partial \Sigma}{\partial \phi}$ points in direction of increasing ϕ . The tangent plane includes both of these vectors so $\Sigma_\theta \times \Sigma_\phi$ naturally gives us the normal. See (319) for more on this.

PROBLEM 15: Find parametrizations for the surfaces described in problems 25 and 26 of section 17.6.

Find the tangent plane at $(3,4,0)$ for #25. Find the tangent plane at $(0,1,3)$ for #26. ~~Find the tangent plane at $(0,0,0)$ for #24.~~

Then plot both the surface and tangent plane for each of the problems using Mathematica. (this should produce two separate graphs to print out, what is the difference between the tangent plane and the surface for #26? There is a difference.)

S17.6 #25 Part of cylinder $y^2 + z^2 = 16$ between $x=0$ and $x=5$.



radius of cylinder is 4.

Consider cross-section $x=x_0 \in [0, 5]$.

At $x=x_0$ we get $y^2+z^2=16$. Parametrize using β ; $y=4\cos\beta$, $z=4\sin\beta$.

We can do this at each x between 0 & 5.

$$\vec{r}(x, \beta) = \langle x, 4\cos\beta, 4\sin\beta \rangle : \begin{cases} 0 \leq x \leq 5 \\ 0 \leq \beta \leq 2\pi \end{cases}$$

Remark: Some of you likely used Θ instead of β . That's perfectly acceptable so long as you understand the usual formulas for Θ must be modified. For example, $\tan \Theta = \bar{z}/y$ (not y/x)

$$\vec{r}_x = \langle 1, 0, 0 \rangle$$

$$\vec{r}_\beta = \langle 0, -4\sin\beta, 4\cos\beta \rangle$$

$$\vec{r}_x \times \vec{r}_\beta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & -4\sin\beta & 4\cos\beta \end{vmatrix} = \langle 0, -4\cos\beta, -4\sin\beta \rangle$$

$$\vec{r}(x_0, \beta_0) = (3, 4, 0) \Rightarrow x_0 = 3 \text{ & } \beta = 0. \text{ Note } \vec{N}(3, 0) = \langle 0, -4, 0 \rangle$$

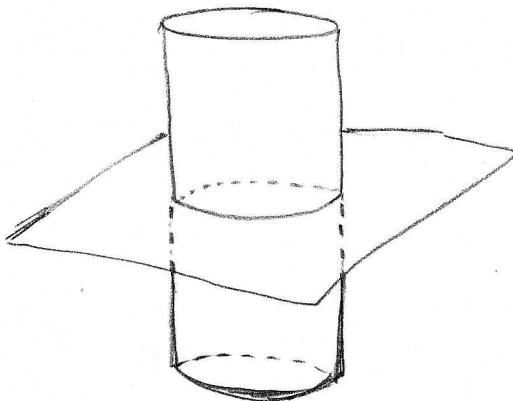
we find tangent plane at $(3, 4, 0)$ is simply $-4(y-4) = 0$ or simply $\boxed{y=4}$.

Remark: We'd find tangent plane $y=4$ at each point on the cylinder with $\beta=0$ since the x -coordinate doesn't appear in $\vec{N}(x, \beta)$. This is also clear geometrically the cylindrical symmetry says we get same geometry at each $x \in [0, 5]$. Of course $x=0$ and $x=5$ are special for this particular example.

Quick Check: $F(x, y, z) = y^2 + z^2 = 16 \rightarrow \nabla F = \langle 0, 2y, 2z \rangle$

and $\nabla F(3, 4, 0) = \langle 0, 8, 0 \rangle$ same direction upto sign.

§17.6 #26] The part of the plane $Z = X + 3$ that lies inside the cylinder $x^2 + y^2 = 1$. We can use X and Y as parameters.



$$\vec{r}(x, y) = \langle x, y, x+3 \rangle$$

$$x^2 + y^2 \leq 1$$

then $\vec{r}_x = \langle 1, 0, 1 \rangle$ & $\vec{r}_y = \langle 0, 1, 0 \rangle$
 thus $\vec{r}_x \times \vec{r}_y = (\hat{i} + \hat{k}) \times \hat{j} = \hat{k} - \hat{i} = \langle -1, 0, 1 \rangle$
 we find tangent plane at $(0, 1, 3)$ is
 $-(X-0) + 0(Y-1) + 1(Z-3) = 0$

$$-X + Z - 3 = 0 \rightarrow Z = X + 3$$

The tangent plane $Z = X + 3$ has no restriction $x^2 + y^2 \leq 1$, it goes on and on in contrast to the elliptical region described by the parametrization $\vec{r}(x, y)$.

Discussion: why it is an ellipse.

The intersection of $x^2 + y^2 = 1$ and $Z = X + 3$ can be parametrized by the polar angle Θ . We have

$$X = \cos \beta$$

$$Y = \sin \beta$$

$$Z = \cos \beta + 3$$

This gives us a "space-curve". Change β to t for the sake of making it more familiar

$$\vec{r}(t) = \langle \cos t, \sin t, \cos t + 3 \rangle$$

What is the arclength of this curve?

$$s = \int_0^{2\pi} |\vec{r}'(t)| dt, \text{ note } \vec{r}'(t) = \langle -\sin t, \cos t, -\sin t \rangle$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + \sin^2 t} dt$$

$$= \int_0^{2\pi} \sqrt{1 + \sin^2 t} dt$$

(this is not elementary,
numerical methods must suffice here)

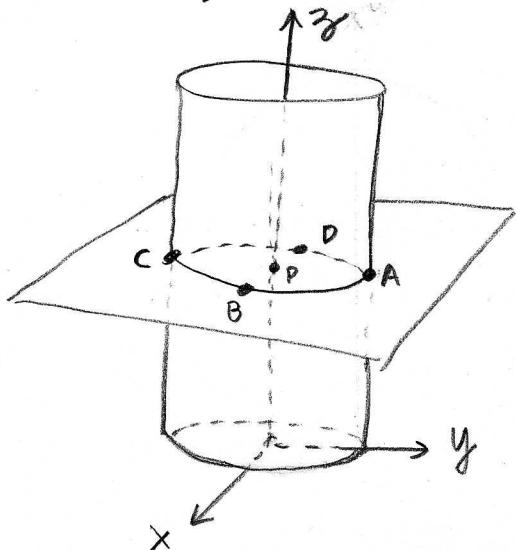
Notice $\sqrt{1 + \sin^2 t} \geq 1$ and there is only equality at $t=0, \pi$ and 2π . This tells us

$$\int_0^{2\pi} \sqrt{1 + \sin^2 t} dt > \int_0^{2\pi} dt = 2\pi$$

The circumference is larger than 2π , its not the unit circle.

Another way to see the surface in #26 is an ellipse,

(18)



I'll draw it as a circle, this picture is not to scale.

Intersection of

$$x^2 + y^2 = 1 \text{ and } z = x + 3.$$

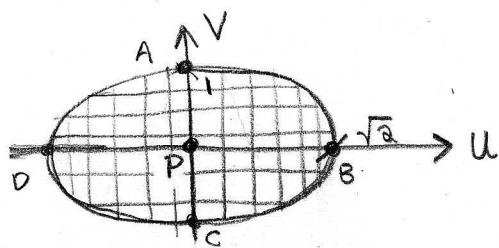
- (A.) $x = 0, y = 1$ we get $z = 3$
- (B.) $x = 1, y = 0$ we get $z = 4$
- (C.) $x = 0, y = -1$ get $z = 3$
- (D.) $x = -1, y = 0$ where $z = 2$.

$$A = (0, 1, 3), B = (1, 0, 4), C = (0, -1, 3), D = (-1, 0, 2), P = (0, 0, 3)$$

$$\text{Notice } |\overline{PA}| = |A-P| = |\langle 0, 1, 0 \rangle| = 1$$

$$|\overline{PB}| = |B-P| = |\langle 1, 0, 1 \rangle| = \sqrt{2}$$

If this were a circle then these should have been equal.



$$v^2 + \frac{u^2}{2} = 1$$

I'm thinking of the surface here. Now how to relate u, v to x, y, z ?

I believe my original parametrization is the same as the one pictured.

$$\vec{r}(u, v) = \langle u, v, u+3 \rangle$$

$$x = u$$

$$y = v$$

$$z = u+3$$

No, because I want $v^2 + u^2/2 = 1$. Somewhat

$$x^2 + y^2 = 1 \rightarrow v^2 + u^2/2 = 1. \text{ Let's try}$$

$$x = u/\sqrt{2}, y = v, z = u/\sqrt{2} + 3$$

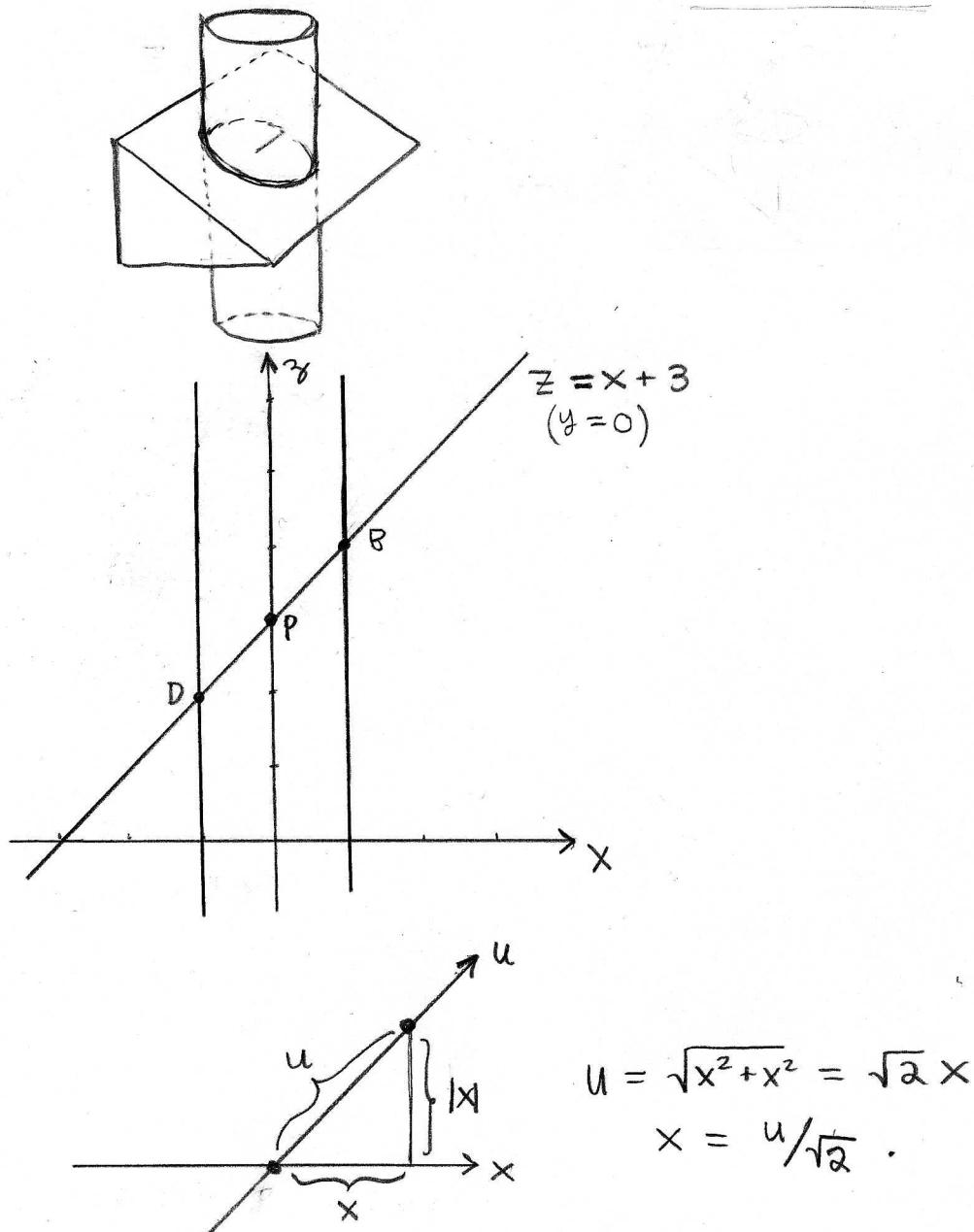
We find $x^2 + y^2 = u^2/2 + v^2 = 1$. Thus

$$\vec{r}_2(u, v) = \langle u/\sqrt{2}, v, u/\sqrt{2} + 3 \rangle$$

Still thinking about the tilted ellipse

$$\vec{r}_2(u, v) = \langle u/\sqrt{2}, v, u/\sqrt{2} + 3 \rangle$$

for $u^2/2 + v^2 \leq 1$ maps to the intersection of $x^2 + y^2 = 1$ and $\underline{z = x + 3}$



Ok, enough about this. (This was a response to a discussion that began in lecture Fall 2008)

PROBLEM 16: Calculate the Jacobian matrices for the various functions discussed in E70 on page 310 of my notes. Work out how the chain rule for the general derivative gives back the chain rules given before in section 15.5. In other words, work out my example so you can see how the Jacobian matrix reproduces the chain rules given previously.

Set-up Let $f(x, y) = x^2 - 3y^2$ and $x = uv$, $y = u + v^2$

We could say $T(u, v) = (uv, u + v^2)$. If $z = f(T(u, v))$ then calculate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using Jacobian Matrix ideas.

$$z = f \circ T$$

$$Dz = D(f \circ T) = (Df) \circ (DT)$$

$$Dz = \left[\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right]$$

$$Df = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] = [2x, -6y]$$

$$DT = \begin{bmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} \end{bmatrix} = \begin{bmatrix} v & 1 \\ u & 2v \end{bmatrix}$$

$$[Dz] = [Df][DT]$$

$$\left[\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right] = [2x, -6y] \begin{bmatrix} v & 1 \\ u & 2v \end{bmatrix} =$$

$$= [2xv - 6yu, 2x - 12yv]$$

$$\therefore \boxed{\begin{aligned} \frac{\partial z}{\partial u} &= 2xv - 6yu = 2uv^2 - 6(u+v^2)u \\ \frac{\partial z}{\partial v} &= 2x - 12yv = 2uv - 12(u+v^2)v \end{aligned}}$$

Remark: If $z = f \circ T \circ H \circ W$ then

$$Dz = Df \circ DT \circ DH \circ DW$$

you can generate many layered chain rules from the general result $D(f \circ g) = Df \circ Dg$.

However, for practical purposes tree diagrams are quite sufficient.

(21)

PROBLEM 17: Solve problems 13 and 29 of section 15.7. Back up any claims via appropriate calculus and theorems from my notes or the text.

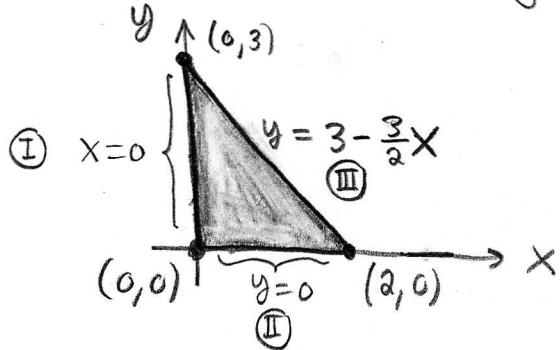
§15.7 #13 Let $f(x, y) = e^x \cos(y)$ find and classify local extrema.

$\nabla f = \langle e^x \cos y, -e^x \sin y \rangle$, exists at all (x, y)
thus any critical point must satisfy $\nabla f = 0$, that gives

$$e^x \cos y = 0 \quad \& \quad -e^x \sin y = 0 \\ \Leftrightarrow \cos y = 0 \quad \& \quad \sin(y) = 0$$

Therefore, there is no critical point, sine and cosine are never simultaneously zero.

§15.7 #29 (optional F08 semester) Let $f(x, y) = 1 + 4x - 5y$ and
Suppose D is the closed triangular region with vertices $(0, 0), (2, 0), (0, 3)$



$\nabla f = \langle 4, -5 \rangle \neq \vec{0}$
there are no local extrema
since there are no critical
points. The extrema on D
must occur on $\partial D = \textcircled{I} \cup \textcircled{II} \cup \textcircled{III}$

I: $x=0, 0 \leq y \leq 3$

$$f(x, y) = f(0, y) = 1 - 5y \equiv g(y)$$

$\frac{dg}{dy} = -5 \neq 0$ no local extrema on I, max/min must occur on
its endpts. $y=0$ or $y=3$. Note $g(0) = 1$ & $g(3) = -14$.

We find max/min of f on I are $f(0, 0) = 1$ & $f(0, 3) = -14$.

II: $0 \leq x \leq 2$ and $y = 0$

$f(x, y) = f(x, 0) = 1 + 4x \equiv h(x)$, Note $h'(x) = 4 \neq 0$. Just
consider endpts. $x=0$ & $x=2$. Have $h(0) = 1$ and $h(2) = 9$. Thus
the extrema of f on II are $f(0, 0) = 1$ and $f(2, 0) = 9$.

III: $y = 3 - \frac{3}{2}x$ for $0 \leq x \leq 2$. $f(x, y) = 1 + 4x - 5\left(3 - \frac{3}{2}x\right) = -14 + \frac{23}{2}x \equiv f(x)$

Again $f'(x) = \frac{23}{2} \neq 0$ just check endpts. $x=0$ and $x=2$. (Already checked in I, II).

We find absolute max of f on D is $f(2, 0) = 9$ and min is $f(0, 3) = -14$

(22)

PROBLEM 18: Let $s = \sqrt{x^2 + y^2}$ and suppose $\beta = \tan^{-1}(\frac{y}{x})$. Calculate ∇s and $\nabla \beta$. Then calculate unit vectors in the same direction (at an arbitrary point), that is calculate

$$\hat{u}_s = \frac{1}{|\nabla s|} \nabla s \quad \hat{u}_\beta = \frac{1}{|\nabla \beta|} \nabla \beta.$$

If will be convenient to invert the equations $s = \sqrt{x^2 + y^2}$ and $\beta = \tan^{-1}(y/x)$ for x and y . Lets begin,

$$\tan(\beta) = y/x \quad \text{and} \quad s^2 = x^2 + y^2$$

$$\Rightarrow y = x \tan \beta$$

$$\Rightarrow s^2 = x^2 + x^2 \tan^2 \beta$$

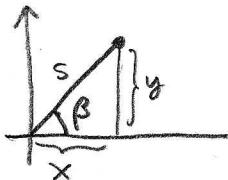
$$\Rightarrow s^2 = x^2(1 + \tan^2 \beta) = x^2 \sec^2 \beta$$

$$\Rightarrow x^2 = s^2 \cos^2 \beta$$

$$\Rightarrow x = \pm s \cos \beta.$$

$$y = \pm s \cos \beta \tan \beta = \pm s \sin \beta.$$

I had in mind polar coordinates, this means β is measured counterclockwise relative to the x -axis.



$x = s \cos \beta$
$y = s \sin \beta$

We choose the (+) solution. (Its not really pinned down by givens, we make a choice here).

$$\nabla s = \left\langle \frac{\partial}{\partial x}(\sqrt{x^2 + y^2}), \frac{\partial}{\partial y}(\sqrt{x^2 + y^2}) \right\rangle = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle = \left\langle \frac{x}{s}, \frac{y}{s} \right\rangle$$

Thus $\nabla s = \langle \cos \beta, \sin \beta \rangle$ and $|\nabla s| = 1$ so $\hat{u}_s = \langle \cos \beta, \sin \beta \rangle$

$$\nabla \beta = \left\langle \frac{\partial}{\partial x}[\tan^{-1}(y/x)], \frac{\partial}{\partial y}[\tan^{-1}(y/x)] \right\rangle$$

$$= \left\langle \frac{1}{1+y^2/x^2} \frac{\partial}{\partial x} \left[\frac{y}{x} \right], \frac{1}{1+y^2/x^2} \frac{\partial}{\partial y} \left[\frac{y}{x} \right] \right\rangle$$

$$= \left\langle \left(\frac{1}{1+y^2/x^2} \right) \left(-\frac{y}{x^2} \right), \left(\frac{1}{1+y^2/x^2} \right) \left(\frac{1}{x} \right) \right\rangle$$

$$= \left\langle -\frac{y}{x^2+y^2}, \frac{1}{x^2+y^2} \right\rangle$$

$$= \left\langle -\frac{y}{s^2}, \frac{1}{s^2} \right\rangle = \frac{1}{s} \langle -\sin \beta, \cos \beta \rangle$$

then we find

$$|\nabla \beta| = 1/s$$

thus

$$\hat{u}_\beta = \langle -\sin \beta, \cos \beta \rangle$$

PROBLEM 19: Show that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is differentiable everywhere then $\nabla f = \langle f_x, f_y \rangle$ can be rewritten as

$$\nabla f = \frac{\partial f}{\partial s} \hat{u}_s + \frac{1}{s} \frac{\partial f}{\partial \beta} \hat{u}_\beta.$$

Here you should use the result of problem 18. The solution will require the use of the chain rule for several variables and some vector algebra.

Observe that if $\hat{u}_s = \hat{i} \cos \beta + \hat{j} \sin \beta$ and $\hat{u}_\beta = -\hat{i} \sin \beta + \hat{j} \cos \beta$
we can invert these and solve for \hat{i} and \hat{j} in terms of \hat{u}_s & \hat{u}_β

$$\hat{i} = A \hat{u}_s + B \hat{u}_\beta$$

$$\hat{j} = C \hat{u}_s + D \hat{u}_\beta$$

But, what should A, B, C, D be? Notice $\hat{u}_s \cdot \hat{u}_\beta = 0$,

$$\hat{i} \cdot \hat{u}_s = A \cancel{\hat{u}_s} \cdot \hat{u}_s + B \cancel{\hat{u}_\beta} \cdot \hat{u}_s = A = \cos \beta$$

$$\hat{i} \cdot \hat{u}_\beta = A \cancel{\hat{u}_s} \cdot \hat{u}_\beta + B \cancel{\hat{u}_\beta} \cdot \hat{u}_\beta = B = -\sin \beta$$

$$\hat{j} \cdot \hat{u}_s = C \hat{u}_s \cdot \hat{u}_s + D \hat{u}_\beta \cdot \hat{u}_s = C = \sin \beta$$

$$\hat{j} \cdot \hat{u}_\beta = C \hat{u}_s \cdot \hat{u}_\beta + D \hat{u}_\beta \cdot \hat{u}_\beta = D = \cos \beta$$

$$\hat{i} = \cos \beta \hat{u}_s - \sin \beta \hat{u}_\beta$$

$$\hat{j} = \sin \beta \hat{u}_s + \cos \beta \hat{u}_\beta$$

Lets keep these in mind as we calculate, also we'll need to use

$$\frac{\partial s}{\partial x} = \frac{x}{s} = \cos \beta \quad \text{and} \quad \frac{\partial s}{\partial y} = \frac{y}{s} = \sin \beta \quad (\text{see } \nabla s \text{ calculation})$$

$$\frac{\partial \beta}{\partial x} = -\frac{y}{s^2} = -\frac{\sin \beta}{s} \quad \text{and} \quad \frac{\partial \beta}{\partial y} = \frac{x}{s^2} = \frac{\cos \beta}{s} \quad (\text{see } \nabla \beta \text{ calculation})$$

Lets derive the ∇f in (s, β) coordinates,

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} \\ &= \left(\frac{\partial f}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial f}{\partial \beta} \frac{\partial \beta}{\partial x} \right) \hat{i} + \left(\frac{\partial f}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial f}{\partial \beta} \frac{\partial \beta}{\partial y} \right) \hat{j} \\ &= \left(f_s \cos \beta - \frac{1}{s} f_\beta \sin \beta \right) \left(\cos \beta \hat{u}_s - \sin \beta \hat{u}_\beta \right) + \left(f_s \sin \beta + \frac{1}{s} f_\beta \cos \beta \right) \left(\sin \beta \hat{u}_s + \cos \beta \hat{u}_\beta \right) \\ &= \left(f_s \cos^2 \beta + f_s \sin^2 \beta \right) \hat{u}_s + \frac{1}{s} \left(f_\beta \sin^2 \beta + f_\beta \cos^2 \beta \right) \hat{u}_\beta - \frac{1}{s} f_\beta \sin \beta \cos \beta \hat{u}_s + \frac{1}{s} f_\beta \cos \beta \sin \beta \hat{u}_s \\ &\quad - f_s \cos \beta \sin \beta \hat{u}_\beta + f_s \sin \beta \cos \beta \hat{u}_\beta \\ &= f_s (\cos^2 \beta + \sin^2 \beta) \hat{u}_s + \frac{1}{s} f_\beta (\sin^2 \beta + \cos^2 \beta) \hat{u}_\beta \\ &= \boxed{\frac{\partial f}{\partial s} \hat{u}_s + \frac{1}{s} \frac{\partial f}{\partial \beta} \hat{u}_\beta} \end{aligned}$$

Remark: it's easier to just check this formula is valid.

PROBLEM 20: Solve problems 8, 12, 16 and 21 of section 16.3. Motivate any change of integration order with appropriate graphs.

§16.3#8 Let $D = \{(x, y) / 0 \leq x \leq 1, 0 \leq y \leq x^2\}$

$$\begin{aligned} \iint_D \frac{y}{x^5 + 1} dA &= \int_0^1 \int_0^{x^2} \frac{y}{x^5 + 1} dy dx \\ &= \int_0^1 \left(\frac{1}{2} \frac{1}{x^5 + 1} y^2 \right) \Big|_0^{x^2} dx \\ &= \int_0^1 \frac{\frac{1}{4} x^4 dx}{x^5 + 1} \\ &= \frac{1}{20} \ln(x^5 + 1) \Big|_0^1 \\ &= \frac{1}{20} [\ln(2) - \ln(1)] \\ &= \boxed{\ln(2)/20} \end{aligned}$$

§16.3#12 $D = \{(x, y) / 0 \leq y \leq 1, 0 \leq x \leq y\}$

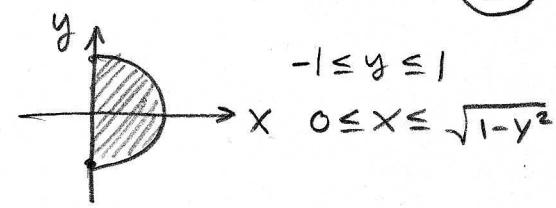
$$\begin{aligned} \iint_D x \sqrt{y^2 - x^2} dA &= \int_0^1 \int_0^y x \sqrt{y^2 - x^2} dx dy \\ &= \int_0^1 \int_{y^2}^0 -\frac{1}{2} \sqrt{u} du dy \\ &= \int_0^1 \left(-\frac{1}{2} \frac{2}{3} u^{3/2} \right) \Big|_{y^2}^0 dy \\ &= \int_0^1 \frac{1}{3} (y^2)^{3/2} dy \\ &= \int_0^1 \frac{1}{3} y^3 dy \\ &= \frac{1}{12} y^4 \Big|_0^1 \\ &= \boxed{\frac{1}{12}}. \end{aligned}$$

$$\begin{aligned} U &= y^2 - x^2, \quad y \text{ constant} \\ du &= -2x dx \\ U(x=y) &= 0 \\ U(x=0) &= y^2 \end{aligned}$$

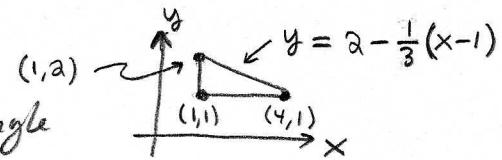
§16.3 #16 D enclosed by $x=0$ and $x=\sqrt{1-y^2}$ ($x^2+y^2=1$)

(25)

$$\begin{aligned}
 \iint_D xy^2 dA &= \int_{-1}^1 \int_0^{\sqrt{1-y^2}} xy^2 dx dy \\
 &= \int_{-1}^1 y^2 \left(\frac{1}{2} x^2 \Big|_0^{\sqrt{1-y^2}} \right) dy \\
 &= \int_{-1}^1 y^2 \frac{1}{2} (\sqrt{1-y^2})^2 dy \\
 &= \int_{-1}^1 \left(\frac{1}{2} y^2 - \frac{1}{2} y^4 \right) dy \\
 &= \frac{1}{6} y^3 \Big|_{-1}^1 - \frac{1}{10} y^5 \Big|_{-1}^1 \\
 &= \frac{1}{6} + \frac{1}{6} - \frac{1}{10} - \frac{1}{10} \\
 &= \frac{1}{3} - \frac{1}{5} \\
 &= \frac{5-3}{15} \\
 &= \boxed{\frac{2}{15}}
 \end{aligned}$$



§16.3 #21 Find volume under $z = xy$ and above triangle
Notice the region is $1 \leq x \leq 4$ and $1 \leq y \leq \frac{2}{3}(x-1)$



$$\begin{aligned}
 V &= \int_1^4 \int_1^{\frac{2}{3}(7-x)} xy dy dx \\
 &= \int_1^4 x \left(\frac{1}{2} y^2 \Big|_1^{\frac{2}{3}(7-x)} \right) dx \\
 &= \int_1^4 \frac{1}{2} x \left(\frac{1}{9}(7-x)^2 - 1 \right) dx \\
 &= \int_1^4 \frac{1}{18} x (49 - 14x + x^2 - 9) dx \\
 &= \int_1^4 \frac{1}{18} (40x - 14x^2 + x^3) dx \\
 &= \frac{1}{18} \left(20x^2 - \frac{14}{3}x^3 + \frac{1}{4}x^4 \Big|_1^4 \right) \\
 &= \frac{1}{18} (20(16-1) - \frac{14}{3}(64-1) + \frac{1}{4}(256-1)) \\
 &= \frac{1}{18} (300 - 14(21) + 64 - \frac{1}{4}) \\
 &= \frac{1}{18} (300 - 294 + 64 - \frac{1}{4}) = \frac{1}{18} (70 - \frac{1}{4}) = \boxed{\frac{31}{8}}
 \end{aligned}$$