On Sections and Curvature for Super Yang-Mills Theory

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In physics, Super Yang-Mill's Theory is expressed in terms of local coordinate dependent expressions on superspace. We show how the classical constructions of Lichernowicz extend to the G-infinity super context. When the curvature is restricted to leaves of a certain foliation it is shown to be trivial. However, the curvature is also nontrivial in other directions. The Bianchi identities have solutions which are used to construct the Lagrangian. Our goal is to show how the theory on the base superspace can be seen as a pull-back of a theory on a super bundle space. These results are derived over an infinite dimensional Banach manifold which possesses a G-infinity supermanifold structure.

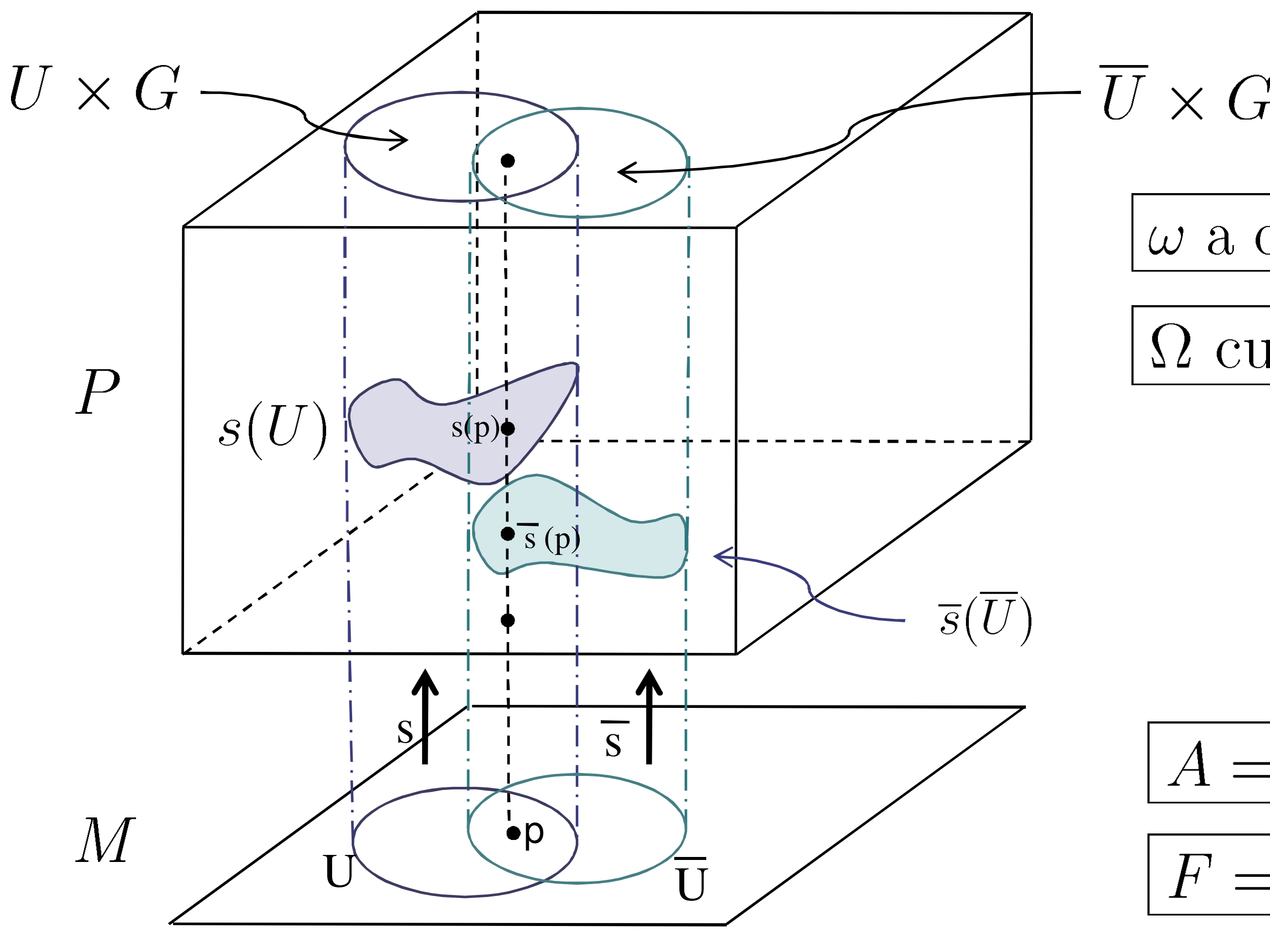
Yang Mills Theory

- In standard Yang Mills theory one considers a princpal fiber bundle P with Lie group G and a Lie-algebra valued one form ω which has curvature $\Omega = D\omega$.
- Given a local section s the connection and curvature pull-back to the base space to give the **potential** $A = s^*\omega$ and **field strength** $F = s^*\Omega$ found in physics.
- The **gauge transformations** *postulated* by physicists arise naturally by comparing the two overlapping local sections. In particular, if $\overline{s} = s \cdot g$ then $\overline{s}^*\Omega = \overline{F} = g^{-1}Fg$ and $\overline{s}^*(\omega) = \bar{A} = g^{-1}Ag + g^{-1}dg$.
- The action is invariant under the Poincare and internal symmetry group

$$\boxed{S_{YM} = \int \frac{1}{4} F_{\mu\nu} F^{\mu\nu} d^4x}$$

Principal Fiber Bundle Geometry of Yang Mills Theory

The base space of the principal fiber bundle is an ordinary manifold. Typically it is either taken to be a 4-dimensional Lorentzian space or a 4-dimensional Euclidean space.



$\boxed{\omega \text{ a connection on } P}$

$\boxed{\Omega \text{ curvature of } \omega}$

$\boxed{A = s^*\omega}$

$\boxed{F = s^*\Omega}$

Superspace and Supersymmetry

- The only known physically reasonable extension of the Poincare algebra is the super Poincare algebra. This algebra generates supersymmetry

$$
\begin{aligned}
&[P_m, P_n] = 0 \\
&[P_m, J_{nk}] = i(\eta_{mn} P_k - \eta_{mk} P_n) \\
&[J_{mn}, J_{lk}] = i(\eta_{nl} J_{mk} - \eta_{ml} J_{nk} + \eta_{mk} J_{nl} - \eta_{nk} J_{ml}) \\
&[Q_\alpha, P_m] = 0 \\
&[\bar{Q}_{\dot\alpha}, P_m] = 0 \\
&[J_{mn}, Q_\alpha] = -i(\sigma_{mn})_\alpha^{\ \beta} Q_\beta \\
&[J_{mn}, \bar{Q}_{\dot\alpha}] = -i(\bar\sigma_{mn})_{\dot\alpha}^{\ \dot\beta} \bar{Q}_{\dot\beta} \\
&\{Q_\alpha, Q_\beta\} = 0 \\
&\{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \\
&\{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^m_{\alpha\dot\beta} P_m
\end{aligned}
$$

- **Superfields** are functions of **superspace**. Superspace consists of space-time x^μ plus 4 anticommuting coordinates θ^α, $\bar\theta^{\dot\alpha}$. Superfields provide a representation of SUSY.

$$
\boxed{Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^m} \qquad \bar{Q}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial x^m}}
$$

- Physicists have yet to experimentally verify SUSY, but

Predictions of Super Yang Mills Theory

If the Minimally Supersymmetric Standard Model is correct then a number of new particles should be found in future experiments:

Table 1.1: Predictions of Supersymmetry

SM Particle	Spin	SUSY	superpartner in MSSM	Spin
electron	1/2	$\leftrightarrows$	selectron	0
photon	1	$\leftrightarrows$	photino	1/2
quark	1/2	$\leftrightarrows$	squark	0
gluon	1	$\leftrightarrows$	gluino	1/2
Higgs	0	$\leftrightarrows$	Higgino	1/2

Superfields for SYM Theory

- A single superfield U typically contains many component fields. The component fields are functions of spacetime.

$$\boxed{U = f + \theta\phi + \bar{\theta}\bar{\mathcal{X}} + \theta\theta m + \bar{\theta}\bar{\theta}n + \theta\sigma^n\bar{\theta}v_n + \theta\theta\bar{\theta}\bar{\Lambda} + \bar{\theta}\bar{\theta}\theta\psi + \theta\theta\bar{\theta}\bar{\theta}d.}$$

- A single superfield contains fields with different spin; superfields, by nature of their construction, provide a balance between bosons and fermions.

scalar fields	f, m, n, d	spin 0	commuting fields
Weyl spinors	$\phi, \bar{\mathcal{X}}, \bar{\lambda}, \psi$	spin 1/2	anticommuting fields
vector field	v_n	spin 1	commuting field

- There are several reductions of the unconstrained superfield U:

  V is a vector superfield if $V^\dagger = V$

  Φ is a chiral superfield if $\bar{D}_{\dot{\alpha}}\Phi = 0$

  $\bar{\Phi}$ is an antichiral superfield if $D_\alpha\bar{\Phi} = 0$

Gauge theory in superfield language

- Physicists postulate that the gauge transformation of e^V is given in terms of a chiral gauge parameter Λ where $\bar{D}_{\dot\alpha}\Lambda = 0$,

$$e^V \longmapsto e^{V'} = e^{-i\bar\Lambda} e^V e^{i\Lambda}$$

The gauge transformation of **spinor superfield** W_α follows from the transformation of e^V,

$$\boxed{W_\alpha = -\tfrac{1}{4}\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} e^{-V} D^\alpha e^V.} \qquad \boxed{W_\alpha \mapsto W'_\alpha = e^{-i\bar\Lambda} W_\alpha e^{i\Lambda}.}$$

- The action is invariant under the "gauge transformations" above,

$$\boxed{S_{SYM} = \int d^4x \; tr\left[W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta\bar\theta}\right]}$$

- Incidentally, the spinor superfields are pairs of (anti)chiral superfields and the component fields of the spinor superfield include the usual YM field strength and more:

$$\boxed{\bar{D}_{\dot\alpha} W_\alpha = 0} \qquad \boxed{W_\alpha = -i\lambda_\alpha + \theta_\alpha D - \sigma^{mn\beta}_{\alpha}\theta_\beta F_{mn} + \theta\theta\sigma^m_{\alpha\dot\beta}\mathcal{D}_m\bar\lambda^{\dot\beta}.}$$

Supermathematics

- Grassmann generators anticommute,

$$\zeta^i\zeta^j = -\zeta^j\zeta^i \qquad \zeta\zeta = 0$$

- Supernumbers are built with Grassmanns

$$Z = Z_o + Z_i\zeta^i + Z_{ij}\zeta^i\zeta^j + \cdots$$

- Supernumbers break into even/odd parts; $\Lambda = {}^0\Lambda \oplus {}^1\Lambda$

$$z = \underbrace{\sum_{p=0}^{\infty}\sum_{I\in\mathcal{I}_{2p}} z_I\zeta^I}_{even} + \underbrace{\sum_{p=0}^{\infty}\sum_{I\in\mathcal{I}_{2p+1}} z_I\zeta^I}_{odd}$$

- Flat superspace of dimension (p|q) provides p-commuting Grassmann variables and q-anticommuting variables

$$\begin{aligned} K^{p|q} &= ({}^0\Lambda)^p \times ({}^1\Lambda)^q \\ &= \{(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q) \mid x^i \in {}^0\Lambda,\ \theta^\alpha \in {}^1\Lambda\} \end{aligned}$$

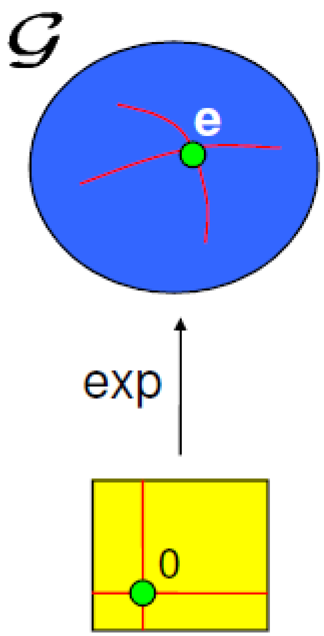
- Superspace is a Banach space thus one can consider Frechet derivatives and smooth functions.

- A function which is smooth and satisfies an additional linearity condition which respects the (p|q) structure is said to be supersmooth. A function is G^1 if

$$f(a+h, b+k) \approx f(a,b) + \sum_{m=0}^{p} h^m \left(\frac{\partial f}{\partial x^m}\right)(a,b) + \sum_{\alpha=0}^{q} k^\alpha \left(\frac{\partial f}{\partial \theta^\alpha}\right)(a,b)$$

G^∞ Super Lie Group

- A supermanifold $\mathcal{G}$ which is also group with G^∞ operations is a super Lie group.



$$\mathfrak{g} = T_e^0\mathcal{G}$$

- Tangent Module at e is $T_e\mathcal{G}$

$T_e^0\mathcal{G}$	$T_e^1\mathcal{G}$

- even: $\partial/\partial x^m \in T_e^0\mathcal{G}$
- odd: $\partial/\partial \theta^\alpha \in T_e^1\mathcal{G}$
- No coordinate basis for $T_e^0\mathcal{G}$

Super PFB, development, and restricted holonomy algebra

- Let $\mathcal{M}$ be a $(p|q)$ dimensional supermanifold and let $\tau : \mathcal{P} \to \mathcal{M}$ be a super principle fiber bundle with group a super Lie group $\mathcal{G}$. We allow $\mathcal{G}$ to be locally modelled on $\mathbb{C}^{r|s}$ or $\mathbb{R}^{r|s}$. We assume that τ and local sections $\mathfrak{s}$ of τ are even mappings. Thus $d\tau$ and $d\mathfrak{s}$ are Λ-linear even mappings at each point.

- ω is a connection on $\mathcal{P}$ and Ω its curvature. We assume that ω is even, i.e., we assume that $\omega(T^i\mathcal{P}) \subseteq \mathfrak{g}^i$, where $\mathfrak{g}$ is the Lie superalgebra of $\mathcal{G}$ and $i = 0, 1$. For each point $u \in \mathcal{P}$ we denote the space of horizontal vectors by H_u and $T_u\mathcal{P} = H_u \oplus V_u$.

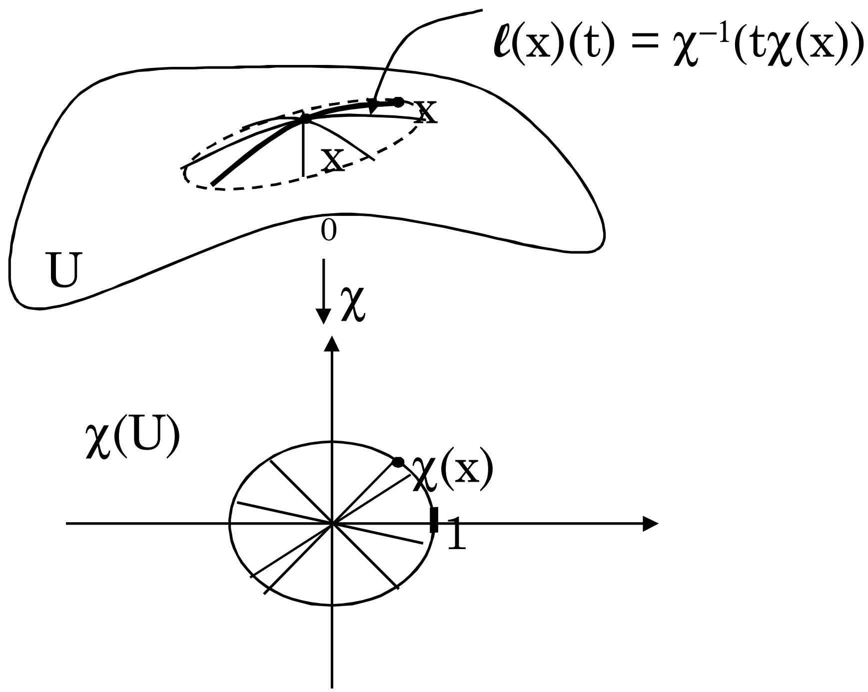
- **Definition 3.1.** *Assume that $\mathfrak{s} : U \to \mathcal{P}$ is a local section of τ and that $\gamma : I \to U$ is a path in U. There exists a unique mapping $g : I \to \mathcal{G}$ such that $\tilde{\gamma}(t) = \mathfrak{s}(\gamma(t))g(t)$ for each $t \in I$. The curve g satisfies the equation*

$$\dot{g}(t)g(t)^{-1} = -(\mathfrak{s}^*\omega)(\dot{\gamma}(t))$$

for all $t \in I$ and is called the development of γ relative to the local section $\mathfrak{s}$. Notice that $\dot{\gamma}(t), \dot{\tilde{\gamma}}(t), \dot{g}(t)$ are all necessarily even tangent vectors for each $t \in I$ and $d\mathfrak{s}$ is necessarily an even mapping.

Special sections of Lichernowicz

- Given $x_o \in \mathcal{M}$ and $z_o \in \mathcal{P}$ such that $\tau(z_o) = x_o$ we follow Lichernowicz [20] who constructs a local section $\mathfrak{s}$ of τ as follows. First choose a chart $\mathcal{X}$ at x_o defined on an open set U containing x_o such that $\mathcal{X}(x_o) = 0$ and $\mathcal{X}(U)$ is an open ball centered at 0 in $\mathbb{R}^{p|q}$ of radius greater than 1. We refer to the curves $\{ \mathcal{X}^{-1}(tu) \mid 0 \le t \le 1 \}$ as "rays" in U. The section $\mathfrak{s}$ is obtained by horizontally lifting each of these rays to z_o. Thus $\mathfrak{s}(U)$ is the union of the horizontal lifts of the paths $t \mapsto \mathcal{X}^{-1}(tu)$ to z_o. A section arising in this manner is called a special section at z_o by Lichernowicz, and we will use this terminology as well.



Super PFB, development, and restricted holonomy algebra

- **Definition 3.5.** *Denote by* $\mathfrak{h}^{\Omega}(U)$ *the Lie sub-superalgebra of* $\mathfrak{g}$ *generated by the set of elements* $\Omega_z(v,w)$ *for* $z \in \tau^{-1}(U)$ *and* $v, w \in T_z^0(\tau^{-1}(U))$.

- **Corollary 3.6.** *If* $\mathfrak{s} : U \to \mathcal{P}$ *is a special local section of* τ *at* $x_o \in U$ *and* $x \in U$ *then* $(\mathfrak{s}^*\omega)_x(v)$ *is in the* <u>*closure*</u> *of* $\mathfrak{h}^{\Omega}(U)$ *for each* $v \in T_x^0 U$.

- **Corollary 3.7.** *If* $\mathfrak{s} : U \to \mathcal{P}$ *is a special section of* τ *at* $x_o \in U$ *and if* $g : I \to \mathcal{G}$ *is the development of an arbitrary loop* $\gamma : I \to U$ *in* U *relative to* $\mathfrak{s}$, *then* $\dot{g}(t)g(t)^{-1} \in \mathfrak{h}^{\Omega}(U)$ *for all* $t \in I$.

- **Corollary 3.8.** *If* $\Omega \equiv 0$, *then each development of each loop in the domain of each special section is trivial, i.e.* $g(t) = e$ *for all* $t \in I$. *Thus the horizontal lift of each loop in* U *is a loop in* $\mathcal{P}$.

Tangent bundles, integrable sub-bundles, Frobenius theorem

If $\mathcal{M}$ is a $(p|q)$ dimensional supermanifold, then vector superbundles E over $\mathcal{M}$ are defined as in the usual case of ordinary Banach manifolds except that the standard fibers of the bundle are $\mathbb{R}^{r|s}$ (for some r,s), all relevant manifolds are supermanifolds, and all relevant mappings are G^∞-mappings (see [19]) and [5]). If $E \hookrightarrow T^0\mathcal{M}$ is a sub-bundle of the even tangent bundle $T^0\mathcal{M} \to \mathcal{M}$, then we say that $E \to \mathcal{M}$ is integrable iff whenever X, Y are sections of $E \to \mathcal{M}$ then so is $[X, Y]$. Since $T^0\mathcal{M} = T\mathcal{BM}$, the sub-bundle $E \to \mathcal{M}$ is an integrable sub-bundle of the tangent bundle $T\mathcal{BM} \to \mathcal{BM}$ of the underlying Banach manifold $\mathcal{BM}$, and so by the Frobenius theorem for Banach manifolds [19], one has a foliation of $\mathcal{BM}$. The leaves of this foliation are initial submanifolds of $\mathcal{BM}$.

Existence of pure gauge solutions

- **Proposition 3.10.** *Let $\tau : \mathcal{P} \to \mathcal{M}$ be a super principal fiber bundle with structure group a super Lie group $\mathcal{G}$. Let $\mathcal{F}_{\mathcal{M}}$ be a regular foliation of $\mathcal{M}$ and $\omega : T\mathcal{P} \to \mathfrak{g}$ an even connection with values in the super Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. Assume that at each point $x_o \in \mathcal{M}$ there exists a local section $\mathfrak{s}_o : U_o \to \mathcal{P}$ of τ at x_o such that the restriction of ${\mathfrak{s}_o}^*\Omega$ to $U_o \cap \mathcal{L}$ is zero for each leaf $\mathcal{L}$ of $\mathcal{F}_{\mathcal{M}}$ having the property that $U_o \cap \mathcal{L} \neq \emptyset$. Then there exists a local section $\mathfrak{s}$ defined on $U \subseteq U_o$ along with distinguished points $x_o^{\mathcal{L}} \in \mathcal{L} \cap U$ such that the horizontal lift of each loop γ in $U \cap \mathcal{L}$ at $x_o^{\mathcal{L}}$ is $\mathfrak{s}_{\mathcal{L}} \circ \gamma$ which is a loop in $\tau^{-1}(U \cap \mathcal{L})$. Here $\mathfrak{s}_{\mathcal{L}} = \mathfrak{s}|(U \cap \mathcal{L})$. In our informal language above, loops in a pancake based at the center of the pancake lift to loops in $\mathcal{P}$ under $\mathfrak{s}$.*

- **Theorem 3.12.** *Assume that $\tau : \mathcal{P} \to \mathcal{M}$ is a super principal fiber bundle with structure group a super Lie group $\mathcal{G}$ where $\mathcal{G}$ acts on the left of the bundle $\mathcal{P}$ (contrary to our convention up to this point). Let $\mathcal{F}_{\mathcal{M}}$ be a regular foliation of $\mathcal{M}$ whose leaves are supermanifolds of dimension $(r|s)$. Let $\omega : T\mathcal{P} \to \mathfrak{g}$ be an even connection on $\mathcal{P}$ whose curvature restricted to $\tau^{-1}(\mathcal{L})$ is zero for each leaf $\mathcal{L}$ of $\mathcal{F}_{\mathcal{M}}$. If $x_o \in \mathcal{M}$, then there exists an open subset U of $\mathcal{M}$ about x_o on which there is defined a local section $\mathfrak{s}$ of τ and a mapping $g : U \to \mathcal{G}$ such that $g(x_o) = e$. Moreover, the mapping g has the property that if $p \in U$ and $\mathcal{L}$ is the leaf of $\mathcal{F}_{\mathcal{M}}$ containing p then $g(p)^{-1} d_p g(v) = -(\mathfrak{s}^*\omega)_p(v)$ for every $v \in T_p^0(U \cap \mathcal{L})$.*

Banach arguments yield even result, then we extend to the odd-part through a standard argument:

- **Corollary 3.13.** *If $g : U \to \mathcal{G}$ is defined as in Theorem 3.12 and if we utilize the convention that in a principal fiber bundle the group $\mathcal{G}$ acts on the left of the bundle, then*

$$g(p)^{-1} d_p g(v) = -(\mathfrak{s}^* \omega)_p(v)$$

for every $v \in T_p(U \cap \mathcal{L})$ where $\mathcal{L}$ is the leaf containing p.

- *Proof.* Let $v \in T_p(U \cap \mathcal{L})$ be an odd tangent vector. For each odd supernumber ζ, ζv is an even tangent vector in $T_p^0(U \cap \mathcal{L})$, consequently

$$dL_{g(p)^{-1}}(d_p g(\zeta v)) = -(\mathfrak{s}^* \omega)_p(\zeta v).$$

But this implies that

$$\zeta dL_{g(p)^{-1}}(d_p g(v)) = -\zeta(\mathfrak{s}^* \omega)_p(v).$$

for all $\zeta \in {}^1\Lambda$. Thus

$$dL_{g(p)^{-1}}(d_p g(v)) = -(\mathfrak{s}^* \omega)_p(v).$$

and the corollary follows. $\square$

Connections and Quotients

- **Definition 3.15.** *If η is a differential form on an open subset U of $\mathcal{M}$, we write $\eta \approx 0$ iff for each leaf $\mathcal{L}$ of $\mathcal{F}_{\mathcal{M}}$ such that $U \cap \mathcal{L} \neq \emptyset$, $i_{\mathcal{L}}^{*}\eta = 0$ where $i_{\mathcal{L}} : U \cap \mathcal{L} \hookrightarrow U$ is the inclusion mapping. If η, ζ are both differential forms on U we write $\eta \approx \zeta$ iff $\eta - \zeta \approx 0$.*

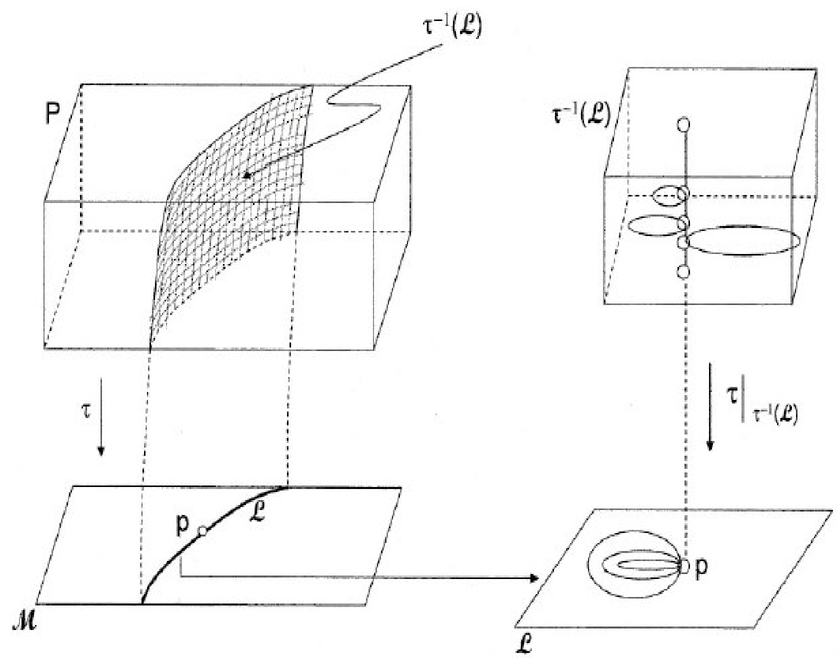
- **Proposition 3.16.** *Suppose $\mathfrak{s}_\alpha$ is a special section on U_α. For each $\alpha \in \mathfrak{I}$ let $\tilde{\mathfrak{s}}_\alpha : U_\alpha \to \mathcal{P}$ be the local section of τ defined by $\tilde{\mathfrak{s}}_\alpha = g_\alpha \mathfrak{s}_\alpha$. Then $\tilde{\mathfrak{s}}_\alpha^{*}\omega \approx 0$.*

- **Theorem 3.20.** *Assume that ω is an even connection on the super principle fiber bundle $\tau : \mathcal{P} \to \mathcal{M}$ such that its curvature Ω satisfies $\Omega \approx 0$ and such that for every vector field X which is tangent to the leaves of $\mathcal{F}_{\mathcal{M}}$, it follows that $\mathcal{L}_{\tilde{X}}\omega = 0$ where $\tilde{X}$ is the ω-horizontal lift of X to $\mathcal{P}$. Then there is a smooth connection $\tilde{\omega}$ on $\tilde{\tau} : \mathcal{P}/\mathcal{F}_{\mathcal{P}} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}$ which is induced by ω in the sense that if $\tilde{\mathfrak{s}} : U \to \mathcal{P}$ is a local section of τ such that $\tilde{\mathfrak{s}}^{*}\omega \approx 0$ then $\hat{\mathfrak{s}}^{*}\tilde{\omega} \circ dq = \tilde{\mathfrak{s}}^{*}\omega$ where $\hat{\mathfrak{s}} : q(U) \to \mathcal{P}/\mathcal{F}_{\mathcal{P}}$ is the local section of $\tilde{\tau}$ defined by $\hat{\mathfrak{s}} \circ q = \rho \circ \tilde{\mathfrak{s}}$.*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\rho} & \mathcal{P}/\mathcal{F}_{\mathcal{P}} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tilde{\tau}} \\ \mathcal{M} & \xrightarrow[q]{} & \mathcal{M}/\mathcal{F}_{\mathcal{M}} \end{array}$$

Geometric set-up for Super Yang Mills Theory

- (1) $\tau : \mathcal{P} \to \mathcal{M}$ is a super principal fiber bundle over the supermanifold $\mathcal{M}$ with structure group a super Lie group $\mathcal{G}$ and an even projection mapping τ,
- (2) $\mathcal{F}_{\mathcal{M}}$ is a regular foliation of $\mathcal{M}$ whose leaves are sub-supermanifolds of dimension $(r|s)$ and there is an induced foliation $\mathcal{F}_{\mathcal{P}}$ of $\mathcal{P}$ whose leaves are $\tau^{-1}(\mathcal{L})$ where $\mathcal{L}$ is a leaf of $\mathcal{F}_{\mathcal{M}}$,
- (3) ω is an even connection on $\mathcal{P}$ with values in the super Lie algebra $\mathfrak{g}$ such that the curvature Ω of ω vanishes on the tangents to the leaves of $\mathcal{F}_{\mathcal{P}}$. Here it suffices to assume that Ω vanishes on <u>even</u> tangent vectors.



Application to SYM Theory

- Assume in this section that the supermanifold $\mathcal{M}$ is locally modeled on the Banach space $\mathbb{R}^{4|4}$. Thus at each point we have a chart whose components are $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$.

- Additionally we assume the existence of four odd vector fields X_1, X_2, Y_1, Y_2 defined on $\mathcal{M}$ such that Y_i is (super)conjugate to X_i for $i = 1, 2$, and we note that for each point $p \in \mathcal{M}$
$$E_p^0 = \{aX_1(p) + bX_2(p) \mid a, b \in {}^1\Lambda_{\mathbb{R}}\}$$
is a ${}^0\Lambda_{\mathbb{R}}$-submodule of $T_p^0\mathcal{M}$. The standard fiber of E^0 is $\mathbb{R}^{0|2}$.

- We assume that at each point $p \in \mathcal{M}$ there exists a chart $(U, (x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}))$ such that at each $q \in U$,
$$X_i(q) = \sum_{\alpha=1}^{2} M_i^\alpha(q) D_\alpha(q)$$
for some (real) supermatrix $M_i^\alpha(q)$. Here $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m}$ (we follow [30], see page 26) so that E^0 is an integrable subbundle of $T^0\mathcal{M}$.

- The conjugate bundle $\bar{E}_q^0$ is generated by Y_1, Y_2 and is locally spanned by $\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\bar{\theta}^{\dot{\alpha}}\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^m}$ for $\dot{\alpha} = 1, 2$ so that it too is an integrable subbundle of $T^0\mathcal{M}$.

Application to SYM Theory

- Each of the foliations $\mathcal{F}_{\mathcal{M}}^{chiral}$, $\mathcal{F}_{\mathcal{M}}^{antichiral}$ give rise to principal fiber bundles,

$$\tau^{chiral} : \mathcal{P}/\mathcal{F}_{\mathcal{P}}^{chiral} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}^{chiral}$$

$$\tau^{antichiral} : \mathcal{P}/\mathcal{F}_{\mathcal{P}}^{antichiral} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}^{antichiral}$$

  where $\tau : \mathcal{P} \to \mathcal{M}$ is any super principal bundle over $\mathcal{M}$ with super Lie group $\mathcal{G}$. Let $E_{\mathcal{P}}$ and $\bar{E}_{\mathcal{P}}$ denote the subbundles of $T\mathcal{P} \to \mathcal{P}$ corresponding to the foliations $\mathcal{F}_{\mathcal{P}}^{chiral}$ and $\mathcal{F}_{\mathcal{P}}^{antichiral}$ respectively, i.e., $E_{\mathcal{P}}$ and $\bar{E}_{\mathcal{P}}$ are the tangent spaces to the leaves of their respective foliations.

- If ω is any even connection on $\mathcal{P}$ such that its curvature Ω_p is zero on pairs of vectors from E_q and such that Ω_p is also zero on pairs of vectors from $\bar{E}_q$ for each $q \in \mathcal{P}$ and if the Lie derivative of ω is zero along horizontal lifts of vectors tangent to leaves of the two foliations, then there are induced connections ω^{chiral} and $\omega^{antichiral}$ on the corresponding quotient bundles defined above.

- We regard these connections as reformulations of the superconnections ϕ, $\tilde{\phi}$ defined by Gieres on page 64 of [12]. Our formulation encodes the chiral and antichiral "pre-gauge transformations", usually regarded as maps $\Sigma, \Pi : U \to \mathcal{G}$ such that $\bar{D}_{\dot{\alpha}}\Sigma = 0$ and $D_{\alpha}\Pi = 0$, as ordinary gauge transformations $\hat{\Sigma}, \hat{\Pi}$ on our quotient bundles.

Application to SYM Theory

- It should be emphasized that to make contact with the physics literature one must continue to work on the bundle $\tau : \mathcal{P} \to \mathcal{M}$; our quotient formalism merely provides a new conceptual framework at this point. The reason for this is that to obtain $N = 1$ super Yang-Mills theory, one must introduce additional constraints on Ω called the "conventional constraints". These constraints require that for $q \in \mathcal{M}$ and $v \in E_q$, $w \in \bar{E}_q$, $\Omega_q(v, w) = 0$.

- We emphasize, however, that even before these extra constraints are imposed, we know from equation (26) that at each point of $\mathcal{M}$ there exists an open set U on which one has coordinates $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ and local sections $\mathfrak{s}_1$, $\mathfrak{s}_2$ of τ defined on U such that for maps $\mathcal{U}, \mathcal{V} : U \to \mathcal{G}$

$$\boxed{(\mathfrak{s}_1^*\omega)_q(v) = -\mathcal{U}(q)^{-1} d_q\mathcal{U}(v), \qquad q \in U,\ v \in E_q}$$

$$\boxed{(\mathfrak{s}_2^*\omega)_q(w) = -\mathcal{V}(q)^{-1} d_q\mathcal{V}(w), \qquad q \in U,\ w \in E_q}$$

$$\boxed{(\mathfrak{s}_1^*\Omega)(v_1, v_2) = 0, \qquad v_1, v_2 \in E_q}$$

$$\boxed{(\mathfrak{s}_2^*\Omega)(w_1, w_2) = 0, \qquad w_1, w_2 \in E_q}$$

Connection on chiral and antichiral quotient

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- We emphasize, however, that even before these extra constraints are imposed, we know from equation (26) that at each point of $\mathcal{M}$ there exists an open set U on which one has coordinates $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ and local sections $\mathfrak{s}_1$, $\mathfrak{s}_2$ of τ defined on U such that for maps $\mathcal{U}, \mathcal{V} : U \to \mathcal{G}$

$$\boxed{(\mathfrak{s}_1^*\omega)_q(v) = -\mathcal{U}(q)^{-1} d_q\mathcal{U}(v), \qquad q \in U,\, v \in \bar{E}_q} \qquad \boxed{(\mathfrak{s}_1^*\Omega)(v_1, v_2) = 0, \qquad v_1, v_2 \in \bar{E}_q}$$

$$\boxed{(\mathfrak{s}_2^*\omega)_q(w) = -\mathcal{V}(q)^{-1} d_q\mathcal{V}(w), \qquad q \in U,\, w \in E_q} \qquad \boxed{(\mathfrak{s}_2^*\Omega)(w_1, w_2) = 0, \qquad w_1, w_2 \in E_q}$$

- We can construct a single gauge $\bar{\mathfrak{s}}$ on U such that the connection ω pulls back to a pure gauge along both the chiral and antichiral leaves

$$(\bar{\mathfrak{s}}^*\omega)|_E = -\mathcal{V}^{-1} d\mathcal{V}|_E \qquad (\bar{\mathfrak{s}}^*\omega)|_{\bar{E}} = -\mathcal{U}^{-1} d\mathcal{U}|_{\bar{E}}$$

The pullback of the curvature Ω vanishes on pairs of chiral and antichiral vectors,

$$(\bar{\mathfrak{s}}^*\Omega)|_{E \times E} = 0 \qquad (\bar{\mathfrak{s}}^*\Omega)|_{\bar{E} \times \bar{E}} = 0.$$

Recovering the local formula from super-PFB

- Define $\mathbb{A} = \mathfrak{s}^*\omega$ and $\mathbb{F} = \mathfrak{s}^*\Omega$. We see that

$$\mathbb{A}_\alpha = \mathbb{A}(D_\alpha) = (\mathfrak{s}^*\omega)(D_\alpha) = -\mathcal{V}^{-1}d\mathcal{V}(D_\alpha) = -\mathcal{V}^{-1}D_\alpha\mathcal{V}$$

$$\mathbb{A}_{\dot{\alpha}} = \mathbb{A}(\bar{D}_{\dot{\alpha}}) = (\mathfrak{s}^*\omega)(\bar{D}_{\dot{\alpha}}) = -\mathcal{U}^{-1}d\mathcal{U}(\bar{D}_{\dot{\alpha}}) = -\mathcal{U}^{-1}\bar{D}_{\dot{\alpha}}\mathcal{U}$$

$$\mathbb{F}_{\alpha\beta} = (\mathfrak{s}^*\Omega)(D_\alpha, D_\beta), \qquad \mathbb{F}_{\dot{\alpha}\dot{\beta}} = (\mathfrak{s}^*\Omega)(\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}), \qquad \mathbb{F}_{\alpha\dot{\beta}} = (\mathfrak{s}^*\Omega)(D_\alpha, \bar{D}_{\dot{\beta}})$$

The constraints $\mathbb{F}_{\alpha\beta} = 0$, $\mathbb{F}_{\dot{\alpha}\dot{\beta}} = 0$ are precisely the conditions $(\mathfrak{s}^*\Omega)(E \times E) = 0$, $(\mathfrak{s}^*\Omega)(\bar{E} \times \bar{E}) = 0$ respectively. The constraint $\mathbb{F}_{\alpha\dot{\beta}} = 0$ is called the "conventional constraint", and it can be stated in our bundle language as $(\mathfrak{s}^*\Omega)(E \times \bar{E}) = 0$.

Interpretation of Giere's local formulas from super-PFB viewpoint

- We believe Giere's definitions of the superconnections ϕ, $\tilde{\phi}$ on page 62 of [12] are slightly flawed. It seems certain what he wants are connections which are zero in chiral and antichiral directions (respectively) but which agree with $\mathbb{A}$ in transverse directions. If we are correct, then he should have

$$\begin{aligned} \phi &= \mathcal{U}\mathbb{A}\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \\ \tilde{\phi} &= \mathcal{V}\mathbb{A}\mathcal{V}^{-1} + (d\mathcal{V})\mathcal{V}^{-1}. \end{aligned} \tag{32}$$

It will then follow, for example, that on $\bar{E}$

$$\phi = \mathcal{U}\mathbb{A}\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \overset{\text{chiral}}{\approx} \mathcal{U}(-\mathcal{U}^{-1})d\mathcal{U}\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \tag{33}$$

hence,

$$\phi \overset{\text{chiral}}{\approx} -(d\mathcal{U})\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} = 0. \tag{34}$$

A similar result applies for $\tilde{\phi}$ over E, namely, $\tilde{\phi} \overset{\text{antichiral}}{\approx} 0$. This follows by replacing $\overset{\text{chiral}}{\approx}$ with $\overset{\text{antichiral}}{\approx}$ along with minor modifications. If our interpretation is correct, then his ϕ, $\tilde{\phi}$ induce our connections ω^{chiral} on $\mathcal{P}/\mathcal{F}_{\mathcal{P}}^{chiral} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}^{chiral}$ and $\omega^{antichiral}$ on $\mathcal{P}/\mathcal{F}_{\mathcal{P}}^{antichiral} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}^{antichiral}$ as defined more generally in previous sections of this paper.

Vector superfield's geometry

- Now notice that if we define local sections of τ by $\mathfrak{s}_{chiral} = \mathcal{U}\mathfrak{s}$ and $\mathfrak{s}_{antichiral} = \mathcal{V}\mathfrak{s}$ then

$$\begin{aligned} \mathfrak{s}^*_{chiral}\omega &= Ad(\mathcal{U})[(\mathfrak{s}^*\omega) + \mathcal{U}^{-1}d\mathcal{U}] \\ &= \mathcal{U}\mathbb{A}\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \\ &= \phi. \end{aligned} \tag{35}$$

Similarly we can derive $\mathfrak{s}^*_{antichiral}\omega = \tilde{\phi}$. Note also that we can connect both of these sections by the equation $\mathfrak{s}_{antichiral} = \mathcal{V}\mathcal{U}^{-1}\mathfrak{s}_{chiral}$ so that if $\mathcal{W} = \mathcal{V}\mathcal{U}^{-1}$, then

$$\begin{aligned} \tilde{\phi} = (\mathfrak{s}^*_{antichiral}\omega) &= Ad(\mathcal{W})[(\mathfrak{s}^*_{chiral}\omega) + \mathcal{W}^{-1}d\mathcal{W}] \\ &= \mathcal{W}\phi\mathcal{W}^{-1} + (d\mathcal{W})\mathcal{W}^{-1} \\ &= \mathcal{W}\phi\mathcal{W}^{-1} - \mathcal{W}d\mathcal{W}^{-1}. \end{aligned} \tag{36}$$

- After discussing some reality conditions, we can show $\mathcal{W} = exp(V)$ where the pre-potential V from the traditional physics literature. Notice that it locally produces a connection form on the supermanifold.

Our model of locally rigid superspace

First, we describe a general class of supermanifolds which are locally diffeomorphic to $\mathbb{R}^{4|4}$. Let (M, g) be a Lorentzian spacetime and assume that M has a spin structure, then there is a double cover SM of the g-orthonormal frame bundle O_gM with structure group $SL(2,\mathbb{C})$ the double cover of $SO^+(1,3)$ (see [4]). If $\Lambda : SL(2,\mathbb{C}) \to SO^+(1,3)$ denotes the covering mapping then $SL(2,\mathbb{C})$ acts on frames $\{e_i\} \in O_gM$ via $\{e_i\} \cdot S \equiv \{e_j \Lambda(S^{-1})^j_i\}$ and acts on $\mathbb{R}^{0|4}$ via $(\theta, \bar\theta) \cdot S \equiv (\theta^\beta S^\alpha_\beta, \bar\theta^{\dot\beta} \bar S^{\dot\alpha}_{\dot\beta})$.

These actions induce an action of $SL(2,\mathbb{C})$ on $O_gM \times \mathbb{R}^{0|4}$ and denoting this induced action by ρ, the associated bundle $\mathcal{M} = O_gM \times_\rho \mathbb{R}^{0|4}$ is a vector bundle with standard fiber $\mathbb{R}^{0|4}$. By Bachelor's Theorem [3] (also see [26] page 92) it has the structure of a supermanifold locally modeled on $\mathbb{R}^{4|4}$.

We assume there exists an atlas $\mathcal{A}$ on $\mathcal{M}$ such that if $(x, \theta, \bar\theta)$ and $(\tilde x, \tilde\theta, \tilde{\bar\theta})$ are two charts with intersecting domains, then

$$\begin{aligned} \tilde x^n &= g^n(x) \\ \tilde\theta^\alpha &= S^\alpha_\beta(x)\theta^\beta \\ \tilde{\bar\theta}^{\dot\alpha} &= \bar S^{\dot\alpha}_{\dot\beta}(x)\bar\theta^{\dot\beta} \end{aligned} \tag{39}$$

where $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^{4|0}$ and $S(x) \in SL(2,\mathbb{C})$. Thus $x \mapsto S^\alpha_\beta(x)$ are transition functions induced from those of the spin bundle SM as in Theorem 8.1.1 of [26].

A mathematical model of N=1 pure SYM Theory

- 1. the super Lie group of the superprincipal bundle must be a super Lie group with super Lie algebra the Grassmann shell of an ordinary Lie algebra whose generators $\{T_a\}$ are real and are subject to the mild condition $[T_a, T_b]^\dagger = [T_b, T_a]$,(see the Appendix)

- 2. at each point of the supermanifold $\mathcal{M}$ there exists a local section $\mathfrak{s}$ of the superprincipal bundle $\mathcal{P} \to \mathcal{M}$ such that $\mathbb{A} \equiv \mathfrak{s}^*\omega$ satisfy the condition $\mathbb{A}^\dagger = -\mathbb{A}$,

- 3. at each point of the supermanifold $\mathcal{M}$ there exists a local section $\tilde{\mathfrak{s}}$ of the subbundle $\mathcal{Z}_0$ of the frame bundle of $\mathcal{M}$.

- 4. The spinor superfields W_α are related to the components of the curvature pulled down to the base supermanifold as follows:

$$W^\alpha = (-i/4)\mathbb{F}_{a\dot{\alpha}}\overline{\sigma}^{a\dot{\alpha}\alpha}$$
$$\overline{W}^{\dot{\alpha}} = (-i/4)\mathbb{F}_{a\alpha}\overline{\sigma}^{a\dot{\alpha}\alpha}.$$

$$S_W = \int d^4x \; tr\left[W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}}\right]$$

Summary of Results

- Superfield gauge transformations also can be interpreted as equations which relate different sections of a super-PFB.

- Spinor superfields are closely related to the curvature of a connection of a super-PFB.