Polynomial Graphing & Factoring & Theorems

Th$^{\underline{m}}$ (Rational Roots or Rational Zeros Th$^{\underline{m}}$)

Let f be polynomial function of degree 1 or higher of form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_0 \neq 0$ and all coefficients $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{Z}$ then if $f(p/q) = 0$ it must be that p is a factor of a_0 and q is a factor of a_n

Application: Look at possible ratios formed by factors of constant coefficient over factors of the leading coefficient... IF there are rational roots for $f(x)$ THEN they'll be in that list.

Example 1: $f(x) = 6x^2 + 7x - 5$ ← GOAL, factor completely.

$a_2 = 1 \cdot 6 = 3 \cdot 2 = (-3)(-2)$ & $a_0 = -5 = (-1)(5) = 1(-5)$

thus should consider,

$\pm 1, \frac{\pm 1}{3}, \frac{\pm 1}{2}, \frac{\pm 1}{6}, \frac{\pm 5}{3}, \frac{\pm 5}{2}, \frac{\pm 5}{6}$

Calculate,

$$f(1/3) = \frac{6}{9} + \frac{7}{3} - 5 = \frac{6 + 21 - 45}{9} \neq 0$$

$$f(-1/3) = \frac{6}{9} - \frac{7}{3} - 5 = \frac{6 - 21 - 45}{9} \neq 0$$

$$f(1/2) = \frac{6}{4} + \frac{7}{2} - 5 = \frac{6 + 14 - 20}{4} = \frac{20 - 20}{4} = 0$$

By the factor theorem we have $(x - \frac{1}{2})$ as factor of $f(x)$, but look for $2x - 1$ is easier,

$$\begin{array}{r} 3x + 5 \\ 2x-1 \overline{\big) 6x^2 + 7x - 5} \\ -(6x^2 - 3x) \\ \hline 10x - 5 \\ -(10x - 5) \\ \hline 0 \end{array}$$

$$\begin{aligned} f(x) &= (2x-1)(3x-5) \\ &= 2 \cdot 3 \left(x - \tfrac{1}{2}\right)\left(x - \tfrac{5}{3}\right) \\ &= 6\left(x - \tfrac{1}{2}\right)\left(x - \tfrac{5}{3}\right). \end{aligned}$$

Example 2: factor $f(x) = 42x^4 - 83x^3 + 53x^2 - 13x + 1$

(once more I'll show what the Rational Roots Th^{m} does)

Notice $a_0 = 1$ and $a_4 = 42 = 3 \cdot 2 \cdot 7 = 6 \cdot 7 = 21 \cdot 2 = 3 \cdot 14 = 42 \cdot 1$

Possible Rational Zeros: $\pm 1, \frac{\pm 1}{2}, \frac{\pm 1}{3}, \frac{\pm 1}{7}, \frac{\pm 1}{14}, \frac{\pm 1}{21}, \frac{\pm 1}{42}$.

Now use factor theorem for guidance, $f(c) = 0 \Leftrightarrow (x-c)$ factor.
Begin with easiest #

GREAT!

$$f(1) = 42 - 83 + 53 - 13 + 1 = 95 - 96 + 1 = 0.$$

So, how to factor out $(x-1)$ from $f(x)$? I'll use long-division,

$$
\begin{array}{r|l}
 & 42x^3 - 41x^2 + 12x - 1 \\
\hline
x-1 & 42x^4 - 83x^3 + 53x^2 - 13x + 1 \\
 & -(42x^4 - 42x^3) \\
\hline
 & -41x^3 + 53x^2 - 13x + 1 \\
 & -(-41x^3 + 41x^2) \\
\hline
 & 12x^2 - 13x + 1 \\
 & -(12x^2 - 12x) \\
\hline
 & -x + 1 \\
 & -x + 1 \\
\hline
 & \boxed{0}
\end{array}
$$

Thus, $f(x) = (x-1)(\underbrace{42x^3 - 41x^2 + 12x - 1}_{f_2(x)})$ ← how to factor this!?

$f_2(1) = 42 - 41 + 12 - 1 = 12 \neq 0$

$f_2(-1) = -42 - 41 - 12 - 1 \neq 0$

Next,

$$
\begin{aligned}
f_2\left(\tfrac{1}{2}\right) &= \frac{42}{8} - \frac{41}{4} + \frac{12}{2} - 1 \\
&= \frac{42 - 2(41) + 12(4) - 8}{8} \\
&= \frac{42 - 82 + 48 - 8}{8} \\
&= 0.
\end{aligned}
$$

$\Rightarrow (x - \frac{1}{2})$ factors $f_2(x)$

So ... look for $(2x - 1)$ to make life easier

Example 2 continued:

Factoring out $2x-1$ from $f_2(x) = 42x^3 - 41x^2 + 12x - 1$,

$$
\begin{array}{r l}
 & 21x^2 - 10x + 1 \\
2x-1 \,\big) & 42x^3 - 41x^2 + 12x - 1 \\
 & -(42x^3 - 21x^2) \\
\hline
 & - 20x^2 + 12x - 1 \\
 & -(-20x^2 + 10x) \\
\hline
 & 2x - 1 \\
 & 2x - 1 \\
\hline
 & \boxed{0}
\end{array}
$$

Thus $f_2(x) = (2x-1)(\underbrace{21x^2 - 10x + 1}_{f_3(x)})$ ← down to a quadratic, we can probably just factor this by sight,

$$21x^2 - 10x + 1 = (3x - 1)(7x - 1)$$

Thus,

$$\underline{f(x) = (x-1)(2x-1)(3x-1)(7x-1)}.$$

Or,

$$\boxed{f(x) = 42(x-1)(x-\tfrac{1}{2})(x-\tfrac{1}{3})(x-\tfrac{1}{7})}$$

Example 3: $f(x) = x^3 - 2x^2 - 9x + 18$

Rational Zeros Possible: $\frac{\text{factors of } 18}{\text{factors of } 1} = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

$$f(1) = 1 - 2 - 9 + 18 = 8 \neq 0$$
$$f(-1) = -1 - 2 + 9 + 18 = 24 \neq 0$$
$$f(2) = 8 - 2\cdot 4 - 18 + 18 = 0$$
$$f(3) = \cancel{27} - 2\cdot\cancel{9} - \cancel{9\cdot 3} + \cancel{18} = 0$$
$$f(-3) = -27 - 2\cdot 9 + 9\cdot 3 + 18 = 0$$

By Factor Th^{m} & common sense, we're done!

Notice,

$$f(2) = 0 \Rightarrow (x-2) \text{ factors } f(x)$$
$$f(3) = 0 \Rightarrow (x-3) \text{ factors } f(x)$$
$$f(-3) = 0 \Rightarrow (x+3) \text{ factors } f(x)$$

Since the leading coefficient is 1 we find,

$$\boxed{f(x) = (x-2)(x-3)(x+3)}$$

Of course, we could do like in Example 2 instead,

$$\begin{array}{r|l} & x^2 - 9 \\ x - 2 & x^3 - 2x^2 - 9x + 18 \\ & -(x^3 - 2x^2) \\ \hline & -9x + 18 \\ & -9x + 18 \\ \hline & \boxed{0} \end{array}$$

$\longrightarrow f(x) = (x-2)(x^2-9)$
$= \underline{(x-2)(x-3)(x+3)}.$

Th^{m} (Descartes' Rule of Signs)

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. Then,

- the # of positive real zeros of f equals the # of variations in the sign of the nonzero coefficients of $f(x)$ <u>or</u> else equals that # less an even integer.
- the # of negative real zeros of f equals the # of variations of the nonzero coeff. of $f(-x)$ or else equals that # less an even integer.

<u>Example</u>: $f(x) = 3x^6 - 4x^4 + 3x^3 + 2x^2 - x - 3$

$+ \quad - \quad + \quad + \quad - \quad -$

$1 \quad 2 \quad 3$

<u>3 variations in sign.</u>

$\Rightarrow$ either one or 3 real positive zeros for $f(x)$.

$f(-x) = 3x^6 - 4x^4 - 3x^3 + 2x^2 + x - 3$

$+ \quad - \quad - \quad + \quad + \quad -$

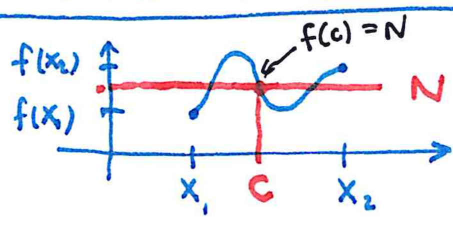
$1 \quad 2 \quad 3$

<u>3 variation in sign</u>

$\Rightarrow$ either one or three real negative zeros for $f(x)$.

Th^{m} / Intermediate Value Th^{m}

If $x_1 < x_2$ and $f(x_1) \le N \le f(x_2)$ then $\exists\, c \in (x_1, x_2)$ for which $f(c) = N$. (same for $f(x_2) \le N \le f(x_1)$)



<u>Typical use</u>:

$f(x_1) < 0 < f(x_2)$

$f(x_1) > 0 > f(x_2)$

then $\exists\, r$ between x_1 & x_2 such that $f(r) = 0$.

<u>Example</u>: for $f(x)$ given in previous example notice

$f(1) = 3 - 4 + 3 + 2 - 1 - 3 = 0$

$f(0) = -3$

$f(-1) = 3 - 4 - 3 + 2 + 1 - 3 = -4$

$f(100) =$ large positive #

$f(-100) =$ large positive #

There must be $r \in (-100, -1)$ s.t. $f(r) = 0$.

Better Example,

Root Finder

Given polynomial $f(x)$ can we find sol^{n} to $f(x) = 0$ on $[a, b]$?

① find $f(a)$ & $f(b)$ if either is zero, done!

② Let $x_1 = \frac{a+b}{2}$ and find $f(x_1)$

③ -if there is a sign-flip from $f(a)$ to $f(x_1)$ then calculate $x_2 = \frac{a+x_1}{2}$ and find $f(x_2)$.

-if there is a sign-flip from $f(x_1)$ to $f(b)$ then calculate $x_2 = \frac{x_1+b}{2}$ and find $f(x_2)$.

continue, each time zooming into where there is a change in sign for $f(x)$

Example: $f(x) = 10x^2 + 8x - 24$. Can we solve $f(x) = 0$ for $0 \le x \le 2$.

$f(0) = -24$ and $f(2) = 40 + 16 - 24 = 32$

Midpoint $x_1 = \frac{2+0}{2} = 1$

$f(1) = 10 + 8 - 24 = -6 \Rightarrow x_2 = \frac{1+2}{2} = 1.5$

$f(1.5) = 10(1.5)^2 + 8(1.5) - 24 = 22.5 + 12 - 24 = 10.5$

$\Rightarrow x_3 = \frac{1+1.5}{2} = 1.25$

$f(1.25) = 10(1.25)^2 + 8(1.25) - 24 = 1.625$

$\Rightarrow x_4 = \frac{1+1.25}{2} = 1.125$

$f(1.125) = 10(1.125)^2 + 8(1.125) - 24 = -2.34 \Rightarrow x_5 = \frac{1.125+1.25}{2}$

$x_5 = 1.1875$

$f(1.1875) = 23.6 - 24 = -0.4$

$\Rightarrow x_6 = \frac{1.1875 + 1.25}{2} = 1.21875$

$f(1.21875) = 24.6 - 24 = 0.6 \Rightarrow x_7 = \frac{1.1875 + 1.21875}{2} \cong 1.203$

$f(1.203) = 0.099$ (actually, $f(1.2) = 0$)