

This chapter is on the "fundamental properties of Lie Groups". While this text is not mathematically rigorous it does have many interesting & quite physical calculations. My goal is to understand what is done in this text then try to compare/contrast it to the more abstract treatment of Lie Groups I've had from DR. FULP.

Example: Rotation Group in 3 dimensions

Its a group of operators, a typical element is

$$\hat{U}_R(\vec{\phi}) = \exp(-i\phi_\mu \hat{J}_\mu) \quad \mu = 1, 2, 3.$$

$$= \exp(-i\vec{\phi} \cdot \vec{J})$$

Thus the group element is given by 3-parameters $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and three fundamental operators $\vec{J} = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$. This is a Lie Group which is defined as a continuous group whose elements depend smoothly on its parameters (see pg. 37 of Greiner).

Differentiate $\hat{U}_R(\vec{\phi})$ to obtain, setting $\phi = 0$.

$$\frac{\partial}{\partial \phi_\nu} [\hat{U}_R(\vec{\phi})] = \frac{\partial}{\partial \phi_\nu} (e^{-i\phi_\nu \hat{J}_\nu}) = e^{-i\phi_\nu \hat{J}_\nu} \frac{\partial}{\partial \phi_\nu} (-i\phi_\nu \hat{J}_\nu) = e^{-i\phi_\nu \hat{J}_\nu} \frac{\partial \phi_\nu}{\partial \phi_\nu} (-i\hat{J}_\nu)$$

$$\Rightarrow \boxed{\hat{J}_\nu = i \left. \frac{\partial \hat{U}_R}{\partial \phi_\nu} \right|_{\vec{\phi}=0}} \quad \left(\begin{array}{l} \text{Because } \frac{\partial \phi_\nu}{\partial \phi_\nu} = \delta_\nu^\nu \text{ and} \\ \hat{U}_R(0) = e^0 = 1. \end{array} \right)$$

• Remark: Its a useful exercise to explicitly calculate $\hat{J}_k = -i \epsilon_{ijk} x^i \frac{\partial}{\partial x^j}$ from the geometrically obvious

rotations in 3-dimensions. Taking $\phi_1 = \text{rotation } \Delta \text{ about } x\text{-axis}$, $\phi_2 = \text{rot. } \Delta \text{ about } y\text{-axis}$ and $\phi_3 = \text{rot. } \Delta \text{ about } z\text{-axis}$.

Defⁿ A continuous group of operators $\hat{U}(\alpha_1, \alpha_2, \dots, \alpha_n; \vec{r})$ which depend on n parameters (coordinates) are called a Lie Group if their elements depend analytically on the n -parameters, and the coordinate \vec{r} (we'll drop the \vec{r} soon.).

Proposition: Let $\hat{U}(0) = \mathbb{1}$ then we can write $\hat{U}(\alpha_1, \alpha_2, \dots, \alpha_n; r) = \exp(-i\alpha_\mu \hat{L}_\mu)$ where $\alpha_\mu \hat{L}_\mu \equiv \sum_{i=1}^n \alpha_i \hat{L}_i$ and the \hat{L}_i are yet unknown operators, we note that $\hat{U} = e^{-i\alpha_\mu \hat{L}_\mu} \Rightarrow \hat{L}_\mu = i \left. \frac{\partial \hat{U}}{\partial \alpha_\mu} \right|_{\alpha=0}$. The \hat{L}_μ are called the generators of the group.

Proof: Define generators to be (easy to see^{why.} in view of comments above)

$$L_\mu = i \left. \frac{\partial \hat{U}}{\partial \alpha_\mu} \right|_{\alpha=0}$$

Then we integrate to find that (meaning solve this differential equation which has solⁿ)

$$U = c_1 e^{-i\alpha_\nu \hat{L}_\nu}$$

we require that $U(0) = \mathbb{1}$, $\Rightarrow c_1 = \mathbb{1}$. Thus

$\hat{U} = \exp(-i\alpha_\mu \hat{L}_\mu)$	← group element
$\hat{L}_\mu \equiv i \left. \frac{\partial \hat{U}}{\partial \alpha_\mu} \right _{\alpha=0}$	← generator

Alternative "Proof": on next page.

Prop: $\hat{U}(\vec{\alpha}, \vec{r}) = \exp(-i \alpha_\mu \hat{L}_\mu)$

Alternative Proof: (following Greiner)

An infinitesimal group transformation near $\mathbb{1}$ is,

$$\hat{U}(\delta\alpha_\mu) = \hat{U}(0) + \left. \frac{\partial \hat{U}}{\partial \alpha_\mu} \right|_{\alpha=0} \delta\alpha_\mu$$

Now again define $\hat{L}_\mu \equiv i \left. \frac{\partial \hat{U}}{\partial \alpha_\mu} \right|_{\alpha=0}$ to obtain using $\hat{U}(0) = \mathbb{1}$,

$$\begin{aligned} \hat{U}(\delta\alpha_\mu) &= \mathbb{1} = i \hat{L}_\mu \delta\alpha_\mu \\ &\equiv \mathbb{1} + d\hat{A} \end{aligned}$$

Now write, for $N \in \mathbb{N}$.

$$d\hat{A} \cong \frac{\hat{A}}{N} = \frac{-i \hat{L}_\mu \alpha_\mu}{N}$$

If we perform N infinitesimal transformations $\hat{U}(\delta\alpha_\mu)$ in order to obtain a finite transformation $\hat{U}(\alpha_\mu)$ this means,

$$\begin{aligned} \hat{U}(\alpha_\mu) &= \lim_{N \rightarrow \infty} \left[\mathbb{1} + \frac{\hat{A}}{N} \right]^N \\ &= e^{\hat{A}} \\ &= \exp(-i \hat{L}_\mu \alpha_\mu) \end{aligned}$$

not the usual defⁿ of the matrix exponential. is this obvious?

Perhaps this proof is better in that it is more constructive, but I'm not convinced.

On (3) we used a 1st order argument to elucidate that $U = \exp(-i\alpha\hat{L}_\mu)$ for $\hat{L}_\mu = i \frac{\partial U}{\partial \alpha_\mu} \Big|_{\alpha=0}$. Now we examine some 2nd order features of the group,

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$$\hat{U}(\delta\alpha_\mu) = \mathbb{1} - i\delta\alpha_\mu \hat{L}_\mu - \frac{1}{2}\delta\alpha_\mu \delta\alpha_\nu \hat{L}_\mu \hat{L}_\nu$$

Where we have utilized the 2nd order Taylor Expansion for a function of many-variables.

Proposition: $\{\hat{L}_\mu\}$ is a linearly independent set.

Proof: Assume otherwise, $L_\mu \delta_\mu = 0$ for $\delta_\mu \neq 0$ then \exists at least some $\delta_0 \neq 0$ for which $L\delta_0 = 0$ yet $L \cdot 0 = 0$ hence we note

$$e^{-iL\delta_0} = e^0 = \mathbb{1} \quad \neq \quad e^{L \cdot 0} = e^0 = \mathbb{1}.$$

However $e^A = e^B \Leftrightarrow A = B$ thus $e^{-iL\delta_0} \neq e^{L \cdot 0}$ are distinct group operators \Rightarrow there is more than one identity for the group, but this is a contradiction we know from elementary group theory course that the group identity is unique.

Proposition: If $\hat{U}(\alpha_\mu)$ is unitary then \hat{L}_μ are hermitian. That is $\hat{U}^\dagger = \hat{U}^{-1} \Rightarrow (\hat{L}_\mu)^\dagger = \hat{L}_\mu$

Proof: (first order)

$$\hat{U}(\delta\alpha_\mu) = 1 - i\delta\alpha_\mu \hat{L}_\mu$$

$$\hat{U}^\dagger(\delta\alpha_\mu) = 1 + i\delta\alpha_\mu \hat{L}_\mu^\dagger$$

We can show that $\hat{U}^{-1}(\delta\alpha_\mu) = \hat{U}(-\delta\alpha_\mu) = 1 + i\delta\alpha_\mu \hat{L}_\mu$ a number of ways,

$$\textcircled{1} \quad \hat{U}(\delta\alpha_\mu) \hat{U}(-\delta\alpha_\mu) = (1 - i\delta\alpha_\mu \hat{L}_\mu)(1 + i\delta\alpha_\mu \hat{L}_\mu) = \mathbb{1} \quad (1^{\text{st}} \text{ order})$$

$$\textcircled{2} \quad \hat{U}(\delta\alpha_\mu) \hat{U}(-\delta\alpha_\mu) = e^{\delta\alpha_\mu \hat{L}_\mu} e^{-\delta\alpha_\mu \hat{L}_\mu} = e^{\delta\alpha_\mu \hat{L}_\mu - \delta\alpha_\mu \hat{L}_\mu + \frac{1}{2} [\delta\alpha_\mu \hat{L}_\mu, \delta\alpha_\mu \hat{L}_\mu] + \dots} = e^0 = \mathbb{1} \quad (2^{\text{nd}} \text{ order})$$

Thus we find that by unitarity $= e^0 = \mathbb{1}$

$$1 + i\delta\alpha_\mu \hat{L}_\mu^\dagger = 1 + \delta\alpha_\mu \hat{L}_\mu$$

$$\Rightarrow \boxed{\hat{L}_\mu^\dagger = \hat{L}_\mu}$$

generator are hermitian

Note: Unitarity $U^{-1} = U^\dagger$ makes it very easy to calculate the inverse of a given group element, we just take the hermitian conjugate. To 2nd order from (4) we obtain

(5)

$$\hat{U}^{-1}(\delta\alpha_\mu) = 1 + i\delta\alpha_\mu \hat{L}_\mu - \frac{1}{2} \delta\alpha_\mu \delta\alpha_\nu \hat{L}_\mu \hat{L}_\nu$$

Taking the parameters to be real $\alpha_\mu^* = \alpha_\mu$.

Now consider the following product in view of the above,

$$\begin{aligned} & U^{-1}(\delta\beta) U^{-1}(\delta\alpha) U(\delta\beta) U(\delta\alpha) = \\ & = \left(1 + i\delta\beta_i L_i - \frac{1}{2} \delta\beta_j \delta\beta_k L_j L_k\right) \left(1 + i\delta\alpha_\ell L_\ell - \frac{1}{2} \delta\alpha_m \delta\alpha_n L_m L_n\right) \cdot \\ & \quad \cdot \left(1 - i\delta\beta_i L_i - \frac{1}{2} \delta\beta_j \delta\beta_k L_j L_k\right) \left(1 - i\delta\alpha_\ell L_\ell - \frac{1}{2} \delta\alpha_m \delta\alpha_n L_m L_n\right) \\ & = \left(1 + i\delta\beta_i L_i + i\delta\alpha_\ell L_\ell - \frac{1}{2} \delta\beta_j \delta\beta_k L_j L_k - \frac{1}{2} \delta\alpha_m \delta\alpha_n L_m L_n - \delta\beta_i \delta\alpha_\ell L_i L_\ell\right) \\ & \quad \cdot \left(1 - i\delta\beta_i L_i - i\delta\alpha_\ell L_\ell - \frac{1}{2} \delta\beta_j \delta\beta_k L_j L_k - \frac{1}{2} \delta\alpha_m \delta\alpha_n L_m L_n - \delta\beta_i \delta\alpha_\ell L_i L_\ell\right) \\ & = 1 + \cancel{i\delta\beta_i L_i} + \cancel{i\delta\alpha_\ell L_\ell} - \frac{1}{2} \delta\beta_i \delta\beta_k \cancel{L_j L_k} - \frac{1}{2} \delta\alpha_m \delta\alpha_n \cancel{L_m L_n} - \delta\beta_i \delta\alpha_\ell L_i L_\ell \\ & \quad - \cancel{i\delta\beta_i L_i} - \cancel{i\delta\alpha_\ell L_\ell} - \frac{1}{2} \delta\beta_j \delta\beta_k \cancel{L_j L_k} - \frac{1}{2} \delta\alpha_m \delta\alpha_n \cancel{L_m L_n} - \delta\beta_i \delta\alpha_\ell L_i L_\ell \\ & \quad + \delta\beta_i \delta\beta_j L_i L_j + \delta\alpha_\ell \delta\alpha_k L_\ell L_k + \delta\beta_i \delta\alpha_\ell L_i L_\ell \\ & \quad + \delta\alpha_\ell \delta\beta_i L_\ell L_i \\ & = 1 - \delta\beta_k \delta\alpha_\ell L_k L_\ell - \delta\beta_k \delta\alpha_\ell L_k L_\ell \\ & \quad + \delta\beta_k \delta\alpha_\ell L_k L_\ell + \delta\beta_k \delta\alpha_\ell L_\ell L_k \\ & = 1 + \delta\alpha_k \delta\beta_m (-L_m L_k + L_k L_m) \\ & = 1 + \delta\alpha_k \delta\beta_m [\hat{L}_k, \hat{L}_m] \end{aligned}$$

Continuing discussion of (5)

Because $U^{-1}(\delta\beta), U^{-1}(\delta\alpha), U(\delta\beta), U(\delta\alpha)$ are all in the group it follows that their product is as well, lets say $U(\delta\gamma)$ is that element, then,

$$\hat{U}(\delta\gamma_r) = \exp(-i\delta\gamma_r \hat{L}_r) = 1 - i\delta\gamma_r \hat{L}_r + \dots$$

Setting $\hat{U}(\delta\gamma_r) = U^{-1}(\delta\beta)U^{-1}(\delta\alpha)U(\delta\beta)U(\delta\alpha)$ and using (5) we find that,

$$1 - i\delta\gamma_r \hat{L}_r = 1 + \delta\alpha_k \delta\beta_m (\hat{L}_k \hat{L}_m - \hat{L}_m \hat{L}_k)$$

Which means that,

$$\delta\alpha_k \delta\beta_m (L_k L_m - L_m L_k) = -i\delta\gamma_r \hat{L}_r$$

Consider then that if we put $\delta\gamma_j$ equal to the following

$$-i\delta\gamma_j = C_{kmj} \delta\alpha_k \delta\beta_m$$

$$\delta\gamma_j \rightarrow 0 \text{ if } \delta\alpha_k \rightarrow 0 \text{ or } \delta\beta_m \rightarrow 0$$

$$\Rightarrow \delta\gamma_j \propto (\delta\alpha_k \delta\beta_m)$$

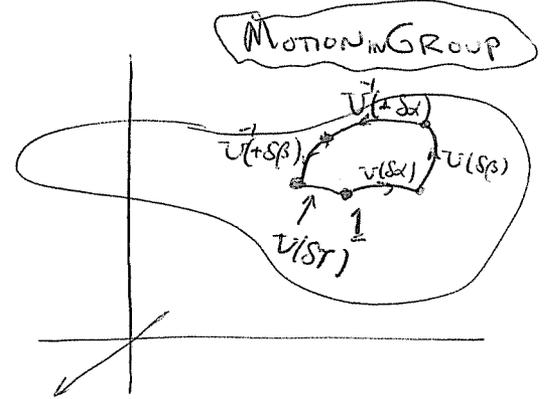
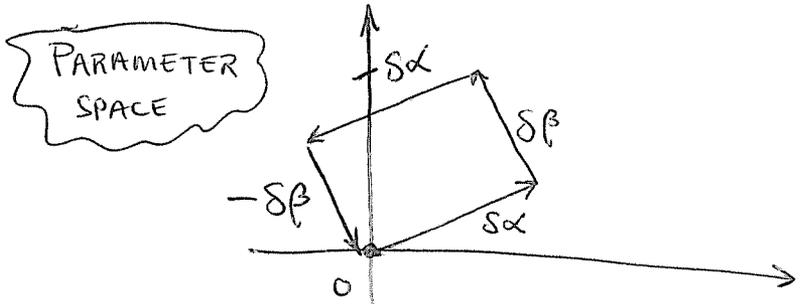
We find that the group properties gives us the closure of the commutator onto the set of generators.

$$\boxed{[\hat{L}_k, \hat{L}_m] = C_{kmj} \hat{L}_j}$$

Where C_{kmj} are the structure constants and the

$$[\hat{L}_i, \hat{L}_j] = -[\hat{L}_j, \hat{L}_i] \Rightarrow C_{kmj} = -C_{mki}$$

antisymmetry of the 1st two indices is revealed by the calculation above.



Some useful identities for the matrix exponential

I.) $e^A e^B = e^{A+B}$ given that $[A, B] = 0$.

$$e^A e^B = \left(\sum_{k=0}^{\infty} \frac{A^k}{k!} \right) \left(\sum_{j=0}^{\infty} \frac{B^j}{j!} \right)$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{A^k}{k!} \frac{B^j}{j!}$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{1}{k!} \frac{1}{(m-k)!} A^k B^{m-k}$$

$m = k + j$
 $k = m - j$ & $j = m - k$
 and flipped the sums!
 needed $AB = BA$ to
 do this?

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{k=0}^m \frac{m!}{k! (m-k)!} A^k B^{m-k} \right)$$

: use the binomial
 th^m , note we need
 $[A, B] = 0$ for
 the binomial th^m
 to hold.

$$= \sum_{m=0}^{\infty} \frac{1}{m!} (A+B)^m$$

$$= e^{A+B}$$

II.) $B e^A B^{-1} = e^{BAB^{-1}}$

$$B e^A B^{-1} = B \left(\sum_{k=0}^{\infty} \frac{A^k}{k!} \right) B^{-1}$$

$$= \sum_{k=0}^{\infty} \frac{B A^k B^{-1}}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(BAB^{-1})^k}{k!}$$

$$= e^{BAB^{-1}}$$

k - factors.

$$(BAB^{-1})^k = \overbrace{(BAB^{-1})(BAB^{-1}) \dots (BAB^{-1})(BAB^{-1})}^k$$

$$= B A \cdot A \dots A B^{-1}$$

$$= B A^k B^{-1}$$

III.) If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of A then it follows that $e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}$ are the eigen values of e^A .

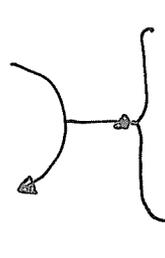
$$\hat{A} x_i = \lambda_i x_i \quad (\text{no summation})$$

$$e^{\hat{A}} x_i = \left(\sum_{k=0}^{\infty} \frac{A^k}{k!} \right) x_i =$$

$$= \sum_{k=0}^{\infty} \frac{A^k x_i}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!}$$

$$= e^{\lambda_i} x_i$$



$$\begin{aligned} A^k x_i &= A^{k-1} (A x_i) \\ &= A^{k-2} A \lambda_i x_i \\ &= A^{k-2} \lambda_i^2 x_i \\ &\vdots \\ &= \lambda_i^k x_i \end{aligned}$$

$$\text{IV) } \det(e^A) = e^{\text{Tr}(A)}$$

$$\det(e^A) = \prod_{i=1}^n (e^{\lambda_i})$$

$$= e^{\left(\sum_{i=1}^n \lambda_i\right)}$$

$$= e^{\text{tr}(A)}$$

: $\det(\text{Matrix}) = \text{product of eigenvalues}$

$$: e^A e^B = e^{A+B}$$

: the trace is the sum of the eigenvalues.

Exercise 3.3: For $L, M \in GL(n)$ prove $e^L M e^{-L} = \sum_{n=0}^{\infty} \frac{1}{n!} [L, M]_{(n)}$

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Where we denote

$$[L, M]_{(0)} \equiv M$$

$$[L, M]_{(1)} = [L, M]$$

$$[L, M]_{(2)} = [L, [L, M]]$$

$$[L, M]_{(n)} = [L, [L, M]_{(n-1)}]$$

Solⁿ: Define a matrix valued function of real-valued parameter

$$\hat{F}(\alpha) = e^{\alpha L} \hat{M} e^{-\alpha L}$$

Now $\hat{F}(\alpha)$ is fairly clearly analytic thus

$$\hat{F}(\alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n \hat{F}}{d\alpha^n} \right|_{\alpha=0} \alpha^n$$

• So calculate the derivative of $\hat{F} = e^{\alpha L} M e^{-\alpha L}$ w.r.t. α , ($M \neq M(\alpha)$!)

$$\frac{dF}{d\alpha} = \frac{d}{d\alpha} (e^{\alpha L}) M e^{-\alpha L} + e^{\alpha L} M \frac{d}{d\alpha} (e^{-\alpha L})$$

$$= e^{\alpha L} \cdot L M e^{-\alpha L} - e^{\alpha L} M L e^{-\alpha L}$$

$$= e^{\alpha L} (LM - ML) e^{-\alpha L}$$

$$= e^{\alpha L} [L, M] e^{-\alpha L}$$

product rule for matrices, just preserve the order, same pt. as usual case.

• Next calculate the 2nd derivative,

$$\frac{d^2 F}{d\alpha^2} = \frac{d}{d\alpha} (e^{\alpha L}) [L, M] e^{-\alpha L} + e^{\alpha L} [L, M] \frac{d}{d\alpha} (e^{-\alpha L})$$

$$= e^{\alpha L} (L [L, M] - [L, M] L) e^{-\alpha L}$$

$$= e^{\alpha L} [L, [L, M]] e^{-\alpha L}$$

Exercise 3.3 Continued

Inductively assume,

$$\frac{d^{n-1}}{d\alpha^{n-1}}(\hat{F}(\alpha)) = e^{\alpha\hat{L}} [\hat{L}, \hat{M}]_{(n-1)} e^{-\alpha\hat{L}}$$

Consider then,

$$\begin{aligned} \frac{d\hat{F}}{d\alpha^n} &= \frac{d}{d\alpha}(e^{\alpha L}) [L, M]_{(n-1)} e^{-\alpha L} + e^{\alpha L} [L, M]_{(n-1)} \frac{d}{d\alpha}(e^{-\alpha L}) \\ &= e^{\alpha L} (L [L, M]_{(n-1)} - [L, M] L) e^{-\alpha L} \\ &= e^{\alpha L} [L, [L, M]_{(n-1)}] e^{-\alpha L} \\ &= e^{\alpha L} [L, M]_{(n)} e^{-\alpha L} \end{aligned}$$

Hence

$$\hat{F}(\alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} [L, M]_{(n)} \alpha^n$$

Therefore $\alpha=1$ yields

$$e^L M e^{-L} = \sum_{n=0}^{\infty} \frac{1}{n!} [L, M]_{(n)}$$

Defⁿ The proper Lorentz Group is all 4×4 matrices $\hat{a} = (a^\mu_\nu)$ subject to

i.) $(a^\mu_\alpha X^\alpha) \eta_{\mu\nu} (a^\nu_\beta X^\beta) = X^\mu \eta_{\mu\nu} X^\nu$ where $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

ii) \hat{a} is continuously connected to $\mathbb{1}$.

As usual make ansatz that $\hat{\alpha} \in L \equiv$ Proper Lorentz Group. has

$$\hat{\alpha}(\vec{\omega}, \vec{\xi}) = \exp(-i\vec{\omega} \cdot \vec{S} - i\vec{\xi} \cdot \vec{K})$$

I don't think this is obvious, unless you identify that L comes from 3-rotations and 3-boosts, alternatively we could count constraints in i.) above, there are in principal $16 = 4^2$ free parameters in 4×4 matrix but ii) amounts to 10 eq^s $= \frac{4(4+1)}{2} = \frac{20}{2} = 10$ hence $16 - 10 = 6$ free parameters.

Infinitesimally: we find,

$$\hat{a}(\delta\omega, \delta\xi) = \mathbb{1} - i\delta\vec{\omega} \cdot \vec{S} - i\vec{\xi} \cdot \vec{K}$$

We say \vec{S} generate rotations while \vec{K} generate boosts.

$$S_1 = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad S_2 = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad S_3 = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

these can be calculated from $S_i = i \frac{\partial U}{\partial \theta_i}$ as we know the group transformations U , Likewise from the Lorentz Transformations arising from boosts are

$$K_1 = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_2 = -i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_3 = -i \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Then define $\hat{L}_i = (\hat{S}_i, \hat{K}_i)$

$$[L_i, L_j] = \begin{cases} [S_i, S_j] = i \epsilon_{ijk} S_k \\ [K_i, K_j] = -i \epsilon_{ijk} S_k \\ [S_i, K_j] = i \epsilon_{ijk} K_k \end{cases}$$

(See Ryder § 2.7 for similar discussion)

Lie Algebra of Generators \implies Lie Group

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Given a set $\{\hat{L}_i\}$ of hermitian generators which are subject to $[\hat{L}_i, \hat{L}_j] = C_{ijk} \hat{L}_k$ we can exponentiate to form a Lie Group (the connected component of the identity really). An infinitesimal group operator is

$$\begin{aligned}\hat{U}(\delta\alpha_\nu) &= 1 - i\delta\alpha_\nu \hat{L}_\nu \\ &= 1 - i \frac{\alpha_\nu}{M} \hat{L}_\nu \quad \text{where } \alpha_\nu = M\delta\alpha_\nu\end{aligned}$$

"Successive M -fold application of these operators in the limit $M \rightarrow \infty$ then yields the operators of the pertinent Lie group"

$$\hat{U}(\alpha_\nu) = \lim_{M \rightarrow \infty} \left(1 - \frac{i\alpha_\nu \hat{L}_\nu}{M} \right)^M = \exp(-i\alpha_\nu \hat{L}_\nu)$$

Obviously this is not a very careful treatment, it completely ignores the subtleties of global topologies.

Defⁿ The rank of a Lie Group is the largest # of generators which commute with each other

Examples

① $\{\hat{P}_x, \hat{P}_y, \hat{P}_z\}$ with $[\hat{P}_i, \hat{P}_j] = 0$ has rank 3.

② $SO(3) : \langle \{\hat{J}_1, \hat{J}_2, \hat{J}_3\} \mid [\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk} \hat{J}_k \rangle$ has rank 1

Let G be a group with H an ABELIAN subgroup then

$$g, h, g^{-1} = h_2 \quad \forall g, \in G \text{ and } h_1, h_2 \in H$$

$$h_1 h_2 = h_2 h_1$$

we say H is an ABELIAN invariant subgroup. If $H < G$ and is invariant but not abelian we say it's an invariant subgroup, duh.

Example: $\{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{T}_1, \hat{T}_2, \hat{T}_3\}$ is the translation-rotation group it has abelian invariant subgroup $\{\hat{P}_1, \hat{P}_2, \hat{P}_3\}$ because

$$\hat{R} \hat{T} \hat{R}^{-1} = \hat{T} \quad \forall \text{ rotations.}$$

Defⁿ A Lie Group is simple if does not possess a continuous invariant subgroup. A Lie Group is semi-simple if it does not possess an abelian invariant subgroup.

Discussion: What condition does the subgroup's algebra get from simpleness?

$$\hat{a}_j = \hat{g} \hat{a}_i \hat{g}^{-1}$$

$$\langle \hat{a}_j \rangle = A \quad \text{groups}$$

$$\langle g \rangle = G \quad \text{groups}$$

$$\hat{a}_l = \hat{a}_j \hat{a}_i^{-1} = \hat{g} a_i \hat{g}^{-1} a_i^{-1}$$

Let A be generated by \hat{A}_j
Let G be generated by \hat{G}_j

For sake of discussion let's say that

$$\hat{g}^{-1} = \hat{U}^{-1}(\delta\alpha)$$

$$a_i^{-1} = \hat{U}^{-1}(\delta\beta)$$

$$\text{so } (\hat{g}^{-1})^{-1} = \hat{g} = \hat{U}(\delta\alpha)$$

$$\hat{U}(\delta\beta) = a_i$$

Hence we note that this is essentially just the same as 3.10 (pg. 5)

$$\hat{g} a_i \hat{g}^{-1} a_i^{-1} = \hat{U}^{-1}(\delta\alpha) \hat{U}^{-1}(\delta\beta) \hat{U}(\delta\alpha) \hat{U}(\delta\beta)$$

$$= \mathbb{1} \equiv \delta\beta_k \delta\alpha_m ([\hat{G}_m, \hat{A}_k]) = a_l = \mathbb{1} - i\delta\chi_m \hat{A}_m$$

Then we can require that for appropriate a_{mkj} that

$$\delta\beta_k \delta\alpha_m a_{mkj} = i\delta\chi_j$$

$$\Rightarrow [\hat{G}_m, \hat{A}_k] = a_{mkj} \hat{A}_j$$

Semi-simple and Simple Lie groups & their algebras

(14)

Let us summarize the preceding discussion. Given Lie Group G generated by \hat{G}_m and a Lie subgroup A generated by \hat{A}_n , where the Lie subgroup is invariant

$$\hat{a}_x = g a_i g^{-1} a_i^{-1} \Rightarrow [\hat{G}_m, \hat{A}_n] = a_{mnl} \hat{A}_l$$

Now because A is a Lie group itself we know that $\{\hat{A}_n\}$ is a Lie Algebra, indeed we say that $\{\hat{A}_n\}$ is an ideal of the Lie Algebra $\{\hat{G}_m\}$.

Defⁿ/ A Lie Algebra is simple if it only possesses itself and $\{0\}$ as ideals. A semisimple Lie Algebra is one which does not possess an Abelian ideal.

Proposition: Simple Lie groups have simple Lie Algebras.
Semisimple Lie groups have semisimple Lie Algebras.

Proof: Group \Rightarrow Algebra is established by argument on (13) & (14) above. The converse still needs some thought. Let \mathfrak{g} be a simple Lie Algebra and let $G = \exp(\mathfrak{g})$ with the obvious meaning, it is then clear that if H was a continuous invariant subgroup then its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ with $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$ that is a non-trivial ideal hence $\mathfrak{h} = \mathfrak{g}$ as \mathfrak{g} is simple, hence $H = G$ as well.

Likewise let \mathfrak{g} be semi-simple then if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$ it follows that $[\mathfrak{h}, \mathfrak{h}] \neq 0$ (it's not Abelian). Now let H be an invariant subgroup of $G = \exp(\mathfrak{g})$ then further suppose that H is abelian then

$$\mathbb{1} = g a_i g^{-1} a_i^{-1} \quad \forall g \in G \ \& \ a_i \in \mathfrak{h}$$

$$\Rightarrow [\hat{G}_m, \hat{A}_n] = 0 \quad \text{since } \mathbb{1} = \exp(0).$$

$$\Rightarrow [\mathfrak{g}, \mathfrak{h}] = 0$$

$$\Rightarrow [\mathfrak{h}, \mathfrak{h}] = 0 \Rightarrow \mathfrak{g} \text{ not-semisimple} \rightarrow \leftarrow.$$

Examples of Lie Algebras

a.) $\mathfrak{G} = \{ \vec{J}, \vec{P} \}$ where \hat{J}_i are the generators of rotations while \hat{P}_i are momentums the generators of translations. The bracket structure is as follows, $\exp(\mathfrak{G}) =$ translation/rotation group.

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk} \hat{J}_k$$

$$[\hat{P}_i, \hat{P}_j] = 0$$

$$[\hat{P}_i, \hat{J}_j] = i\epsilon_{ijk} \hat{P}_k$$

Notice $\mathfrak{h} = \{ \hat{P}_i \}$ forms an invariant Abelian ideal hence \mathfrak{G} is not semisimple. (or simple)

b.) $\mathfrak{G} = \{ \hat{J}_i \}$ forms the rotation group $G = \exp(\mathfrak{G}) \cong \text{so}(3)$. which is simple since $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk} \hat{J}_k \Rightarrow \nexists$ any non-trivial ideal.

c.) DIRECT PRODUCT OF SIMPLE GROUPS IS SEMI-SIMPLE. For instance

$$\text{so}(3) \times \text{so}(3) = \{ \exp(-i\phi \cdot J) \times \exp(-i\phi' \cdot J') \}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[J'_i, J'_j] = i\epsilon_{ijk} J'_k$$

$$[J_i, J'_k] = 0$$

\swarrow independent generators
 \searrow act in different spaces.

Clearly $\text{so}(3) \times \mathbb{1} \neq \mathbb{1} \times \text{so}(3)$ independently form a non-abelian ideal hence $\text{so}(3) \times \text{so}(3)$ is not simple, but is semisimple.

Th^m / Any semisimple Lie group can be expressed as the direct product of simple groups.

Proof: hmmm....

Examples of Lie Algebra Continued

d.) While simple Lie groups possess no invariant continuous subgroups it may be the case \exists discrete inv. subgroups. (these subgroups are finite and as such are not continuous). Consider $SU(2)$ where $U_R = \exp(-i\phi \cdot \vec{\sigma}) = \exp(-\frac{1}{2}i\phi \cdot \hat{\sigma})$ rotates a spinor (see pg. 46-47 greiner) let $\vec{\phi} = \phi \hat{n}$ then (see below)

$$U_R = \mathbb{1} \cos(\frac{1}{2}\phi) - i\hat{n} \cdot \hat{\sigma} \sin(\frac{1}{2}\phi) \text{ where } \hat{\sigma} \text{ are the pauli-matrices}$$

Notice that $\phi = 0$ and $\phi = \pi$ give the following,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^{\dagger}$$

These are both in $SU(2)$.

We defer the explicit proof that this forms an invariant subgroup of $SU(2)$.

e.) Heisenberg Lie Algebra $\{\hat{P}, \hat{Q}, \hat{E}\}$ where \hat{P} is momentum and \hat{Q} is the coordinate on the same direction, \hat{E} is the identity in introductory quantum mechanics we found

$$[\hat{E}, \hat{P}] = [\hat{E}, \hat{Q}] = 0$$

$$[\hat{P}, \hat{Q}] = -i\hat{E}$$

Notice that $\{\hat{E}, \hat{P}\}$ forms an abelian invariant subalgebra. This algebra is neither simple nor semisimple.

Ex 3.8 Show that: $\exp(-\frac{1}{2}i\hat{n} \cdot \hat{\sigma}) = \mathbb{1} \cos(\frac{1}{2}\phi) - i\hat{n} \cdot \hat{\sigma} \sin(\frac{1}{2}\phi)$,

$$\exp(-i\beta \hat{n} \cdot \hat{\sigma}) = \sum_{k=0}^{\infty} \frac{(-i\beta \hat{n} \cdot \hat{\sigma})^k}{k!} \quad (\beta = \frac{1}{2}\phi)$$

$$= \sum_{k=0}^{\infty} \frac{(-i\beta)^{2k}}{(2k)!} (\hat{n} \cdot \hat{\sigma})^{2k} + \sum_{k=0}^{\infty} \frac{(-i\beta)^{2k+1}}{(2k+1)!} (\hat{n} \cdot \hat{\sigma})^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (\beta)^{2k}}{(2k)!} \mathbb{1} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} (\hat{n} \cdot \hat{\sigma}) \beta^{2k+1} \quad (\hat{n} \cdot \hat{\sigma})^{2k} = \mathbb{1}$$

$$= \mathbb{1} \cos(\frac{1}{2}\phi) - (\hat{n} \cdot \hat{\sigma}) \sin(\frac{1}{2}\phi)$$

• Where $(\hat{n} \cdot \hat{\sigma})^2 = \hat{n}_m \hat{\sigma}_m \hat{n}_n \hat{\sigma}_n = \hat{n}_m \hat{n}_n \hat{\sigma}_m \hat{\sigma}_n = \hat{n}_m \hat{n}_n (i \epsilon_{mnpk} \hat{\sigma}_k + \delta_{mn} \mathbb{1}) = \mathbb{1} \hat{n} \cdot \hat{n} = \mathbb{1}$
 $\therefore (\hat{n} \cdot \hat{\sigma})^2 = \mathbb{1}$

i^{2k}	k
$i^2 = -1$	1
$i^4 = 1$	2
$i^6 = -1$	3

Cartan's Criteria & the Killing Form

Defⁿ/ $g_{\alpha\lambda} = C_{\alpha\rho\tau} C_{\lambda\tau\rho}$ where $[\hat{L}_i, \hat{L}_j] = C_{ijk} \hat{L}_k$
is the Killing Form (or "metric" tensor) of $\{\hat{L}_i\} = \mathfrak{g}$.

This amounts to tracing the adjoint rep. of \mathfrak{g}

$$\begin{aligned} g_{ij} &= (\hat{L}_i, \hat{L}_j) \\ &= \text{Tr}(\text{ad}(L_i) \circ \text{ad}(L_j)) \\ &= C_{ilm} C_{jml} \end{aligned}$$

why? let's try to elucidate this

As: $\text{ad}(L_i)(L_m) = [L_i, L_m] = C_{imn} L_n \Rightarrow [\text{ad}(L_i)]_{kl} = C_{ikl}$
 $\text{ad}(L_j)(L_m) = [L_j, L_m] = C_{jmk} L_k$

~~What~~ What do we mean by $\text{Tr}(\text{ad}(L_i) \text{ad}(L_j))$?
Well as usual this is to be understood in terms of a matrix calculation

$$\begin{aligned} \text{Tr}(\text{ad}(L_i) \text{ad}(L_j)) &= \text{Tr}([\text{ad}(L_i)]_{kl} [\text{ad}(L_j)]_{ln}) \\ &= [\text{ad}(L_i)]_{kl} [\text{ad}(L_j)]_{lk} \\ &= \boxed{C_{ikl} C_{jlk} = g_{ij}} \end{aligned}$$

Defⁿ/ $\text{ad}(\mathfrak{g}) = \{l \in \mathfrak{gl}(\mathfrak{g}) \mid l(x) = [l, x] \quad \forall x \in \mathfrak{g}\}$
this is the adjoint representation of \mathfrak{g} .

Cartan's Criteria:

$$\mathfrak{g} \text{ is semisimple} \iff g_{ij} \text{ is invertible, that is } \det(g_{ij}) \neq 0$$

Proof: See next page.

Cartan's Criteria: G is semisimple $\iff \det(\mathfrak{g}) \neq 0$
 where $G = \exp(\mathfrak{g})$ and $\mathfrak{g} = \{\hat{L}_i\}$ and \mathfrak{g} is the Killing Form

Proof: Let $\mathfrak{g} = \{\hat{L}_i\}$ and $\mathfrak{h} = \{\hat{L}_{i'}\}$ generate an abelian ideal
 meaning that $C_{i'jk'} = 0$ and $C_{ijk} = \begin{cases} 0 & k \neq k' \\ \text{not zero} & k' \end{cases}$ as $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$
 Consider them with this compact notation in mind,

$$\begin{aligned} g_{\sigma\lambda'} &= C_{\sigma\rho\tau} C_{\lambda'\tau\rho} \\ &= C_{\sigma\rho'\tau} C_{\lambda'\tau\rho'} && : \mathfrak{h} \text{ is an ideal.} \\ &= C_{\sigma\rho'\tau'} C_{\lambda'\tau'\rho'} && : \text{again } \mathfrak{h} \text{ is an ideal so } C_{\sigma\rho'\tau} = 0 \\ &= 0 && \text{for } \tau \neq \tau' \\ & && : C_{\lambda'\tau'\rho'} = 0 \text{ } \mathfrak{h} \text{ is abelian.} \end{aligned}$$

This holds for arbitrary σ and the particular λ' relative to $\mathfrak{h} = \{\hat{L}_{\lambda'}\}$
 anyway this says at least one row of g_{ij} is zero $\therefore \det(\mathfrak{g}) = 0$.
 (Hence not semisimple \Rightarrow not $\det(\mathfrak{g}) \neq 0$). Conversely see (19)

$$\det(\mathfrak{g}) = 0 \Rightarrow \exists a \text{ with } \mathfrak{g} a^\sigma \mathfrak{g}_\sigma = 0 \text{ for } a^\sigma \neq 0$$

Now $\hat{L}_i' \equiv a_i^\sigma \hat{L}_\sigma$ where these are the sol^2 in the kernel of \mathfrak{g}

Notice that

$$\begin{aligned} \text{Tr}([L_i', L_j'] L_k) &= \text{Tr}(L_i' L_j' L_k - L_j' L_i' L_k) \\ &= \text{Tr}(L_i' L_j' L_k - L_i' L_k L_j') \\ &= \text{Tr}(L_i' [L_j', L_k]) \end{aligned}$$

With this in mind calculate

$$\begin{aligned} \text{Tr}([L_i', L_j'] L_k) &= a_i^\sigma a_j^\nu \text{Tr}(L_\sigma [L_\nu, L_k]) \\ &= a_i^\sigma a_j^\nu C_{\nu km} \text{Tr}(L_\sigma L_m) \\ &= a_i^\sigma a_j^\nu C_{\nu km} \mathfrak{g}_{\sigma m} \\ &= 0 \quad (\text{because } a_i^\sigma \mathfrak{g}_{\sigma m} = 0) \end{aligned}$$

Now since L_k was arbitrary this means that we can conclude

$$[L_i', L_j'] = 0 \quad \therefore \text{it is an Abelian subalgebra}$$

But is it an ideal? Need to show

$$[L_i', L_k] = C_{ikm} L_m \quad \text{why must } L_m \in \langle \hat{L}_i' \rangle ?$$

$$\begin{aligned} L_i' L_k - L_k L_i' &= a_i^\sigma (L_\sigma L_k - L_k L_\sigma) \\ &= a_i^\sigma \underbrace{C_{\sigma km} L_m}_{\in \text{span}\{\hat{L}_i'\}} \end{aligned}$$

just a linear combination.
and each is of the form we desire.

Example: $\mathfrak{so}(3)$ has $C_{ijk} = i\epsilon_{ijk}$ thus notice that $\epsilon_{iml}\epsilon_{jml} = \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}$ ----

$$\begin{aligned} \mathfrak{g}_{ij} &= -\epsilon_{ilm}\epsilon_{jml} \\ &= +\epsilon_{iml}\epsilon_{jml} \\ &= 2\delta_{ml} \quad \therefore \det(\mathfrak{g}_{ij}) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 \neq 0 \end{aligned}$$

Thus $\mathfrak{so}(3)$ is semisimple, in fact we already knew it was simple from previous discussion.

Compact Lie Groups and their algebras:

(20)

A Lie group is compact if the parameter space consists of a finite # of bounded regions. Otherwise the group is non compact. Any compact Lie group has a semisimple Lie algebra.

INVARIANT OPERATORS, aka CASIMIR OPERATORS (pg. 101 GAEBNER & MÜLLER)

Discussion: The spherical harmonics $Y_{lm}(\theta, \phi)$ are characterized by the quantum #'s l and m . This is because Y_{lm} is a simultaneous eigen function of \hat{L}^2 and \hat{L}_3 which are connected to the generators of the rotation group for spinless fields by ($J = L + S$ & $S = 0$)

$$\hat{L}^2 = \sum_{i=1}^3 \hat{J}_i^2 = \hat{J}^2, \quad \hat{L}_3 = \hat{J}_3$$

Notice that \hat{J}^2 is not a generator, it's a bilinear function of the generators that commutes with all the generators

$$[\hat{J}^2, \hat{J}_i] = 0$$

Hence using linearity of bracket,

$$[\hat{J}^2, \hat{U}_R(\phi)] = 0$$

That is \hat{J}^2 commutes with the whole group. (hmm. should this be \hat{J} instead of G ?)

Now \hat{J}^2 has $(2j+1)$ eigenvectors for each j . These eigenvectors represent the multiplets of the rotation group.

$j = 0$ has 1 state (singlet)

$j = \frac{1}{2}$ has 2 states (doublet)

$j = 1$ has 3 states (triplet)

The rotation group for spinless fields ($SO(3)$ perhaps or is it $SU(2)$?) has rank 1 thus it has 1 Casimir operator, this is the claim of the "theorem of Racah" see (21).

THEOREM OF RACAH:

For any semisimple Lie Group of rank λ there exist a set of λ CASIMIR operators. These are functions $\hat{C}_\lambda(\hat{L}_1, \hat{L}_2, \dots, \hat{L}_\lambda)$ (for $\lambda = 1, 2, 3, \dots, \lambda$) of the generators \hat{L}_i and commute with every operator of the group and themselves; $[\hat{C}_\lambda, \hat{L}_i] = 0 \neq [\hat{C}_\lambda, \hat{C}_\rho] = 0$. Additionally, the eigenvalues of \hat{C}_λ uniquely characterizes the multiplets of the group. (which are irreducible invariant subspaces with respect to the group operators)

Example: The orthogonal vectors $\{Y_{00}, Y_{11}, Y_{10}, Y_{1-1}\}$ form a 4-dim'l invariant subspace with respect to the rotation group, because any of the generators $\hat{J}_n = \hat{L}_n$ only change m of Y_{lm} but not l . Consequently \hat{J}_n transform these vectors among themselves, just as $\hat{U}_R(\phi)$ do because they're just made of the generators!

$$\hat{J}_\pm \equiv \hat{J}_1 \pm i \hat{J}_2 \quad (\text{ladder operators})$$

As we know (from chapters 1 & 2 of GREINER and many other places)

$$\hat{J}_\pm Y_{lm} = \sqrt{l(l+1) - m(m \pm 1)} Y_{l, m \pm 1}$$

$$\hat{J}_3 Y_{lm} = m Y_{lm}$$

The space $\{Y_{00}, Y_{11}, Y_{10}, Y_{1-1}\}$ is reducible into a triplet & singlet

$\{Y_{11}, Y_{10}, Y_{1-1}\}$ triplet
 $\{Y_{00}\}$ singlet

We can write $\{Y_{00}, Y_{11}, Y_{10}, Y_{1-1}\} = \{Y_{11}, Y_{10}, Y_{1-1}\} \oplus \{Y_{00}\}$ notice the matrix elements have block-diagonal form.

$$\langle Y_{00} | \hat{U}_R(\phi) | Y_{lm} \rangle = 0 \quad (l \neq 0)$$

Which means that we have block-diag. form,

$$[\hat{U}_R(\phi)] = \left[\begin{array}{ccc|c} Y_{11} & Y_{10} & Y_{1-1} & Y_{00} \\ * & * & * & 0 \\ * & * & * & \\ * & * & * & \\ \hline 0 & & & * \end{array} \right]$$

• Next we'll see how to generate each multiplet by just operating on any state in the multiplet say ψ_0 by the group, as we operate we'll fill out the multiplet.

Constructing a Multiplet from scratch

Begin with some normalized state completely within a multiplet
 lets call it ψ_0 . Then notice that (can also take linear combinations)

$$\psi_\alpha = \hat{U}(\alpha) \psi_0$$

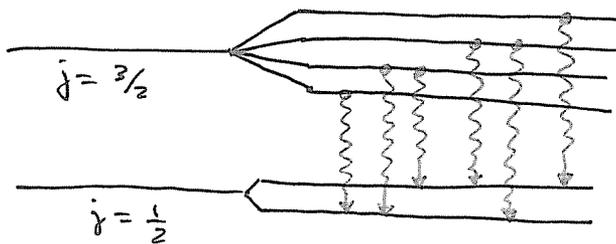
If we operate by group then ψ_α is still in same multiplet. These will form a sphere for the rotation group. Taking linear combinations of such ψ_α form an invariant subspace, they are a multiplet.

Examples: Atomic Spectroscopy is the origin of the word "multiplet"
 these states are characterized by n, l and j

$2P_{3/2}$	\longleftrightarrow	$\left[\begin{array}{l} n=2 \quad (\text{principal or electronic quantum \#}) \\ l=1 \quad (\text{P-state } Y_{lm}, \text{ orbital ang. mom}) \\ j=\frac{3}{2} \quad (\text{total ang. momentum}) \end{array} \right]$
$3d_{5/2}$	\longleftrightarrow	$\left[\begin{array}{l} n=3 \quad (\text{principal qnt. \#}) \\ l=2 \quad (\text{d-state by convention } Y_{3m}) \\ j=\frac{5}{2} \quad (\text{tot. ang. momentum}) \end{array} \right]$

The $2P_{3/2}$ is $2j+1 = 3+1 = 4$ times degenerate
 While $3d_{5/2}$ has $2j+1 = 5+1 = 6$ - fold degenerate.

This degeneracy can be removed by the presence of an external B-field.



these transitions have energies understood in view of Zeeman Effect ($B_{ext} \neq 0$)

Remark: Only for continuous semisimple groups do we have the theorem of Racah & the related structures of multiplet building with the help of the Casimir operators.
 In general there is no method to find representations (irred.)

In this section we will examine the physical significance of invariant subspaces in the Hamiltonian formalism of Q.M. we set $\hbar = 1$ so Schrödinger's $E\psi = \hat{H}\psi$ becomes:

$$i \frac{\partial}{\partial t} \psi = \hat{H} \psi \quad \text{---} \quad E\psi = \hat{H}\psi \quad \text{---} \quad \textcircled{1}$$

A system (described by \hat{H}) is invariant w.r.t. \hat{U} if any rotated (or whatever the group operation may be) state $\hat{U}\psi$ is also a solⁿ to the Sch. $E\psi$, $\psi' = \hat{U}(\alpha)\psi$,

$$i \frac{\partial}{\partial t} \psi' = \hat{H} \psi' \quad \text{---} \quad \textcircled{2}$$

Now $\hat{U}(\alpha)$ is not a function of time so $\textcircled{1}$ becomes,

$$i \frac{\partial}{\partial t} (\hat{U}(\alpha)\psi) = \hat{U}(\alpha) \hat{H} \psi \\ = \hat{U}(\alpha) \hat{H} \hat{U}^{-1}(\alpha) \hat{U}(\alpha)\psi$$

Recognize this means in terms of $\psi' = \hat{U}(\alpha)\psi$ that,

$$i \frac{\partial}{\partial t} (\psi') = \hat{U}(\alpha) \hat{H} \hat{U}^{-1}(\alpha) \psi' \\ = \hat{H} \psi' \quad \text{: using } \textcircled{2}$$

Hence we find that

$$\hat{U}(\alpha) \hat{H} \hat{U}^{-1}(\alpha) = \hat{H} \quad \Leftrightarrow \quad \hat{U}(\alpha) \hat{H} = \hat{H} \hat{U}(\alpha) \\ \Leftrightarrow \quad [\hat{H}, \hat{U}(\alpha)] = 0 \quad \text{---} \quad \textcircled{3}$$

Consequently, we find $[\hat{H}, \hat{L}_i] = 0$ for all generators of group. Conversely if we know that \hat{H} commutes with a group $\hat{U}(\alpha)$ then $\psi' = \hat{U}(\alpha)\psi$ is a solⁿ when ψ is a solⁿ. Because,

$$[\hat{H}, \hat{L}_i] = 0 \quad \Rightarrow \quad [\hat{H}, \hat{U}(\alpha)] = 0 \quad \Rightarrow \quad \hat{U}(\alpha) \hat{H} \hat{U}^{-1}(\alpha) = \hat{H}$$

And if $\hat{H}\psi_0 = E_0\psi_0$ consider $\psi' = \hat{U}(\alpha)\psi_0$

$$\hat{H}\psi' = \hat{H} \hat{U}(\alpha)\psi_0 \\ = \hat{U}(\alpha) \hat{H} \psi_0 \\ = \hat{U}(\alpha) E_0 \psi_0 \\ = E_0 \psi'$$

all states $\psi' = \hat{U}(\alpha)\psi_0$ share the same eigenvalue E_0 for the Hamiltonian H . The Hamiltonian is degenerate on Multiplets of a group it commutes with

Hamiltonians and symmetry group's multiplet

Continuing our thoughts on (23) each multiplet can be characterized with the help of the Casimir operators. We know for a semisimple Lie group there are l Casimir operators that commute with the Lie algebra. Indeed consider

$$[\hat{C}_\lambda(\hat{L}_1, \hat{L}_2, \dots, \hat{L}_n), \hat{L}_i] = 0 \quad : \lambda = 1, 2, \dots, l$$

From which it follows,

$$[\hat{C}_\lambda, \hat{C}_{\lambda'}] = 0$$

And if \hat{U} is a symmetry of \hat{H} we have:

$$[\hat{H}, \hat{C}_\lambda] = 0 \quad \forall \lambda \in \{1, 2, \dots, l\}.$$

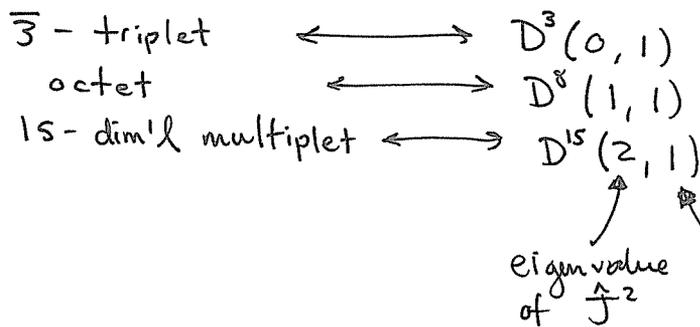
Commuting operators can be simultaneously diagonalized and consequently share the same eigen-functions. Thus \hat{C}_λ is also degenerate on each multiplet which \hat{H} is degenerate on. Thus for a multiplet say $\{\psi_0\}$ we have

$$H\psi_0 = E_0\psi_0, \quad \hat{C}_1\psi_0 = C_1\psi_0, \dots, \hat{C}_l\psi_0 = C_l\psi_0$$

And each is also true for $U(a)\psi_0$ replacing ψ_0 in the above. Another multiplet will necessarily get different eigenvalues for the Casimir operators, we see the eigenvalues of the l -Casimir's uniquely characterize the multiplets of a semisimple Lie Group.

Example: The rotation group has rank 1 and Casimir \hat{J}^2 . The multiplets are uniquely characterized by $j(j+1)$ the e.v. of \hat{J}^2 .

Example: $SU(3)$ has rank 2, it has two Casimirs one is \hat{J}^2 the other Casimir we'll discuss later (207-209 in particular). Some multiplets of $SU(3)$ are



each multiplet uniquely labeled by eigenvalues of the Casimirs.

Constructing Invariant Operators

(25)

For $SU(n)$ groups it was shown by L.C. Biedenharn: J. Math. Phys. 4, 436 (1963) that the Casimir operators have to be simple homogeneous polynomials in the generators,

$$\hat{C}_\lambda = \sum_{i,j} a_{ij}^\lambda \dots \hat{L}_i \hat{L}_j \dots \quad (\lambda - \text{factors}) \leftarrow (?) \text{ oh, I see.}$$

Where \hat{C}_1 is quadratic, \hat{C}_2 is third order in \hat{L}_i and so on. There is a general procedure to construct \hat{C}_1 , but $\hat{C}_2, \hat{C}_3, \dots$ there is no general method. We will find \hat{C}_1 and \hat{C}_2 for $SU(3)$ later.

• Unique Casimirs? Well no if G has rank l and $\hat{C} \neq \hat{C}'$ are casimirs then so are the following:

$$\hat{C} \neq \hat{C}' \quad \hat{C} \neq \hat{C}'$$

Hence for a unitary group where $U^{-1}(\alpha) = U^\dagger(\alpha)$ we have hermitian $\hat{L}_i^\dagger = \hat{L}_i$ as previously discussed, it is possible and convenient to also choose \hat{C} to be hermitian. We can do this because:

\hat{C} some invariant operator of unitary group G

$$\hat{C} \hat{U} = \hat{U} \hat{C}$$

$$\Rightarrow \hat{U}^\dagger \hat{C}^\dagger = \hat{C}^\dagger \hat{U}^\dagger$$

$$\Rightarrow \hat{U}^{-1} \hat{C}^\dagger = \hat{C}^\dagger \hat{U}^{-1}$$

Now this holds $\forall \hat{U}^{-1} \in G$ which says that \hat{C}^\dagger is also a Casimir Operator

Thus we can replace the arbitrary Casimir \hat{C} with the hermitian casimir \hat{C}''

$$\hat{C}'' = \hat{C} + \hat{C}^\dagger$$

Thus by convention we may assume $\hat{C} = \hat{C}^\dagger$ for the remainder of our discussion wlog.

Constructing the 1st Casimir invariant \hat{C}_1

Let $g^{\rho\sigma}$ be the inverse of the Killing form for a semisimple Lie Group (where $\det(g) \neq 0$) then \hat{C}_1 can be found by

$$\hat{C}_1 = g^{\rho\sigma} \hat{L}_\rho \hat{L}_\sigma$$

Let us now prove that \hat{C}_1 is a Casimir using the above construction.

$$\begin{aligned} [\hat{C}_1, \hat{L}_\tau] &= g^{\rho\sigma} [\hat{L}_\rho \hat{L}_\sigma, \hat{L}_\tau] \\ &= g^{\rho\sigma} (L_\rho [L_\sigma, L_\tau] + [L_\rho, L_\tau] L_\sigma) \quad : \text{Q.M. Identity.} \\ &= g^{\rho\sigma} (C_{\sigma\tau}^\lambda L_\rho L_\lambda + C_{\rho\tau}^\lambda L_\lambda L_\sigma) \quad : [L_i, L_j] \equiv C_{ij}^k L_k. \\ &= g^{\rho\sigma} C_{\sigma\tau}^\lambda (L_\rho L_\lambda + L_\lambda L_\rho) \end{aligned}$$

Following Greiner we'll show the above vanishes due to the $(\rho\lambda)$ antisymmetry of the $g^{\rho\sigma} C_{\sigma\tau}^\lambda \equiv a_{\tau}^{\rho\lambda}$ tensor. Let $b_{\sigma\rho\nu} \equiv g_{\sigma\lambda} C_{\rho\nu}^\lambda$ where $g_{\sigma\lambda}$ is the Killing form itself; $g_{\sigma\lambda} = C_{\sigma\rho}^\tau C_{\lambda\tau}^\rho$ then

$$\begin{aligned} b_{\sigma\rho\nu} &\equiv g_{\sigma\lambda} C_{\rho\nu}^\lambda \\ &= C_{\sigma\rho}^\tau C_{\lambda\tau}^\rho C_{\rho\nu}^\lambda \\ &= C_{\sigma\rho}^\tau C_{\rho\nu}^\lambda C_{\lambda\tau}^\rho \\ &= -C_{\sigma\rho}^\tau (C_{\nu\tau}^\lambda C_{\lambda\rho}^\rho + C_{\tau\rho}^\lambda C_{\lambda\nu}^\rho) \quad : C_{ijm} C_{mkn} + C_{jkm} C_{min} + C_{kim} C_{mijn} = 0 \\ &= C_{\sigma\rho}^\tau C_{\nu\tau}^\lambda C_{\lambda\rho}^\rho + C_{\rho\sigma}^\tau C_{\tau\rho}^\lambda C_{\lambda\nu}^\rho \quad C_{ijm} \mapsto C_{ij}^m \text{ in our notation here.} \end{aligned}$$

Notice that $b_{\sigma\rho\nu} = b_{\nu\rho\sigma}$ and so on for cyclic permutations. But notice that $b_{\sigma\rho\nu} = -b_{\sigma\nu\rho}$ thanks to $C_{\rho\nu}^\lambda = -C_{\nu\rho}^\lambda$ thus $b_{\sigma\rho\nu}$ is antisymmetric in all pairs of indices. Finally consider

$$a_{\tau}^{\rho\lambda} = g^{\rho\sigma} C_{\sigma\tau}^\lambda = g^{\rho\sigma} g^{\nu\lambda} b_{\nu\sigma\tau}$$

Hence, converting the $[\hat{C}_1, \hat{L}_\tau] =$ above to a "b" expression,

$$g^{\rho\sigma} g^{\nu\lambda} b_{\nu\sigma\tau} (L_\rho L_\lambda + L_\lambda L_\rho) = 0 \quad \therefore \boxed{[\hat{C}_1, \hat{L}_\tau] = 0}$$

$\underbrace{g^{\rho\sigma} g^{\nu\lambda}}_{\text{sym. in } (\sigma\nu)}$ $\underbrace{b_{\nu\sigma\tau}}_{\text{Vanishes because } (\nu\sigma) \text{ anti-sym.}}$ $\underbrace{(L_\rho L_\lambda + L_\lambda L_\rho)}_{\text{symmetric in } (\rho\lambda)}$

Non-semisimple example

The translation-rotation (Euclidean) group is generated by $\{J_1, J_2, J_3, P_1, P_2, P_3\}$ it has the following Lie algebra,

$$[J_i, J_j] = \epsilon_{ijk} J_k \quad [P_i, J_j] = \epsilon_{ijk} P_k$$
$$[P_i, P_j] = 0$$

Again we note the momentum generate translations which form an invariant abelian subgroup hence this group is not semisimple. Despite this the Casimir operators are simply

$$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2, \quad \hat{P} \cdot \hat{J} = \hat{P}_1 \hat{J}_1 + \hat{P}_2 \hat{J}_2 + \hat{P}_3 \hat{J}_3, \quad \hat{P}^2 = \hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2$$

• Additionally we can make the general remark that if G is ABELIAN \Rightarrow each generator is a Casimir operator. So the Th¹² of Racah still applies to this trivial case.

§3.11 Completeness Relation for Casimir Operators (pg. 110)

The l casimir op. for a Lie group which is semisimple with rank l are not uniquely determined, but they form a "complete set", meaning:

- Each \hat{A} which commutes with \mathfrak{G} is necessarily a function of the l casimir operators; $\hat{A} = \hat{A}(\hat{C}_\alpha)$

That is the casimirs form the largest set of independent operators which commute with the group (well algebra would be better.)

$$\hat{A} = \hat{A}(\hat{L}_i) \quad \text{with} \quad [\hat{A}, \hat{L}_i] = 0 \quad (\text{of course } \hat{A} = \hat{A}(\hat{L}_i), \text{ they are the generators there had better not be an operator inside } \mathfrak{G} \text{ not coming from the generating set!})$$

Anyway, recall $\psi_\alpha = \hat{U}(\alpha) \psi_0$

$$\hat{A} \psi_\alpha = \hat{A} \hat{U}(\alpha) \psi_0 = \hat{U}(\alpha) \hat{A} \psi_0$$

Then since \hat{A} commutes with \hat{L}_i they have a common eigenstate

$$\hat{L}_i \psi_0 = l_i \psi_0 \quad \text{and} \quad \hat{A} \psi_0 = a \psi_0$$

Consequently ψ_α is an eigenstate of \hat{A} with e.v. "a".

$$\hat{A} \psi_\alpha = a \hat{U}(\alpha) \psi_0 = a \psi_\alpha$$

Hence it lies in the ψ_0 multiplet as well. Thus as every eigenstate in the \hat{A} multiplet is reached via group operators $\hat{U}(\alpha)$ this says that \hat{A} must be some combination of these group operators, moreover \hat{A} behaves as an invariant operator, it is diagonal within the multiplet \Rightarrow either \hat{A} is a Casimir or some function of them. But Racah's Th¹² says \exists but l casimirs $\therefore A$ must be a function of those one or on of them.

• Remark: If H has symmetry G then it must be built from the casimirs of G because H commutes with G (it's a symmetry)

a.) Hamiltonian for spinless particles in a central field has spherical symmetry, hence it must commute with operators of the rotation group $\hat{U}_R(\phi)$ thus $(r^2$ and p^2 give length of \vec{r} and \vec{p})

$$\begin{aligned} \hat{H} &= T(r^2, p^2) + f(r^2, p^2) \hat{L}^2 \\ &= T(r^2, p^2) \mathbb{1} + f(r^2, p^2) \hat{L}^2 \end{aligned}$$

b.) Hamiltonian invariant under translations should contain the Casimirs of the translation group $\{P_1, P_2, P_3\}$ for example

$$\hat{H} = \alpha \mathbb{1} + \beta \cdot \hat{P} + \gamma \hat{P}^2 + \dots$$

c.) The isospin group of 2-particle systems is (for $\hat{T} = \{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \}$)

$$\hat{T}^2 = \frac{1}{4} (\hat{T}_1 + \hat{T}_2)^2 = \frac{1}{4} (6 \cdot \mathbb{1} + 2 \hat{T}_1 \cdot \hat{T}_2) \quad \text{the isospin vector}$$

the paulimatrices act in isospin space. A Hamiltonian with isospin symmetry can be (More complicated examples are possible in all of the above)

$$\hat{H} = f(r) \cdot \mathbb{1} + G(r) \hat{T}_1 \cdot \hat{T}_2$$

Summary of Examples

Group	Generators	Rank	Invariant Operators	Type
Translations	$\hat{P} = (\hat{P}_1, \hat{P}_2, \hat{P}_3)$	3	$\hat{P} = (\hat{P}_1, \hat{P}_2, \hat{P}_3)$	ABELIAN
Rotations	$\hat{J} = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$	1	$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$	simple \Rightarrow semisimple
Rotation-Translation (Euclidean)	\hat{J} and \hat{P}	3	$\hat{P}^2 = P_1^2 + P_2^2 + P_3^2$ $\hat{J} \cdot \hat{P} = J_1 P_1 + J_2 P_2 + J_3 P_3$ (\hat{J}^2 not included why?)	not simple nor semisimple
Inversion Invariance	\hat{P} where $\hat{P} f(\vec{r}) = f(-\vec{r})$	1	\hat{P}	Discrete (not Lie type)
Rotation & Inversion	\hat{J} and \hat{P}	2	$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$ \hat{P}	unconnected
Isotopic Spin	$\hat{T} = (\hat{T}_1, \hat{T}_2, \hat{T}_3)$	1	$\hat{T}^2 = \hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2$	simple \Rightarrow semisimple.

Consider a Lie group of coordinate transformations $x' = Ux$ or

$$x' = f(x; a)$$

where x and x' are spatial vectors in a n -dim'l space and a is the set of r -group parameters. Less compactly

$$x'_i = f_i(x_1, x_2, \dots, x_n; a_1, a_2, \dots, a_r) \quad i=1, 2, \dots, n$$

As usual choose parameters so that $a=0$ gives identity trans.

$$x = f(x; 0)$$

So consider an infinitesimal "rotation" da starting from the identity

$$x' = x + dx = f(x; da)$$

$$\Rightarrow dx = f(x; da) - x = f(x, da) - f(x, 0)$$

So we identify that

$$dx = \left[\frac{\partial}{\partial a} f(x, a) \right] \Big|_{a=0} \cdot da$$

$$dx_i = \left[\frac{\partial}{\partial a_\mu} f_i(x, a) \right] \Big|_{a=0} da_\mu$$

Using the abbreviation $u(x) = \left[\frac{\partial}{\partial a} f(x, a) \right] \Big|_{a=0} \Rightarrow dx = u(x) \cdot da$
or simply $dx_i = u_{i\mu} da_\mu$ for $i=1, 2, \dots, n$ and μ summed $1, \dots, r$.

Nothing profound so far lets consider the change of $F(x)$ under the infinitesimal rotation da ,

$$dF = \frac{\partial F(x)}{\partial x} \cdot dx$$

$$= \frac{\partial F}{\partial x} \cdot u(x) \cdot da$$

$$= \sum_{\mu, i} da_\mu \left\{ u_{i\mu}(x) \frac{\partial}{\partial x_i} \right\} F(x) \quad : \quad u_{i\mu}(x) = \left[\frac{\partial f_i(x, a)}{\partial a_\mu} \right] \Big|_{a=0}$$

$$= -i \sum_{\mu, i} da_\mu \hat{L}_\mu(x) F(x)$$

$$\Rightarrow u_{i\mu}(x) \frac{\partial}{\partial x_i} = \frac{\partial f}{\partial a_\mu} \Big|_{a=0} \frac{\partial}{\partial x_i}$$

(notation a bit lacking here.)

Exercise 3.16: Given group of transformations $x' = ax + b = f(x; a, b)$ calculate the generators and their algebra

$$\hat{L}_a = i \frac{\partial f}{\partial a} \bigg|_{x=0} \frac{\partial}{\partial x} = ix \frac{\partial}{\partial x}$$

$$\hat{L}_b = i \frac{\partial f}{\partial b} \bigg|_{x=0} \frac{\partial}{\partial x} = i \frac{\partial}{\partial x}$$

$$[\hat{L}_a, \hat{L}_b] = -x \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial}{\partial x} x \frac{\partial}{\partial x} = \frac{\partial}{\partial x} = -i \hat{L}_b \quad (\text{Product rule cancels with 1st term})$$

Exercise 3.17: Find the generators and infinitesimal operators of $SO(n)$.

We digress from pg. 116 for a time.

$$SO(n) \equiv \{A \in GL(n) \mid A^T = -A \text{ and } \det(A) = 1\}$$

What then is the structure of $so(n)$ where $\exp(so(n)) = SO(n)$?

$$\det(\exp(so(n))) = e^{\text{tr}(so(n))} = 1 \quad \therefore \boxed{\text{tr}(so(n)) = 0}$$

This is not too surprising after all $sl(n) = \{A \in gl(n) \mid \text{tr}(A) = 0\}$ and

$$SO(n) \subset SL(n) \implies so(n) \subset sl(n).$$

We need to understand what $A^T = -A$ means for the algebra recall it is the same as $A^T A = -I$ then take a curve $\gamma(t)$ thru I with tangent B we can parametrize $\gamma(t)$ by

$$\gamma(t) = e^{Bt} \begin{cases} \gamma(0) = I \\ \gamma'(0) = B e^{Bt} \big|_{t=0} = B \end{cases}$$

Next consider the curve lies inside $so(n)$,

$$\begin{aligned} \gamma(t)^T \gamma(t) &= I \implies \gamma'(t)^T \gamma(t) + \gamma(t)^T \gamma'(t) = 0 \\ &\implies B^T I + I^T B = 0 \quad (t=0) \\ &\implies \boxed{B = -B^T} \end{aligned}$$

$$\text{Thus } \boxed{so(n) = \{B \in gl(n) \mid \text{Tr}(B) = 0 \text{ AND } B = -B^T\}}$$

(Here we have ignored the physics convention of placing an i in the matrix exponential)

A basis for $so(n)$ is simply

$$\{E_{ij} - E_{ji} \mid i < j, 1 \leq i, j \leq n\}$$

$$E_{ij} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \downarrow i$$

$$\boxed{\dim(so(n)) = \frac{1}{2} n(n-1)}$$

$$\{E_{12} - E_{21}, E_{13} - E_{31}, \dots, E_{1n} - E_{n1}, E_{23} - E_{32}, \dots, E_{n-1/n-2} - E_{(n-2)/(n-1)}\}$$

Exercise 3.17 continued: Generators of $SO(n)$

We were just remarking that the basis of $SO(n)$ is simply

$$\{E_{ij} - E_{ji} \mid 1 \leq i < j \leq n\} \leftarrow \text{the span of this will give } SO(n)$$

This is the mathematicians custom of connecting G and \mathfrak{G} according to

$$G = \exp(\mathfrak{G})$$

Lie Group Lie Algebra

Whereas in physics (as in Greiner)

$$G = \exp(-i\hat{\mathfrak{G}})$$

Then in "math" we found for $SO(n)$ that the generators were antisymmetric $B = -B^T$. Let \hat{L} be the corresponding generator from the "physics" convention $B = -i\hat{L} \Rightarrow \hat{L} = iB$.

$$\hat{L}^\dagger = L \Rightarrow -iB^T = iB$$

Hermitian. Antisymmetric.

Let $x' = Ax$ where $A \in SO(n)$ then consider (as on 116) this to be infinitesimal $x' = x + \delta A \cdot x$ then to be a rotation we need the dot-product to be preserved

$$\begin{aligned} x' \cdot y' &= (x + \delta Ax) \cdot (y + \delta Ay) \\ &= x^T y + x^T \delta Ay + (\delta Ax)^T y + \cancel{(\delta Ax)^T \delta Ay} \\ &= x \cdot y + x^T (\delta A + (\delta A)^T) y \end{aligned}$$

2nd order small.

Thus $\delta A = -(\delta A)^T$ in order that δA be an infinitesimal isometry, just as we found for $SO(n)$ on (30). The generators \hat{S}_{pr} for $1 \leq p < r \leq n$ are

$$\hat{S}_{pr} = i \sum_{j=1}^n \left[\frac{\partial}{\partial a_{pr}} f_j(x, a) \right]_{a=0} \frac{\partial}{\partial x^j} \leftarrow \text{see (32) for a better exposition}$$

Where the group parameters are $a_{12}, a_{13}, \dots, a_{1n}, a_{23}, \dots$

$$\hat{S}_{pr} = i \frac{\partial}{\partial a_{pr}} \begin{pmatrix} 0 & a_{12} & a_{13} & \dots \\ -a_{12} & 0 & \dots & \dots \\ -a_{13} & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 \end{pmatrix} = i \underbrace{(E_{pr} - E_{rp})}_{\text{matrix}}$$

$$\left(\hat{S}_{pr} \right)_{ij} = i (\delta_{ip} \delta_{jr} - \delta_{ir} \delta_{jp}) \leftarrow \text{component form of the above.}$$

Exercise 3.17 continued: Generators of $SO(n)$

For small transformations near I in $SO(n)$ we found

$$X' = AX$$

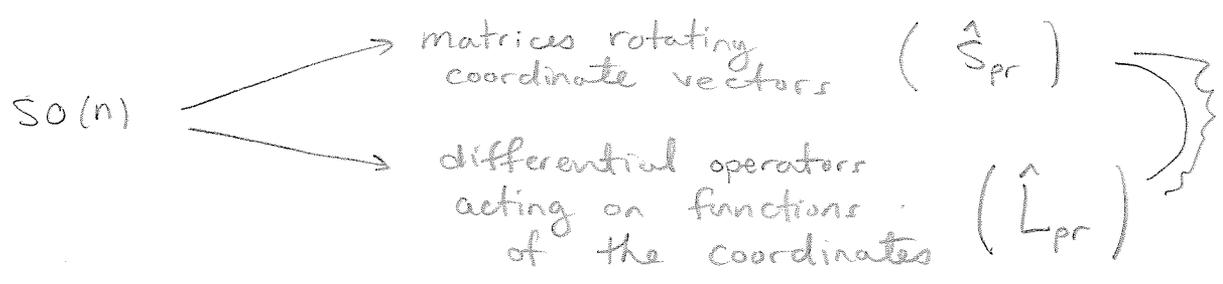
In components; where j is summed from 1 to n .

$$X'_k = A_{kj} X_j = X_k - i \sum_{j=1}^n \sum_{p=1}^n (a_{pr} \hat{S}_{pr})_{kj} X_j \quad \left(\begin{array}{l} \text{infinitesimal} \\ \text{coordinate} \\ \text{transformation} \\ \text{by } SO(n) \end{array} \right)$$

Where $(\hat{S}_{pr})_{ij} = i(\delta_{ip}\delta_{jr} - \delta_{ir}\delta_{jp})$ from (30).

Now the corresponding infinitesimal operators are

$$\begin{aligned} \hat{L}_{pr} &= i \sum_{j=1}^n \frac{\partial}{\partial a_{pr}} X'_j \Big|_{\delta A=0} \frac{\partial}{\partial X^i} \\ &= i \sum_{j=1}^n (-i) \sum_{k=1}^n (\hat{S}_{pr})_{jk} X_k \frac{\partial}{\partial X^i} \\ &= \sum_{j=1}^n \sum_{k=1}^n i (\delta_{pj}\delta_{rk} - \delta_{rj}\delta_{pk}) X_k \frac{\partial}{\partial X^i} \\ &= \boxed{i \left(X_r \frac{\partial}{\partial X^p} - X_p \frac{\partial}{\partial X^r} \right) = \hat{L}_{pr}} \end{aligned}$$



Remark: $SO(3)$ is rotations in 3-dim'l space. Using the above we see the generators are simply,

$$\begin{aligned} \hat{L}_{xy} &= i \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) : \text{generates rotations about } z\text{-axis} \\ \hat{L}_{yz} &= i \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right) : \text{gen. rot. about } x\text{-axis} \\ \hat{L}_{zx} &= i \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right) : \text{gen. rot about } y\text{-axis} \end{aligned}$$

One can contrast this approach to the earlier technique of differentiating the finite transformations near I .

Exercise 3.18: Show that $\hat{S}_{\alpha\beta} = i(\hat{E}_{\alpha\beta} - \hat{E}_{\beta\alpha})$ form a representation of the Lie algebra of spin-1 objects (Also $L_{\alpha\beta}$ discuss)

Notice that $so(n)$ has dimension $\frac{n(n-1)}{2}$, there are that many generators to $so(n)$. Only in 3 dimensions does it happen that $so(n)$ has the same dimension as the space it acts on (for coordinate transformations)

$$\dim(so(3)) = \frac{1}{2} 3(3-1) = \frac{1}{2}(6) = 3.$$

$$\dim(so(4)) = \frac{1}{2} 4(4-1) = \frac{1}{2}(12) = 6.$$

In Greiner we call $so(n)$'s generators the generalized angular momentum operators, Consider first the infinitesimal operator version of $so(n)$,

$$\hat{L}_{\alpha\beta} \equiv i(x_\beta \partial_\alpha - x_\alpha \partial_\beta)$$

Let's calculate the algebra, (factoring i^2 to the LHS to begin)

$$\begin{aligned} -[\hat{L}_{\alpha\beta}, \hat{L}_{\gamma\nu}] &= [x_\beta \partial_\alpha - x_\alpha \partial_\beta, x_\nu \partial_\gamma - x_\mu \partial_\nu] \\ &= [x_\beta \partial_\alpha, x_\nu \partial_\gamma] - [x_\beta \partial_\alpha, x_\mu \partial_\nu] - [x_\alpha \partial_\beta, x_\nu \partial_\gamma] + [x_\alpha \partial_\beta, x_\mu \partial_\nu] \\ &= x_\beta [\partial_\alpha, x_\nu \partial_\gamma] + [x_\beta, x_\nu \partial_\gamma] \partial_\alpha \\ &\quad - x_\beta [\partial_\alpha, x_\mu \partial_\nu] - [x_\beta, x_\mu \partial_\nu] \partial_\alpha \\ &\quad - x_\alpha [\partial_\beta, x_\nu \partial_\gamma] - [x_\alpha, x_\nu \partial_\gamma] \partial_\beta \\ &\quad + x_\alpha [\partial_\beta, x_\mu \partial_\nu] + [x_\alpha, x_\mu \partial_\nu] \partial_\beta \\ &= x_\beta x_\nu \cancel{[\partial_\alpha, \partial_\nu]}^0 + x_\beta [\partial_\alpha, x_\nu] \partial_\gamma + x_\nu [x_\beta, \partial_\mu] \partial_\alpha + \cancel{[x_\beta, x_\nu]}^0 \partial_\mu \partial_\alpha \\ &\quad - x_\beta [\partial_\alpha, x_\mu] \partial_\nu - x_\mu [x_\beta, \partial_\nu] \partial_\alpha \\ &\quad - x_\alpha [\partial_\beta, x_\nu] \partial_\gamma - x_\nu [x_\alpha, \partial_\mu] \partial_\beta \\ &\quad + x_\alpha [\partial_\beta, x_\mu] \partial_\nu + x_\mu [x_\alpha, \partial_\nu] \partial_\beta \\ &= \delta_{\alpha\nu} x_\beta \partial_\mu - \delta_{\beta\gamma} x_\nu \partial_\alpha - \delta_{\alpha\mu} x_\beta \partial_\nu + \delta_{\beta\nu} x_\mu \partial_\alpha \\ &\quad - \delta_{\beta\nu} x_\alpha \partial_\mu + \delta_{\alpha\mu} x_\nu \partial_\beta + \delta_{\beta\mu} x_\alpha \partial_\nu - \delta_{\alpha\nu} x_\mu \partial_\beta \\ &= -i i \delta_{\alpha\nu} (x_\beta \partial_\mu - x_\mu \partial_\beta) + i^2 \delta_{\beta\gamma} (x_\nu \partial_\alpha - x_\alpha \partial_\nu) \\ &\quad + i i \delta_{\alpha\mu} (x_\beta \partial_\nu - x_\nu \partial_\beta) - i^2 \delta_{\beta\nu} (x_\mu \partial_\alpha - x_\alpha \partial_\mu) \\ &= -i \delta_{\alpha\nu} \hat{L}_{\mu\beta} + i \delta_{\beta\gamma} \hat{L}_{\alpha\nu} + i \delta_{\alpha\mu} \hat{L}_{\nu\beta} - i \delta_{\beta\nu} \hat{L}_{\alpha\mu} \\ &= i (\delta_{\alpha\nu} \hat{L}_{\beta\mu} + \delta_{\beta\gamma} \hat{L}_{\alpha\nu} + \delta_{\alpha\mu} \hat{L}_{\nu\beta} + \delta_{\beta\nu} \hat{L}_{\mu\alpha}) \end{aligned}$$

$$\begin{aligned} [\partial_\alpha, x_\beta] f &= \\ &= \partial_\alpha(x_\beta f) - x_\beta \partial_\alpha f \\ &= \delta_{\alpha\beta} f + x_\beta \partial_\alpha f - x_\beta \partial_\alpha f \end{aligned}$$

dropping
 $[x_\mu, x_\nu] = 0$
 $[\partial_\mu, \partial_\nu] = 0$
 terms

Modulo a typo this is what GREINER has in (4) on p.118.

$$[L_{\alpha\beta}, L_{\mu\nu}] = i(\delta_{\alpha\mu} L_{\beta\nu} + \delta_{\beta\nu} L_{\alpha\mu} - \delta_{\alpha\nu} L_{\beta\mu} - \delta_{\beta\mu} L_{\alpha\nu})$$

Construct the Casimir Operator by guessing it's the \sum of the squares

$$\Lambda^2 = \frac{1}{2} \delta^{\alpha\mu} \delta^{\beta\nu} \hat{L}_{\alpha\beta} \hat{L}_{\mu\nu}$$

Just as in the $so(3)$ case where $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$, lets check that Λ^2 is in fact a Casimir operator,

$$\begin{aligned} [\Lambda^2, L_{\sigma\tau}] &= \frac{1}{2} \delta^{\alpha\mu} \delta^{\beta\nu} (L_{\alpha\beta} [L_{\mu\nu}, L_{\sigma\tau}] + [L_{\alpha\beta}, L_{\sigma\tau}] L_{\mu\nu}) \\ &= \frac{i}{2} \delta^{\alpha\mu} \delta^{\beta\nu} [L_{\alpha\beta} (\delta_{\mu\sigma} L_{\nu\tau} + \delta_{\nu\tau} L_{\mu\sigma} - \delta_{\mu\tau} L_{\nu\sigma} - \delta_{\nu\sigma} L_{\mu\tau}) \\ &\quad + (\delta_{\alpha\sigma} L_{\beta\tau} + \delta_{\beta\tau} L_{\alpha\sigma} - \delta_{\alpha\tau} L_{\beta\sigma} - \delta_{\beta\sigma} L_{\alpha\tau}) L_{\mu\nu}] \\ &= \frac{i}{2} (\delta^{\alpha\sigma} \delta^{\beta\nu} L_{\alpha\beta} L_{\nu\tau} + \delta^{\alpha\mu} \delta^{\beta\tau} L_{\alpha\beta} L_{\mu\sigma} - \delta^{\alpha\nu} \delta^{\beta\sigma} L_{\alpha\beta} L_{\nu\sigma} - \delta^{\alpha\mu} \delta^{\beta\sigma} L_{\alpha\beta} L_{\mu\tau} \\ &\quad + \delta^{\sigma\tau} \delta^{\beta\nu} L_{\beta\tau} L_{\mu\nu} + \delta^{\alpha\tau} \delta^{\beta\nu} L_{\alpha\sigma} L_{\mu\nu} - \delta^{\sigma\mu} \delta^{\beta\nu} L_{\beta\sigma} L_{\mu\nu} - \delta^{\alpha\mu} \delta^{\sigma\nu} L_{\alpha\tau} L_{\mu\nu}) \\ &= \frac{i}{2} (L_{\alpha\beta} L_{\nu\tau} + L_{\mu\sigma} L_{\beta\tau} - L_{\nu\sigma} L_{\beta\tau} - L_{\mu\tau} L_{\beta\sigma} \\ &\quad + L_{\nu\tau} L_{\beta\sigma} + L_{\mu\sigma} L_{\beta\tau} - L_{\nu\sigma} L_{\beta\tau} - L_{\mu\tau} L_{\beta\sigma}) \\ &= 0 \quad (\text{may be some typo above}) \end{aligned}$$

The explicit form of Λ^2 can be found from inserting the definition $L_{\alpha\beta} = i(x_\beta \partial_\alpha - x_\alpha \partial_\beta)$ into our formula for Λ^2 ,

$$\begin{aligned} \Lambda^2 &= \frac{1}{2} \delta^{\alpha\mu} \delta^{\beta\nu} i^2 (x_\beta \partial_\alpha - x_\alpha \partial_\beta) (x_\nu \partial_\mu - x_\mu \partial_\nu) \\ &= -\frac{1}{2} \delta^{\alpha\mu} \delta^{\beta\nu} [x_\beta \partial_\alpha x_\nu \partial_\mu - x_\beta \partial_\alpha x_\mu \partial_\nu - x_\alpha \partial_\beta x_\nu \partial_\mu + x_\alpha \partial_\beta x_\mu \partial_\nu] \\ &= -\frac{1}{2} (x_\nu \partial_\mu x_\nu \partial_\mu - x_\nu \partial_\mu x_\mu \partial_\nu - x_\mu \partial_\nu x_\nu \partial_\mu + x_\mu \partial_\nu x_\mu \partial_\nu) \\ &= -\frac{1}{2} (x_\mu \partial_\mu + x_\nu x_\nu \partial_\mu \partial_\mu - x_\nu \delta_{\mu\mu} \partial_\nu - x_\nu x_\mu \partial_\mu \partial_\nu \\ &\quad - x_\mu \delta_{\nu\nu} \partial_\mu - x_\mu x_\nu \partial_\nu \partial_\mu + x_\nu \partial_\nu + x_\mu x_\mu \partial_\nu \partial_\nu) \\ &= - (x^\mu \partial_\mu + x^\nu x_\nu \partial^\mu \partial_\mu - n x^\nu \partial_\nu - x^\nu x^\mu \partial_\mu \partial_\nu) \end{aligned}$$

Define the "homogeneous" Euler operator" $J_e = x^\mu \partial_\mu$ and notice

$$\begin{aligned} J_e^2 &= x^\mu \partial_\mu x^\nu \partial_\nu \\ &= x^\mu \delta_\mu^\nu \partial_\nu + x^\mu x^\nu \partial_\mu \partial_\nu \\ &= J_e + x^\mu x^\nu \partial_\mu \partial_\nu \end{aligned}$$

Consequently we can rewrite Λ^2 see (34),

$$\begin{aligned} \Lambda^2 &= -J_e - x^\nu x_\nu \partial^\mu \partial_\mu + N J_e + J_e^2 - J_e \\ &= - (x^\nu x_\nu \partial^\mu \partial_\mu - J_e (J_e + N - 2)) \end{aligned}$$

Now define a Hilbert space which consists of l -degree homo. poly. s.t.

$$H_l = \{ f \mid f(\lambda x) = \lambda^l f(x), \partial^\mu \partial_\mu f = 0 \}$$

Consider then the "eigenwertspektrum" of Λ^2 , let $f \in H_l$

$$\begin{aligned} \Lambda^2 f &= - (x^\nu x_\nu \partial^\mu \partial_\mu - J_e (J_e + N - 2)) f \\ &= J_e (J_e + N - 2) f \quad \text{: since } \partial^\mu \partial_\mu f = 0 \\ &= l(l + N - 2) f \end{aligned}$$

← ? ? ? How

why should $J_e f = l f$?

$$\partial_\mu [f(\lambda x)] = f'(\lambda x) \partial_\mu (\lambda x) = f'(\lambda x) \cdot \lambda$$

Exercise 3.18 continued

Now that we have explored the operator version of $\mathfrak{so}(n)$ we now continue to examine the matrix version of $\mathfrak{so}(n)$ generated by $\hat{S}_{\alpha\beta}$

$$\hat{S}_{\alpha\beta} = i(E_{\alpha\beta} - E_{\beta\alpha})$$

$$\begin{aligned} ([\hat{S}_{\alpha\beta}, \hat{S}_{\mu\nu}])_{ik} &= (\hat{S}_{\alpha\beta})_{im} (\hat{S}_{\mu\nu})_{mk} - (\hat{S}_{\mu\nu})_{im} (\hat{S}_{\alpha\beta})_{mk} \\ &= [(\delta_{\alpha i} \delta_{\beta m} - \delta_{\beta i} \delta_{\alpha m})(\delta_{\mu m} \delta_{\nu k} - \delta_{\nu m} \delta_{\mu k}) \\ &\quad - (\delta_{\mu i} \delta_{\nu m} - \delta_{\nu i} \delta_{\mu m})(\delta_{\alpha m} \delta_{\beta k} - \delta_{\beta m} \delta_{\alpha k})] i^2 \\ &= i^2 [\delta_{\alpha i} \delta_{\beta m} \delta_{\mu m} \delta_{\nu k} - \delta_{\alpha i} \delta_{\beta m} \delta_{\nu m} \delta_{\mu k} \\ &\quad - \delta_{\beta i} \delta_{\alpha m} \delta_{\mu m} \delta_{\nu k} + \delta_{\beta i} \delta_{\alpha m} \delta_{\nu m} \delta_{\mu k} \\ &\quad - \delta_{\mu i} \delta_{\nu m} \delta_{\alpha m} \delta_{\beta k} + \delta_{\mu i} \delta_{\nu m} \delta_{\beta m} \delta_{\alpha k} \\ &\quad + \delta_{\nu i} \delta_{\mu m} \delta_{\alpha m} \delta_{\beta k} - \delta_{\nu i} \delta_{\mu m} \delta_{\beta m} \delta_{\alpha k}] \\ &= i^2 [\delta_{\alpha i} \delta_{\beta\mu} \delta_{\nu k} - \delta_{\alpha i} \delta_{\beta\nu} \delta_{\mu k} - \delta_{\beta i} \delta_{\mu\alpha} \delta_{\nu k} + \delta_{\beta i} \delta_{\alpha\nu} \delta_{\mu k} \\ &\quad - \delta_{\mu i} \delta_{\nu\alpha} \delta_{\beta k} + \delta_{\mu i} \delta_{\nu\beta} \delta_{\alpha k} + \delta_{\nu i} \delta_{\mu\alpha} \delta_{\beta k} - \delta_{\nu i} \delta_{\mu\beta} \delta_{\alpha k}] \\ &= i^2 [\delta_{\beta\mu} (\delta_{\alpha i} \delta_{\nu k} - \delta_{\nu i} \delta_{\alpha k}) - \delta_{\beta\nu} (\delta_{\alpha i} \delta_{\mu k} - \delta_{\mu i} \delta_{\alpha k}) \\ &\quad - \delta_{\alpha\mu} (\delta_{\beta i} \delta_{\nu k} - \delta_{\nu i} \delta_{\beta k}) + \delta_{\alpha\nu} (\delta_{\beta i} \delta_{\mu k} - \delta_{\mu i} \delta_{\beta k})] \\ &= i (\delta_{\beta\mu} (\hat{S}_{\alpha\nu})_{ik} - \delta_{\beta\nu} (\hat{S}_{\alpha\mu})_{ik} - \delta_{\alpha\mu} (\hat{S}_{\beta\nu})_{ik} + \delta_{\alpha\nu} (\hat{S}_{\beta\mu})_{ik}) \end{aligned}$$

$$\therefore [\hat{S}_{\alpha\beta}, \hat{S}_{\mu\nu}] = i (\delta_{\beta\mu} \hat{S}_{\alpha\nu} - \delta_{\beta\nu} \hat{S}_{\alpha\mu} - \delta_{\alpha\mu} \hat{S}_{\beta\nu} + \delta_{\alpha\nu} \hat{S}_{\beta\mu})$$

Similarly we can find the Casimir S^2 like Λ^2

$$S^2 = \frac{1}{2} S^{\alpha\beta} S_{\alpha\beta} = (N-1)\mathbb{1} \quad (\text{see pg. 121})$$

In analogy to the generalized angular momentum Casimir relation $\Lambda^2 f = l(l+N-2)f$ ($N=3$ has $\Lambda^2 f = l(l+1)f$) we define a quantity s with

$$S^2 f = s(s+N-2)\mathbb{1}f$$

Comparing with the above we see $s=1$. We have thus found the spin one matrices in arbitrary dimension.

Translation in 1-dim'l case is $x' = x + a$ is generated by

$$\hat{P} = i \frac{\partial}{\partial a} (f(x; a)) \Big|_{a=0} \frac{\partial}{\partial x} = -i \frac{\partial}{\partial x}$$

(I don't understand the minus signs in (3).) Guess it just the below

$$F'(x) = F(x') = F(\hat{T}^{-1}x) = F(x-a) = F(f(x; a))$$

Where $\hat{T}x = x+a$ thus $\hat{T}^{-1}x = x-a$.

The generators of E_3 are momentum and angular momentum

$$\hat{P} = -i \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$$

$$\hat{L} = -i \left\{ (y\partial_z - z\partial_y), (z\partial_x - x\partial_z), (x\partial_y - y\partial_x) \right\}$$

$$[\hat{P}_i, \hat{P}_j] = 0, \quad [\hat{L}_i, \hat{L}_j] = i \epsilon_{ijk} \hat{L}_k, \quad [\hat{P}_i, \hat{L}_j] = i \epsilon_{ijm} \hat{P}_m$$

The above generate finite translations via the exponential,

$$\hat{U}_T(\vec{a}) = e^{-i\vec{a} \cdot \vec{P}}$$

$$\hat{U}_R(\vec{\phi}) = e^{-i\vec{\phi} \cdot \vec{L}}$$

This chapter was quite sketchy & repeated earlier better thought-out things in chapter 3. Let's just hit the highlights

- Given a symmetry group which is semisimple with rank l and dimension $n > l$ we choose hermitian generators \hat{L}_i and know by Racah $\exists l$ Casimirs \hat{C}_α . The system which is subject to \hat{H} which commutes with $\hat{U}(\alpha) = \exp(-i\alpha_i \hat{L}_i)$ $\forall \alpha$ has $2l$ "good" quantum #'s.
 - l come from generators \hat{L}_i (can choose l generators which commute)
 - l come from Casimirs \hat{C}_α (these l operators commute.)
- Quantum numbers are conserved because the Hamiltonian is degenerate on the multiplet, I mean we cannot change the quantum #'s by the time-evolution generated by the Hamiltonian.

Def: A multiplet is a set of states which have the same quantum numbers $(C_1, C_2, C_3, \dots, C_l)$.

- The Casimirs $\hat{C}_1, \hat{C}_2, \dots, \hat{C}_l$ uniquely determine the multiplets. The Hamiltonian \hat{H} commutes with all the Casimirs thus it has the same energy for all the states in a multiplet. A multiplet is a degenerate set of Energy Eigen states. We see that the symmetry group will shift states within a multiplet but not out of the multiplet.
- Transition from multiplet to multiplet forbidden if symmetry unbroken
 Let $\langle C_{\lambda'} |$ and $\langle C_\lambda |$ be in different multiplets $\lambda \neq \lambda'$ we know $[\hat{H}, \hat{C}_\lambda] = 0 = \hat{H}\hat{C}_\lambda - \hat{C}_\lambda\hat{H}$ thus

$$0 = \langle C_{\lambda'} | \hat{C}_\lambda \hat{H} - \hat{H} \hat{C}_\lambda | C_\lambda \rangle$$

$$= \langle C_{\lambda'} | \hat{C}_\lambda \hat{H} | C_\lambda \rangle - \langle C_{\lambda'} | \hat{H} \hat{C}_\lambda | C_\lambda \rangle$$

$$= (C_\lambda - C_{\lambda'}) \langle C_{\lambda'} | \hat{H} | C_\lambda \rangle \Rightarrow \boxed{\langle C_{\lambda'} | \hat{H} | C_\lambda \rangle = 0}$$
 Thus the transition between multiplets is impossible.
 (w/o some breaking of the symmetry!)

Schur's Lemma

Any operator \hat{H} which commutes with all group operators $\hat{U}(\alpha)$ (and generators \hat{L}_i ; necessarily) has every state of a multiplet as an eigenvector and is degenerate on every multiplet.

$$[\hat{H}, \hat{U}(\alpha)] = 0 \Leftrightarrow [\hat{H}, \hat{L}_i] = 0 \Rightarrow [\hat{H}, \hat{C}_\alpha] = 0$$

Proof: (it seems to me we have been using this all along and have proved it before).
Anyway let ψ be an eigenstate of \hat{H}

$$\hat{H}\psi = E\psi$$

Then since $HU = UH$ it follows after multiplying by U that,

$$\hat{H}(U\psi) = E(U\psi)$$

Thus the symmetry shifted state $\psi' = U\psi$ also has energy E .
Additionally as $[\hat{H}, \hat{C}_\alpha] = 0$ we can simultaneously diagonalize \hat{C}_α and \hat{H} , then ψ being an eigenstate of $\hat{H} \Rightarrow \psi$ an eig. state of \hat{C}_α

Hence ψ' is also an eigenstate of \hat{C}_α and

$$\begin{aligned} \hat{C}_1 \psi &= c_1 \psi \quad \dots \quad \hat{C}_\alpha \psi = c_\alpha \psi \\ \hat{C}_1 \psi' &= c_1 \psi' \quad \dots \quad \hat{C}_\alpha \psi' = c_\alpha \psi' \end{aligned} \quad \text{for } \psi' = U(\alpha)\psi$$

Hence the shifted state has same $(c_1, c_2, \dots, c_\alpha)$ as ψ and by definition it is in the same multiplet.

Questions: Why does $[\hat{H}, \hat{C}] = 0 \Rightarrow \hat{H}$ and \hat{C} can be simultaneously diagonalized.
Why does simult. diag of \hat{H} and \hat{C} share eigenstates?