

Math 231 Homework Project III: Vector Calculus:

Follow instructions. Be careful to answer all the questions raised in each part. Please turn in neat work with problems clearly labeled and your name on each page. Thanks. Some of these problems are hard. You are not alone. If you get stuck, phone a friend. Or email me, or stop by office hours. Start early though, otherwise we may not be able to resolve your questions in time. I plan for there to be 30 total problems. This assignment has 10 problems, this means it is worth 10pts of your final grade.

Remark: Cartesian coordinates are sufficient to describe almost any geometry, however the formulas can be needlessly complicated for certain cases. If we use coordinates which mirror the symmetry of the problem often complicated Cartesian calculations collapse to simple calculations in cylindrical or spherical coordinates. I have spent some effort to derive the gradient, curl and divergence in Spherical and Cylindrical coordinates, please make use of those results where it helps in the problems of this Homework Project.

PROBLEM 21: Let $f(x, y) = y^2 - x^2$. Plot four level curves of this function in the xy -plane; that is create a contour plot for this function. Calculate the ∇f . Then redraw or copy that plot and add little vectors that illustrate the direction of ∇f through-out representative points in your plot.

PROBLEM 22: Let $f(r, \theta) = r$. Plot four level curves of this function in the xy -plane with polar coordinates r, θ . Calculate ∇f in polar coordinates (see the notes for the formula, or think about Homework Project 18-19 where $s = r, \beta = \theta$). Then redraw the contour plot and add little vectors that illustrate the direction of ∇f throughout the plot.

PROBLEM 23: We argued in lecture that ∇F gives the normal vector field to $F(x, y, z) = k$. Use a two-dimensional analogue to that argument to show that ∇f is a field of vectors in the plane that are everywhere orthogonal to the level curves $f(x, y) = k$.

PROBLEM 24: Let

$$f(x, y, z) = \tan^{-1}\left(\frac{y}{x}\right) + \sin^{-1}\left(\frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}\right) + (x^2 + y^2 + z^2)^3.$$

Calculate ∇f and $\nabla^2 f$. There is an easy way to do this, and there is a hard way. Choose your path.

PROBLEM 25: Assume that $a, b, c > 0$. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Parametrize this surface via elliptical spherical coordinates β, γ where $0 \leq \beta \leq 2\pi$ and $0 \leq \gamma \leq \pi$ and

$$x = a \cos(\beta) \sin(\gamma), \quad y = b \sin(\beta) \sin(\gamma), \quad z = c \cos(\gamma).$$

Find the vector area element $d\vec{A}$ in terms of these modified spherical coordinates. Find the surface area of this ellipsoid. Assume that $Q_0 \in \mathbb{R}$. Calculate the flux of the vector field $\vec{E} = (4\pi abc Q_0) e_\rho$ through the ellipsoid. If \vec{E} is the electric field then what is the charge enclosed by the ellipsoid?

PROBLEM 26: Section 17.3#7, 13, 27.

PROBLEM 27: Section 17.4 #11 and 13.

PROBLEM 28: Section 17.7# 43 and 47.

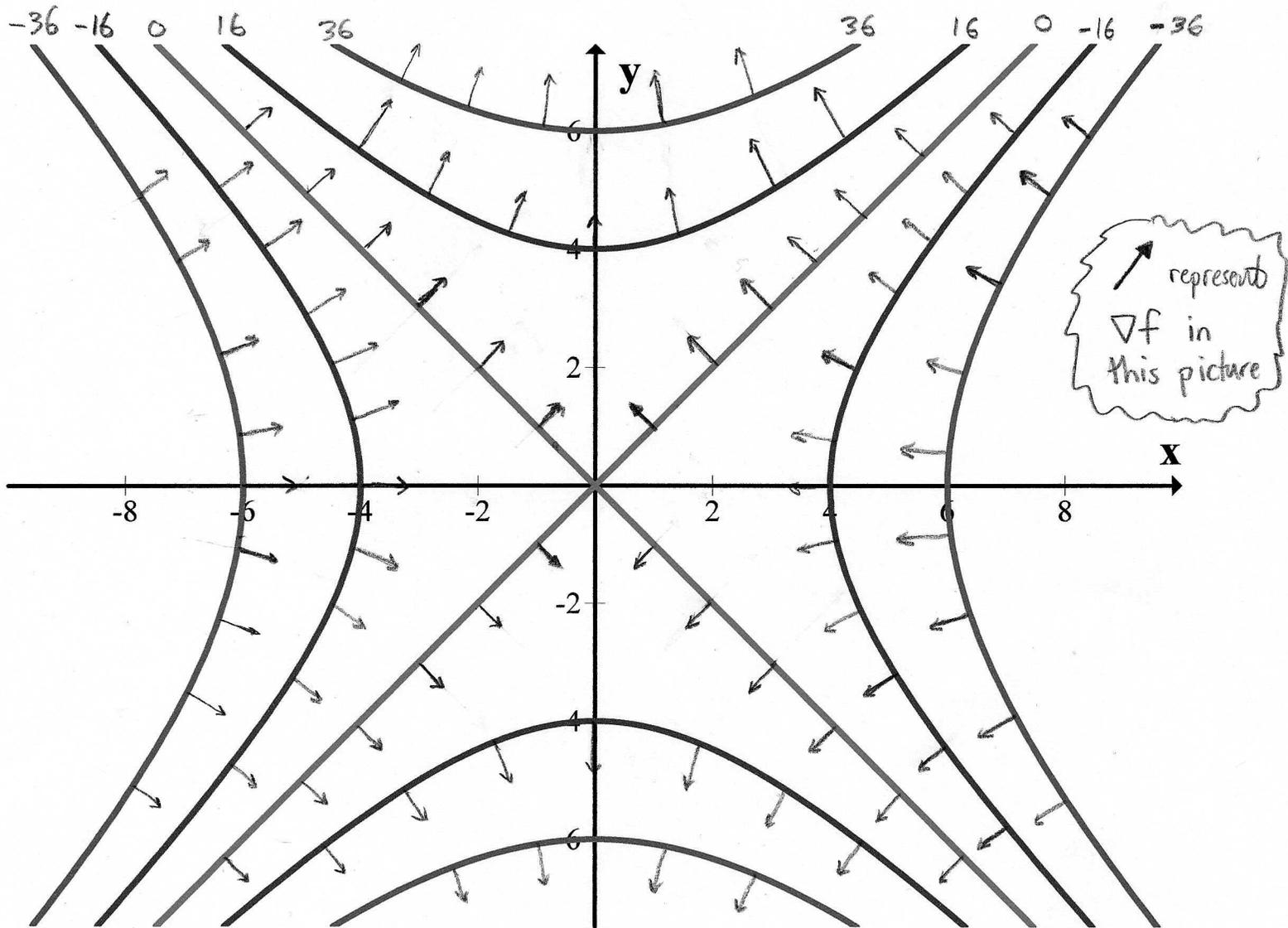
PROBLEM 29: Section 17.8#6 and 7.

PROBLEM 30: Section 17.9# 10 and 13.

HOMEWORK PROJECT III SOLUTION, CALCULUS III, FALL 2008

PROBLEM 21 Consider $f(x,y) = y^2 - x^2$. Plot several level curves for the function. Then plot the grad(f) along those curves. Note $\nabla f = \langle -2x, 2y \rangle = 2\langle -x, y \rangle$.

Below I've plotted $f(x,y) = y^2 - x^2 = k$ for the cases $k = 0, 16, 36, -16, -36$. The numbers at the top are the values of k along those branches. Each value gives two disconnected branches.



Sample Calculations for plotting ∇f

$$\nabla f(1,1) = 2\langle -1, 1 \rangle = \langle -2, 2 \rangle$$

$$\nabla f(0,4) = 2\langle 0, 4 \rangle = \langle 0, 8 \rangle$$

$$\nabla f(-1,-1) = 2\langle 1, -1 \rangle = \langle 2, -2 \rangle$$

$$\nabla f(-1,1) = \langle 2, 2 \rangle$$

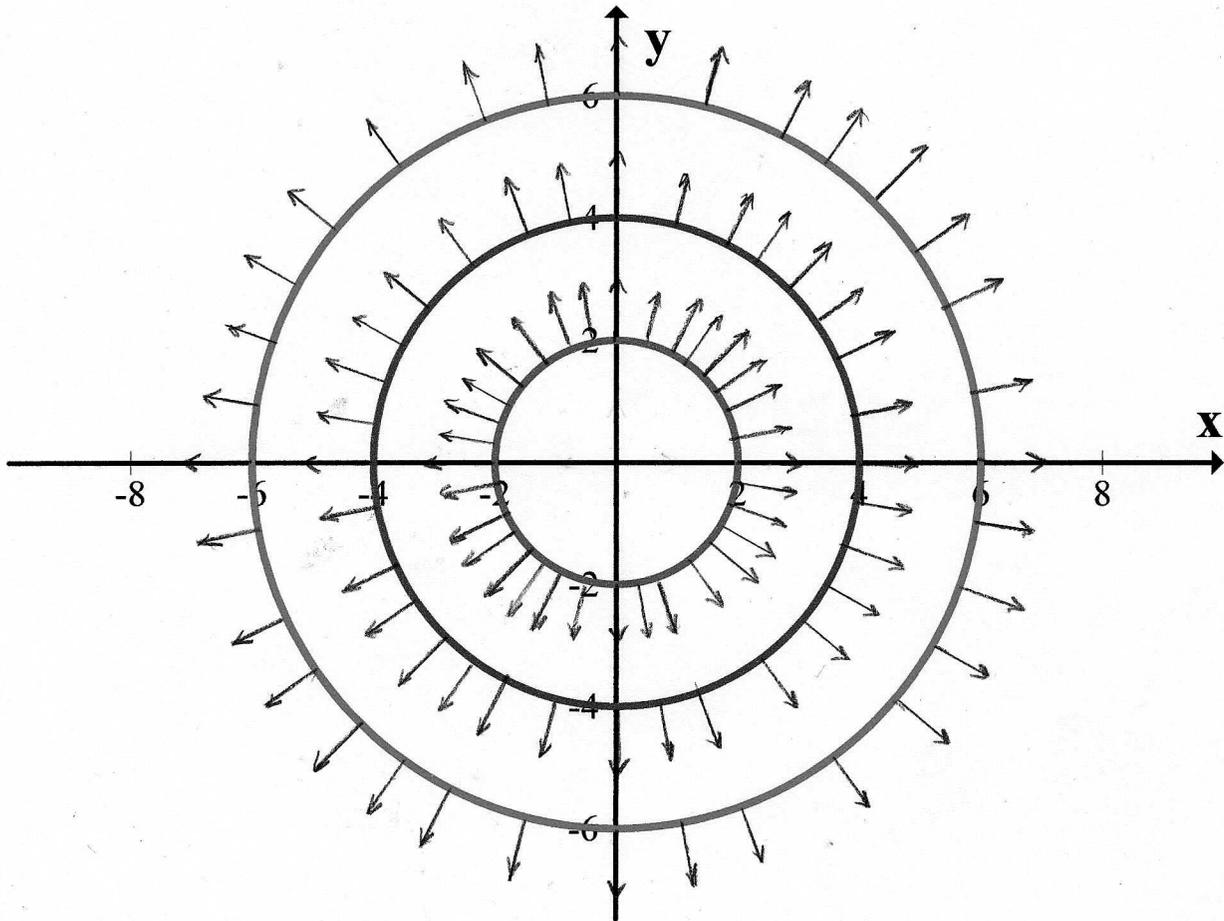
$$\nabla f(1,-1) = \langle -2, -2 \rangle$$

$$\nabla f(2,2) = \langle -4, 4 \rangle = 2\langle -2, 2 \rangle$$

I'll scale these down to make the picture pretty. In fact, I've given up on scale, the little ∇f arrows indicate direction only. Note $|\nabla f|(p) = 2|p|$. The length of ∇f is twice ρ .

PROBLEM 22 Consider $f(r, \theta) = r$ where $r = \sqrt{x^2 + y^2}$ and θ is the standard polar angle in the xy -plane. I've plotted several level curves for f below; $f(r, \theta) = r = k$ yield circles. I chose $k = 2, 4$ and 6 . Notice that

$$\nabla f = \left\langle \frac{\partial}{\partial x}(\sqrt{x^2 + y^2}), \frac{\partial}{\partial y}(\sqrt{x^2 + y^2}) \right\rangle = \frac{1}{\sqrt{x^2 + y^2}} \langle x, y \rangle = e_r$$
the length of ∇f is constant in this problem; $|\nabla f| = 1$.



Remark: We can graph $z = f(x, y)$ from the information given in the plots we've assembled in Problems 21 & 22. It's not hard to see Problem 22 gives a cone, problem 21 gives a hyperboloid

PROBLEM 23 Consider $f(x, y) = k$. Let $\vec{r}(t)$ parametrize this level curve, this means

$$f(\vec{r}(t)) = k \quad \text{for } \vec{r}(t) = \langle x(t), y(t) \rangle$$

Differentiate with respect to t ,

$$\begin{aligned} \frac{d}{dt} [f(\vec{r}(t))] &= \frac{d}{dt} [f(x(t), y(t))] \\ &= \frac{\partial f}{\partial x}(x(t), y(t)) \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t)) \frac{dy}{dt} \\ &= \nabla f(x(t), y(t)) \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \\ &= \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \end{aligned}$$

Note $\vec{r}(t)$ parametrizes the level curve so $\frac{d}{dt} [f(\vec{r}(t))] = \frac{d}{dt}(k) = 0$.

Thus, $\nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} = 0$. Thus ∇f is \perp to the tangents to the level curve, we can apply this to each level curve as we look at k varying. These curves will fill out the plane and ∇f will be \perp everywhere to the tangents of those level curves.

PROBLEM 24 $f(x, y, z) = \tan^{-1}\left(\frac{y}{x}\right) + \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right) + (x^2+y^2+z^2)^3$ then

$f(\rho, \theta, \phi) = \theta + \phi + \rho^6$. Now I'll use the notes where I derived the formula for ∇f in sphericals,

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial \rho} e_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} u_\phi + \frac{1}{\rho \sin \phi} \frac{\partial f}{\partial \theta} u_\theta \\ &= \boxed{6\rho^5 e_\rho + \frac{1}{\rho} u_\phi + \frac{1}{\rho \sin \phi} u_\theta}\end{aligned}$$

Next, $\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot (6\rho^5 e_\rho + \frac{1}{\rho} u_\phi + \frac{1}{\rho \sin \phi} u_\theta)$

So we identify $F_\rho = 6\rho^5$, $F_\phi = 1/\rho$ and $F_\theta = 1/\rho \sin \phi$

for $\vec{F} = \nabla f = F_\rho e_\rho + F_\phi e_\phi + F_\theta e_\theta$. Using my notes (379)

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} [\rho^2 F_\rho] + \frac{1}{\rho \sin \phi} \frac{\partial}{\partial \phi} [\sin \phi F_\phi] + \frac{1}{\rho \sin \phi} \frac{\partial F_\theta}{\partial \theta} \\ &= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} [6\rho^7] + \frac{1}{\rho \sin \phi} \frac{\partial}{\partial \phi} \left[\frac{\sin \phi}{\rho} \right] + \frac{1}{\rho \sin \phi} \frac{\partial}{\partial \theta} \left[\frac{1}{\rho \sin \phi} \right] \rightarrow 0 \\ &= \frac{42\rho^6}{\rho^2} + \frac{1}{\rho \sin \phi} \frac{\cos \phi}{\rho} \\ &= \boxed{42\rho^4 + \frac{\cos \phi}{\rho^2 \sin \phi} = \nabla^2 f}\end{aligned}$$

PROBLEM 25 Assume $a, b, c > 0$. Consider the ellipsoid
 $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. We parametrize the ellipsoid
 by the following where $0 \leq \gamma \leq \pi$ and $0 \leq \beta \leq 2\pi$,

$$\vec{X}(\gamma, \beta) = \langle a \cos \beta \sin \gamma, b \sin \beta \sin \gamma, c \cos \gamma \rangle$$

$$\frac{\partial \vec{X}}{\partial \gamma} = \langle a \cos \beta \cos \gamma, b \sin \beta \cos \gamma, -c \sin \gamma \rangle$$

$$\frac{\partial \vec{X}}{\partial \beta} = \langle -a \sin \beta \sin \gamma, b \cos \beta \sin \gamma, 0 \rangle$$

$$\frac{\partial \vec{X}}{\partial \gamma} \times \frac{\partial \vec{X}}{\partial \beta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos \beta \cos \gamma & b \sin \beta \cos \gamma & -c \sin \gamma \\ -a \sin \beta \sin \gamma & b \cos \beta \sin \gamma & 0 \end{vmatrix}$$

$$= \langle -bc \cos \beta \sin^2 \gamma, ac \sin \beta \sin^2 \gamma, ab (\cos^2 \beta \sin \gamma \cos \gamma + \sin^2 \beta \sin \gamma \cos \gamma) \rangle$$

$$= \sin \gamma \langle bc \cos \beta \sin \gamma, ac \sin \beta \sin \gamma, ab \cos \gamma \rangle$$

Thus $d\vec{S} = \sin \gamma \langle bc \cos \beta \sin \gamma, ac \sin \beta \sin \gamma, ab \cos \gamma \rangle d\beta d\gamma$

To find surface area we integrate $dS = |d\vec{S}|$ over the ellipsoid.

$$|d\vec{S}| = \sin \gamma \sqrt{b^2 c^2 \cos^2 \beta \sin^2 \gamma + a^2 c^2 \sin^2 \beta \sin^2 \gamma + a^2 b^2 \cos^2 \gamma} d\beta d\gamma$$

Thus we find

$$A_S = \int_0^{2\pi} \int_0^\pi \sin \gamma \sqrt{c^2 \sin^2 \gamma (b^2 \cos^2 \beta + a^2 \sin^2 \beta) + a^2 b^2 \cos^2 \gamma} d\gamma d\beta$$

In the special case $a = b = c = R$ we can calculate this integral,

$$A_S = \int_0^{2\pi} \int_0^\pi R^2 \sin \gamma d\gamma d\beta$$

$$= 2\pi R^2 \int_0^\pi \sin \gamma d\gamma$$

$$= 2\pi R^2 (-\cos(\pi) + \cos(0))$$

$$= \underline{4\pi R^2}. \quad (\text{surface area of sphere of radius } R)$$

PROBLEM 25 Continued

Calculate the flux of $\vec{E} = Qz \hat{k}$ through the ellipsoid.

$$\begin{aligned} \iint_S \vec{E} \cdot d\vec{S} &= \iint_S Qz \sin\gamma \langle 0, 0, 1 \rangle \cdot \langle bc \cos\beta \sin\gamma, ac \sin\beta \sin\gamma, ab \cos\gamma \rangle d\beta d\gamma \\ &= \int_0^{2\pi} \int_0^\pi (Qz \sin\gamma) (ab \cos\gamma) d\gamma d\beta \quad (\text{where } z = c \cos\gamma) \\ &= \int_0^{2\pi} \int_0^\pi Qabc \cos^2\gamma \sin\gamma d\gamma d\beta \\ &= 2\pi Qabc \int_1^{-1} u^3 (-du) \quad \leftarrow \begin{array}{|l} u = \cos\gamma \\ du = -\sin\gamma d\gamma \end{array} \begin{array}{|l} u(0) = 1 \\ u(\pi) = -1 \end{array} \\ &= \frac{2\pi Qabc}{3} u^3 \Big|_{-1}^1 \\ &= \boxed{\frac{4\pi Qabc}{3}} \end{aligned}$$

If \vec{E} is the electric field then $\Phi_{\vec{E}} = \text{flux of } \vec{E} \text{ (relative to the ellipsoid)}$ gives the charge enclosed divided by ϵ_0

$$\Phi_{\vec{E}}(S) = \iint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{4\pi Qabc}{3}$$

$$\therefore \boxed{Q_{\text{enc}} = \frac{4\pi Qabc \epsilon_0}{3}}$$

Notice that $\nabla \cdot \vec{E} = \frac{\partial}{\partial z} (Qz) = Q$ we can check my answer with the assistance of the Divergence Th^m

$$\iint_S \vec{E} \cdot d\vec{S} = \iiint_V (\nabla \cdot \vec{E}) dV = Q \iiint_V dV = Q \left(\frac{4}{3} \pi abc \right)$$

\vec{E}
with
 $\partial E = S$

Remark: the field $\vec{E} = (4\pi abc Q_0) \hat{e}_r$ is not so easy to calculate $\iint_S \vec{E} \cdot d\vec{S}$, I changed it to make it reasonable. (need $a=b=c=R$ to make other reasonable)

PROBLEM 26, §17.3# 7, 13, 27

§17.3#7] Is the following vector field conservative?

$$\vec{F}(x, y, z) = \langle ye^x + \sin(y), e^x + x \cos(y) \rangle$$

Yes, because $\vec{F} = \nabla f$ for $f(x, y) = ye^x + x \sin(y)$.

How did I find f ? Simple, integrate and check. (see my notes for a more systematic method.)

§17.3#13] Calculate $\int_C \vec{F} \cdot d\vec{r}$ along the

$$\text{curve } C: \vec{r}(t) = \langle t + \sin\left(\frac{\pi t}{2}\right), t + \cos\left(\frac{\pi t}{2}\right) \rangle \quad 0 \leq t \leq 1$$

for $\vec{F}(x, y) = \langle xy^2, x^2y \rangle$. Notice that C goes from $\vec{r}(0) = \langle 0, 0 \rangle$ to $\vec{r}(1) = \langle 2, 1 \rangle$. Moreover,

$$\vec{F} = \nabla f \quad \text{for} \quad f(x, y) = \frac{1}{2}x^2y^2$$

Thus by FTC for line integrals,

$$\int_C \vec{F} \cdot d\vec{r} = f(2, 1) - f(0, 0) = \frac{1}{2}(4)(1) = \boxed{2}$$

§17.3#27] Show that if $\vec{F} = \langle P, Q, R \rangle$ is conservative and P, Q, R have continuous 1st order partial derivatives then

Proof: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$.

Since \vec{F} is conservative, there exists f with $\vec{F} = \nabla f$ thus,

$$P = \frac{\partial f}{\partial x} \quad Q = \frac{\partial f}{\partial y} \quad R = \frac{\partial f}{\partial z}$$

Then Clairaut's Th^m applies as the partials are continuous

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial Q}{\partial x}$$

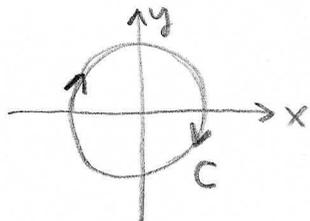
$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z} \right) = \frac{\partial R}{\partial x}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} \right) = \frac{\partial R}{\partial y}$$

PROBLEM 27; §17.4#11 #13

§17.4#11, See Homework 15 S.1².

§17.4#13 | $F(x, y) = \langle e^x + x^2y, e^y - xy^2 \rangle$. Calculate $\int_C \vec{F} \cdot d\vec{r}$ where C is the circle $x^2 + y^2 = 25$ oriented clockwise.



need to apply Green's Th^m to $-C$ which has positive orientation.

$$\int_{-C} \vec{F} \cdot d\vec{r} = \iint_{x^2+y^2 \leq 25} \left[\frac{\partial}{\partial x}(e^y - xy^2) - \frac{\partial}{\partial y}(e^x + x^2y) \right] dA$$

$$= \iint_{x^2+y^2 \leq 25} [-y^2 - x^2] dA, \quad \text{use polar coordinates } dA = r dr d\theta.$$

$$= \int_0^{2\pi} \int_0^5 -r^3 dr d\theta$$

$$= -2\pi \frac{r^4}{4} \Big|_0^5$$

$$= -\frac{\pi}{2} 5^4$$

$$= -\frac{625\pi}{2}$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \frac{625\pi}{2}$$

PROBLEM 28, §17.7 #43 & 47

§17.7#43 | Use Gauss' Law to find charge enclosed in the solid hemisphere $x^2 + y^2 + z^2 \leq a^2$ for $z \geq 0$, given the electric field \vec{E} is

$$\vec{E}(x, y, z) = \langle x, y, 2z \rangle$$

$$\nabla \cdot \vec{E} = 1 + 1 + 2 = 4 = \rho / \epsilon_0 = \frac{1}{\epsilon_0} \frac{dQ}{dV}$$

$$Q_{\text{enc}} = 4\epsilon_0 \iiint_V dV$$

we've calculate the volume of a sphere is $\frac{4}{3}\pi a^3$ thus we take $\frac{1}{2}$ of $\frac{4}{3}$ that here

$$= 4\epsilon_0 \left(\frac{2}{3} \pi a^3 \right)$$

$$= \boxed{\frac{8\pi\epsilon_0 a^3}{3}} = Q_{\text{enc}}$$

Remark: frankly $\vec{E} = \langle x, y, 2z \rangle$ is nonsense. If we're putting in ϵ_0 and pretending to care about units then we ought to have $\vec{E} = \epsilon_0 \langle x, y, 2z \rangle$ where the units of ϵ_0 are in Newtons/coulombs (Electric Fields have $\frac{N}{C}$ units so $\vec{F} = q\vec{E}$ makes sense)

§17.7#47 | Let $\vec{F} = \frac{c}{r^3} \vec{r}$ for c a constant and $\vec{r} = \langle x, y, z \rangle$.

Show that $\iint_{S_R} \vec{F} \cdot d\vec{S} = 4\pi c$ for sphere of radius R centered at $(0, 0, 0)$ is independent of the radius R .

$$\iint_{S_R} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^\pi \left(\frac{c\vec{r}}{r^3} \cdot R^2 \sin\theta d\theta d\phi \hat{r} \right) \Big|_{r=R}$$

$d\vec{S} = R^2 \sin\theta d\theta d\phi \hat{r}$
physics notation

$$= \int_0^{2\pi} \int_0^\pi c R^2 \sin\theta \frac{R}{R^3} (\hat{r} \cdot \hat{r}) d\theta d\phi$$

$\vec{r} = r \hat{r} = R \hat{r}$
on sphere of radius R

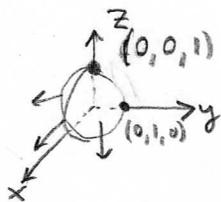
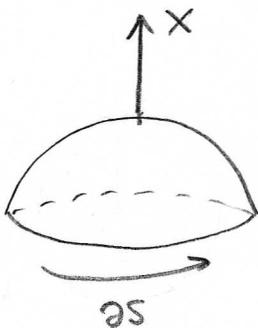
$$= c \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta$$

$$= \boxed{4\pi c} \text{ independent of } R.$$

Remark: go see (383) - (384) on why $0 \leq \theta \leq \pi$ & $0 \leq \phi \leq 2\pi$ in my solⁿ here.

PROBLEM 29, §17.8 #6 & 7

§17.8 #6 Calculate $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$ for $\vec{F}(x,y,z) = \langle e^{xy} \cos(z), x^2 z, xy \rangle$
 for S the hemisp here $x = \sqrt{1-y^2-z^2}$ a.k.a $x^2+y^2+z^2=1, x \geq 0$.

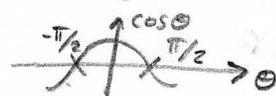


$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
 $0 \leq \phi \leq \pi$ } we can check that these reproduce $x \geq 0$

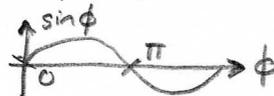
just checking my geometrically motivated claim.

Remember $x = \cos \theta \sin \phi$
 $y = \sin \theta \sin \phi$
 $z = \cos \phi$

Now $\cos \theta \geq 0$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

just look at 

and $\sin \phi \geq 0$ for $0 \leq \phi \leq \pi$



Let's use Stoke's Th^m,

$$\begin{aligned} \iint_S \nabla \times \vec{F} &= \int_{\partial S} \vec{F} \cdot d\vec{r} \\ &= \int_0^{2\pi} (\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}) dt \end{aligned}$$

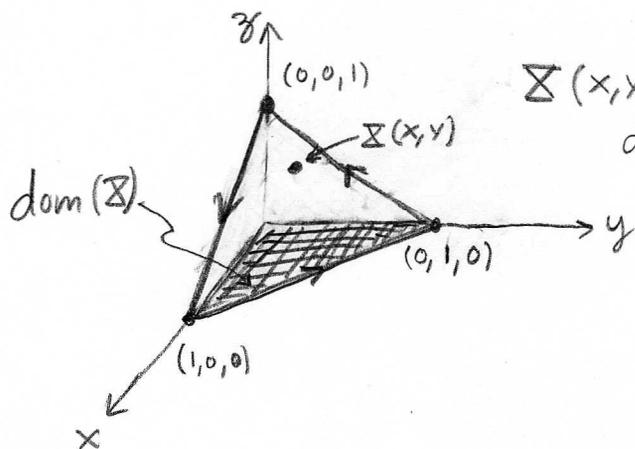
$\partial S \mid \vec{r}(t) = \langle 0, \cos t, \sin t \rangle$ $0 \leq t \leq 2\pi$
 note $\vec{r}(0) = \langle 0, 1, 0 \rangle$ & $\vec{r}(\pi/2) = \langle 0, 0, 1 \rangle$
 so ∂S has desired orientation.
 We have $x=0, y=\cos t, z=\sin t$

$$= \int_0^{2\pi} \langle \cos(\sin t), 0, 0 \rangle \cdot \langle 0, -\sin t, \cos t \rangle dt$$

$$= \int_0^{2\pi} (0) dt$$

$$= \boxed{0}$$

§17.8#7] Calculate $\int_C \vec{F} \cdot d\vec{r}$ where C is oriented counter-clockwise as viewed from above. Let C be the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and $\vec{F}(x, y, z) = \langle x+y^2, y+z^2, z+x^2 \rangle$



$$\Sigma(x, y) = \langle x, y, 1-x-y \rangle \text{ with } \text{dom}(\Sigma) = 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1-x.$$

[this parametrization follows from observing $x+y+z=1$ is the given plane. You can check $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$ are on this plane.]

Calculate

$$\begin{aligned} \nabla \times \vec{F} &= \left\langle \frac{\partial}{\partial y}(z+x^2) - \frac{\partial}{\partial z}(y+z^2), \frac{\partial}{\partial z}(x+y^2) - \frac{\partial}{\partial x}(z+x^2), \frac{\partial}{\partial x}(y+z^2) - \frac{\partial}{\partial y}(x+y^2) \right\rangle \\ &= \langle -2z, -2x, -2y \rangle \end{aligned}$$

We should calculate $d\vec{S}$ for this triangle,

$$d\vec{S} = \left(\frac{\partial \Sigma}{\partial x} \times \frac{\partial \Sigma}{\partial y} \right) dx dy = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} dx dy = \langle 1, 1, 1 \rangle dx dy$$

Thus, letting S' be the triangle with $\partial S' = C$,

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_{S'} (\nabla \times \vec{F}) \cdot d\vec{S} \\ &= \int_0^1 \int_0^{1-x} \langle -2z, -2x, -2y \rangle \cdot \langle 1, 1, 1 \rangle dy dx, \quad \underline{z=1-x-y} \\ &= \int_0^1 \int_0^{1-x} -2(z+x+y) dy dx, \quad \underline{z=1-x-y} \\ &\quad \text{on } S'. \\ &= \int_0^1 \int_0^{1-x} -2(1-x-y+x+y) dy \\ &= \int_0^1 \int_0^{1-x} -2 dy dx \\ &= \int_0^1 -2(1-x) dx \\ &= \int_0^1 (2x-2) dx = (x^2-2x) \Big|_0^1 = \boxed{-1} \end{aligned}$$

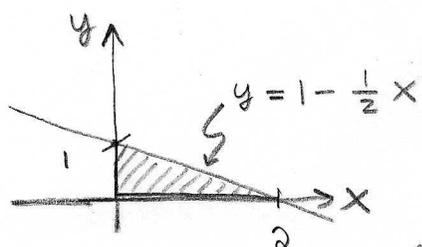
PROBLEM 30, §17.9 #10 \$13

§17.9 #10) Calculate $\iint_S \vec{F} \cdot d\vec{S}$,

$\vec{F}(x,y,z) = \langle x^2y, xy^2, 2xyz \rangle$ for S the surface of the tetrahedron bounded by the planes $x=0, y=0, z=0, x+2y+z=2$. Calculate the divergence of \vec{F} , $\nabla \cdot \vec{F} = 2xy + 2xy + 2xy = 6xy$. Apply the Divergence Th^m; notice $S = \partial E$ for E the solid tetrahedron,

$$z = 2 - x - 2y \quad \rightarrow \quad 0 \leq z \leq 2 - x - 2y.$$

Now $z=0$ has $2 - x - 2y = 0$
 $x = 2 - 2y$ thus $0 \leq x \leq 2 - 2y$



from picture, $0 \leq y \leq 1$.

Then, using $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{F}) dV$ we find,

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} 6xy \, dz \, dx \, dy$$

$$= \int_0^1 \int_0^{2-2y} 6xy(2-x-2y) \, dx \, dy$$

$$= \int_0^1 \int_0^{2-2y} (12y(1-y)x - 6x^2y) \, dx \, dy$$

$$= \int_0^1 (6y(1-y)(2-2y)^2 - 2(2-2y)^3 y) \, dy$$

$$= \int_0^1 (24y(1-y)^3 - 16y(1-y)^3) \, dy$$

$$= \int_0^1 8y(1-y)^3 \, dy \quad \begin{array}{l} u = 1-y \quad u(0) = 1 \\ y = 1-u \quad u(1) = 0 \end{array}$$

$$= \int_1^0 8(1-u)u^3 (-du)$$

$$= \int_0^1 (8u^3 - 8u^4) du = \left(2u^4 - \frac{8}{5}u^5 \right) \Big|_0^1 = 2 - \frac{8}{5} = \boxed{\frac{2}{5}}$$

§17.9#13 Calculate $\iint_S \vec{F} \cdot d\vec{S}$ for $\vec{F}(x,y,z) = \langle 4x^3z, 4y^3z, 3z^4 \rangle$
for S a sphere with radius R centered on the origin.

$$\begin{aligned}\nabla \cdot \vec{F} &= 12x^2z + 12y^2z + 12z^3 \\ &= 12z(x^2 + y^2 + z^2) \\ &= 12\rho^3 \cos\phi\end{aligned}$$

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^\pi \int_0^R (12\rho^3 \cos\phi) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= 24\pi \int_0^\pi \cos\phi \sin\phi \, d\phi \int_0^R \rho^5 \, d\rho \\ &= 12\pi \int_0^\pi \sin(2\phi) \, d\phi \left(\frac{1}{6} R^6\right) \\ &= 2\pi R^6 \left(\frac{-1}{2} \cos(2\phi)\right) \Big|_0^\pi \\ &= 2\pi R^6 \left(-\frac{1}{2} \cos(2\pi) + \frac{1}{2} \cos(0)\right) \\ &= \boxed{0}\end{aligned}$$

Remark

$$\iiint_S 12z(x^2+y^2+z^2) = \int_{-1}^1 \underbrace{\left(\iint_{x^2+y^2 \leq 1} 12z(x^2+y^2+z^2) \, dx \, dy\right)}_{g(z)} \, dz = 0.$$

clearly $g(-z) = -g(z)$