16.2. SEPARATION OF VARIABLES

This section begins with a short proof of the method. In summary, separating variables is u-substitution. We almost did these last semester, what differs is notation and presentation of the problem. At the conclusion of this section I have a number of examples which show the wide and diverse application of first order ODEs. Let's begin.

Certain 1st order ode's can be seen to have the form

$$\frac{dy}{dx} = g(x)\,f(y)$$

In which case we can seperate the variables

$$\frac{dy}{f(y)} = g(x)\,dx$$

And then integrate

$$\int \frac{dy}{f(y)} = \int g(x)\,dx$$

When you can actually do these integrals this will implicitly (and sometimes explicitly) define y in terms of x; that means you found a sol^n!

Pf/ $\displaystyle\int \frac{dy}{f(y)} = \int \frac{g(x)f(y)\,dx}{f(y)}$ ∴ Changing variables to x: $dy = g(x)f(y)\,dx$ by assumption

$$= \int g(x)\,dx \;//$$

Example 16.2.1

$\boxed{E1}$ $\displaystyle\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k\,dt$ then integrate

$\displaystyle\int \frac{dy}{y} = \int k\,dt \Rightarrow \underline{\ln|y| = kt + c}$ (implicit sol^n)

Thus $y(t) = \pm e^{kt+c} = \pm e^c e^{kt} = \boxed{y_o e^{kt} = y(t)}$ (explicit sol^n)

If $y(0) = 3$ find the sol^n

$y(0) = y_o e^{k(0)} = \boxed{y_o = 3} \Rightarrow \underline{y(t) = 3e^{kt}}$

(This is why y_o is good notation here)

Remark: $k > 0$ exponential growth eq^n. $k < 0$ exponential decay eq^n	(More on this later)

Example 16.2.2

ITC

E2 $\frac{dy}{dx} = a^{x+y}$: find sol^{n} thru sep. of variables

$\int a^{-y}dy = \int a^{x}dx$: seperate then integrate.

$\frac{-1}{\ln(a)} a^{-y} = \frac{1}{\ln(a)} a^{x} + \tilde{c}$

$a^{-y} = -a^{x} - \ln(a)\tilde{c} = c - a^{x}$: want to solve for y.

$\ln(a^{-y}) = \ln(c - a^{x})$: it is crucial to insure we are taking the ln of positive quantites! that is why we moved the minus sign to the other side.

$-y \ln(a) = \ln(c - a^{x})$

$$\boxed{y = \frac{-\ln(c - a^{x})}{\ln(a)}}$$

Notice that c is arbitrary and can only be specified if we supply further demands (an intial or boundary condition)

Example 16.2.3

E3 $\frac{du}{d\theta} = \frac{2\theta + \sec^2\theta}{2u}$ find sol^{n} with $u(0) = -5$

$2u\,du = (2\theta + \sec^2\theta)d\theta$: seperated variables, now integrate,

$u^2 = \theta^2 + \tan\theta + C$: an implicit sol^{n}.

$\therefore\ u = \pm\sqrt{\theta^2 + \tan\theta + c}$; an explicit sol^{n}.

$u(0) = \pm\sqrt{c} = -5$

$\Rightarrow -\sqrt{c} = -5$ (Must choose negative sqrt. sol^{n})

$\Rightarrow c = 25$

$$\boxed{u(\theta) = -\sqrt{\theta^2 + \tan\theta + 25}}$$

Example 16.2.4

(173)

E4 Find implicit sol^{n} to $\frac{dy}{dx} = \frac{\cos(x)}{y^4+y^2+23}$. Also find Equilbrium sol^{n}'s.

$$\int (y^4 + y^2 + 23)\,dy = \int \cos(x)\,dx$$

$$\boxed{\frac{1}{5}y^5 + \frac{1}{3}y^3 + 23y = \sin(x) + C} \leftarrow$$ this implicitly solves the DE$_q$^{n}.

Here we cannot simply solve for y as a fnct. of x. The sol^{n} is still useful because it describes how x & y are related along the sol^{n} curves. Next find Equilibrium Sol^{n}'s

$$\frac{dy}{dx} = 0 \Rightarrow \frac{\cos(x)}{y^4+y^2+23} = 0$$

$$\Rightarrow \cos(x) = 0$$

$$\Rightarrow \boxed{x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}.}$$

Example 16.2.4b (each time, separate then integrate)

§2.2#7 $\frac{dy}{dx} = \frac{1-x^2}{y^2} \Rightarrow y^2 dy = (1-x^2)dx$

integrating both sides: $\Rightarrow \boxed{y^3/3 = x - x^3/3 + C}$ (implicit sol^{n} for DE$_q$^{n})

§2.2#8 $\frac{dy}{dx} = \frac{1}{xy^3} \Rightarrow \int y^3 dy = \int \frac{dx}{x} \Rightarrow \boxed{\frac{1}{4}y^4 = \ln|x| + C}$

§2.2#9 $\frac{dy}{dx} = y(2+\sin(x)) \Rightarrow \int \frac{dy}{y} = \int (2+\sin(x))dx \Rightarrow \boxed{\ln|y| = 2x - \cos(x) + C}$

§2.2#10 $\frac{dx}{dt} = 3xt^2 \Rightarrow \int \frac{dx}{x} = \int 3t^2 dt \Rightarrow \boxed{\ln|x| = t^3 + C}$

Example 16.2.4c (each time, separate then integrate)

§2.2#11 $\frac{dy}{dx} = \frac{\sec^2(y)}{1+x^2} \Rightarrow \int \frac{dy}{\sec^2(y)} = \int \frac{dx}{1+x^2}$ note $\frac{1}{\sec^2 y} = \cos^2(y)$

$$\int \frac{dy}{\sec^2 y} = \int \cos^2 y\, dy \overset{(*)}{=} \int \tfrac{1}{2}(1+\cos(2y))dy = \tfrac{1}{2}(y + \tfrac{1}{2}\sin(2y)) + C$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$$

(*) trig identity

Thus we find $\boxed{\tfrac{1}{2}y + \tfrac{1}{4}\sin(2y) = \tan^{-1}(x) + C}$

Example 16.2.4d

§2.2#12 $x\frac{dv}{dx} = \frac{1-4v^2}{3v} \Rightarrow \frac{3v\,dv}{1-4v^2} = \frac{dx}{x}$ then consider,

$$\int \frac{3v\,dv}{1-4v^2} = \int \frac{3\frac{-1}{8}du}{u} = -\frac{3}{8}\ln|u| + C = -\frac{3}{8}\ln|1-4v^2| + C$$

$u = 1-4v^2$
$du = -8v\,dv$

thus $\boxed{-\frac{3}{8}\ln|1-4v^2| = \ln|x| + C}$

Example 16.2.4e

§2.2#20 $\frac{dy}{dx} = \frac{3x^2+4x+2}{2y+1} \Rightarrow (2y+1)dy = (3x^2+4x+2)dx$

$$\Rightarrow y^2 + y = x^3 + 2x^2 + 2x + C$$

$y(0) = -1 \Rightarrow (-1)^2 - 1 = 0 + 2(0)^2 + 2(0) + C \Rightarrow \boxed{C = 0}$

therefore the sol^{n} with $y(0) = -1$ is $\boxed{y^2 + y = x^3 + 2x^2 + x}$

Example 16.2.4f

§2.2#21 $\frac{dy}{dx} = 2\sqrt{y+1}\cos(x) \Rightarrow \int \frac{dy}{\sqrt{y+1}} = \int 2\cos(x)\,dx$

$$\int \frac{dy}{\sqrt{y+1}} = \int \frac{du}{\sqrt{u}} = \frac{u^{\frac{1}{2}}}{1/2} + C = 2\sqrt{y+1} + C$$

letting $u = y+1$
so $du = dy$

$\therefore\ 2\sqrt{y+1} + C = 2\sin(x)$

We know $y(\pi) = 0 \Rightarrow 2\sqrt{0+1} + C = 2\sin(\pi)$

$\Rightarrow 2 + C = 0 \Rightarrow C = -2$

$\Rightarrow \boxed{2\sqrt{y+1} - 2 = 2\sin(x)}$

Example 16.2.4g

§2.2#22 $x^2 dx + 2y\,dy = 0 \Rightarrow \int x^2 dx = \int -2y\,dy$

thus $\frac{1}{3}x^3 = -y^2 + C$. We know $y(0) = 2 \Rightarrow 0 = -4 + C \therefore \underline{C = 4}$

Hence $\boxed{\frac{1}{3}x^3 = y^2 + 4}$.

Example 16.2.4h

§2.2#23 $\frac{dy}{dx} = 2x\cos^2(y) \Rightarrow \int \frac{dy}{\cos^2 y} = \int 2x\,dx \overset{(*)}{\Rightarrow} \tan(y) = x^2 + C$.

$(*)\left(\frac{1}{\cos^2 y} = \sec^2 y \text{ and } \int \sec^2 y\,dy = \tan(y) + c\right)$ then $y(0) = \frac{\pi}{4} \Rightarrow \underline{\tan(\pi/4) = 0 + C}$

Now $\tan(\pi/4) = 1$ thus $\boxed{\tan(y) = x^2 + 1}$

Example 16.2.4i

§2.2#24 $\frac{dy}{dx} = 8x^3 e^{-2y} \Rightarrow e^{2y} dy = 8x^3 dx$ H ⑥

$\Rightarrow \frac{1}{2}e^{2y} = 2x^4 + C$

$y(1) = 0 \Rightarrow \frac{1}{2}e^{(0)} = 2(1)^4 + C \Rightarrow \frac{1}{2} = 2 + C \Rightarrow \underline{C = -3/2}$

thus $\boxed{\frac{1}{2}e^{2y} = 2x^4 - \frac{3}{2}}$ ok I'll find the explicit sol^{c}

for this one, $e^{2y} = 4x^4 - 3 \Rightarrow 2y = \ln(4x^4 - 3)$

$\Rightarrow \boxed{y = \ln\left(\sqrt{4x^4 - 3}\right)}$.

Example 16.2.4j

§2.2#25 $\frac{dy}{dx} = x^2(1+y) \Rightarrow \int \frac{dy}{1+y} = \int x^2 dx \Rightarrow \ln|1+y| = \frac{x^3}{3} + C$.

Now $y(0) = 3 \Rightarrow \ln|4| = 0^3/3 + C \therefore C = \ln(4)$ hence

$\boxed{\ln|1+y| = \frac{1}{3}x^3 + \ln(4)}$

Example 16.2.4k

§2.2#26 $\sqrt{y}\,dx + (1+x)dy = 0 \Rightarrow \int \frac{dy}{\sqrt{y}} = \int \frac{-dx}{1+x}$

thus integrating yields $2\sqrt{y} = -\ln|1+x| + C$

Then $y(0) = 1 \Rightarrow 2 = -\ln(1) + C = C \therefore \boxed{2\sqrt{y} = -\ln|1+x| + 2}$

Or if you want $\boxed{y = \left[\ln\left(\frac{1}{\sqrt{|1+x|}}\right) + 2\right]^2}$

Example 16.2.5 (I often cover this in calculus I)

$\boxed{E5}$ $mg = ma = m\frac{dV}{dt} = m\frac{dx}{dt}\frac{dV}{dx} = mV\frac{dV}{dx}$

Lets solve $mg = mV\frac{dV}{dx}$ to find velocity as function of x

Cancel m to begin

$$g = V\frac{dV}{dx}$$

$$\int g\,dx = \int V\,dV$$

$$gx = \frac{1}{2}V^2 + C$$

Now if $V(x_o) = V_o$ then $gx_o = \frac{1}{2}V_o^2 + C \therefore C = gx_o - \frac{1}{2}V_o^2$

Yielding that,

$$gx = \frac{1}{2}V^2 + gx_o - \frac{1}{2}V_o^2$$

relates velocity & position w/o reference to time.

$$g(x - x_o) = \frac{1}{2}(V^2 - V_o^2) \Rightarrow \boxed{V^2 = V_o^2 + 2g(x - x_o)}$$

Remark: Many physics problems boil down to solving some differential equation. Sometimes it is actually a *system of differential equations*. You've probably heard of Newton's Second Law; $\vec{F} = m\vec{a}$. This is actually a special case where the mass is constant. The general form of the Second Law is $\vec{F} = \frac{d\vec{P}}{dt}$ where the momentum $\vec{P} = m\vec{v}$. In the special case m is constant we find this simplifies to $\vec{F} = m\vec{a}$ since $\frac{d\vec{P}}{dt} = \frac{d}{dt}m\vec{v} = m\frac{dv}{dt} = m\vec{a}$. We'll learn the details of such calculations in calculus III, for now (other than this discussion) we have to study one-dimensional motion because I don't assume you understand the nuts and bolts of vector math. Notice that the Second Law is a system of differential equations since if $\vec{P} = < P_x, P_y, P_z >$ then

$$\vec{F} = \frac{d\vec{P}}{dt} \iff < F_x, F_y, F_z > = < \frac{dP_x}{dt}, \frac{dP_y}{dt}, \frac{dP_z}{dt} >$$
$$\iff F_x = \frac{dP_x}{dt}, \ F_y = \frac{dP_y}{dt}, \ F_z = \frac{dP_z}{dt}.$$

For example, if $\vec{F} = < 1, t, t^2 >$ then we'd have to solve all three differential equations at once in order to find the momentum: assuming the initial momentum is zero,

$$1 = \frac{dP_x}{dt}, \quad t = \frac{dP_y}{dt}, \quad t^2 = \frac{dP_z}{dt} \implies P_x = t, \quad P_y = \frac{1}{2}t^2, \quad P_z = \frac{1}{3}t^3$$

This discussion is **not** part of the required material, I include it in the hopes of giving a better context to the other examples.

Example 16.2.6 (an example with variable mass)

$\boxed{E6}$ The Falling Raindrop: Imagine a drop falling thru a cloud gathers water as it falls, let $m(t)$ be it's varying mass. Further assume as the drop gets bigger it gathers more & more mass proportionate to it's mass; $\frac{dm}{dt} = km$ for $k > 0$.

$F = ma$ is more generally $F = \frac{dP}{dt}$ when m varies.

$$F = \frac{dP}{dt} = \frac{d}{dt}(mv) = mg \quad (\text{fall's due to gravity})$$

$$\frac{dm}{dt} v + m\frac{dv}{dt} = mg$$

$$kmv + m\frac{dv}{dt} = mg$$

$$\frac{dv}{dt} = \frac{mg - kmv}{m} = g - kv$$

$$\therefore \frac{dv}{kv - g} = -dt \Rightarrow \frac{dv}{v - g/k} = -k\,dt$$

Integrate both sides, $\ln|v - g/k| = -kt + \tilde{c}$ and exponentiate,

$$v - g/k = e^{-kt + \tilde{c}} = c\,e^{-kt}$$

$$\boxed{v(t) = ce^{-kt} + g/k}$$

The terminal velocity would be

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(c\underbrace{e^{-kt}}_{0} - g/k\right) = \boxed{g/k}$$

Remark: We took down as positive direction & Assumed no friction besides the water growth. Physically this amounts to water friction!

$$m\frac{dv}{dt} = ma = mg - kmv$$

When $v = g/k$ we have $mg - km\left(\frac{g}{k}\right) = 0$

terminal velocity happens when the forces balance

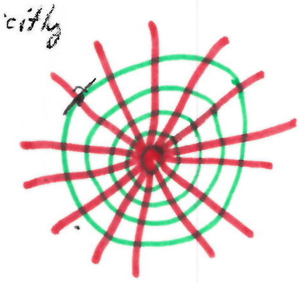
Orthogonal Trajectories (OT)

The word "orthogonal" means perpendicular as it applies to Euclidean geometry. To find a trajectory (just another word for a curve) which is perpendicular to a given curve $y = f(x)$ we would want a new curve $y = g(x)$ such that $g'(x) = \frac{-1}{f'(x)}$ wherever the curves interect. If the given curve $y = f(x)$ was a solution to a differential equation then the OT must be a solution to a different (but related) DEqn. Specifically,
if $\frac{dy}{dx} = F(x, y)$ has solution f then $\frac{dy}{dx} = \frac{-1}{F(x,y)}$ has solutions which are orthogonal to f.

Example 16.2.7

$\boxed{E7}$ $x^2 + y^2 = R^2$ defines a circle for each $R>0$. Let's find the orthogonal trajectories to this family of curves, diff. implicitly

$$2x + 2y\frac{dy}{dx} = 0 \quad \therefore \quad \frac{dy}{dx} = \frac{-x}{y} \quad \text{(for circles)}$$



Now then the O.T must have $\frac{dy}{dx} = \frac{-1}{-x/y} = \frac{y}{x}$ hence

$$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln|y| = \ln|x| + c$$

$$\Rightarrow y = \pm e^{\ln|x| + c} = \pm e^{\ln|x|}e^{c} = \pm x e^{c} = mx$$

This shows that orthogonal trajectories to circles are lines through the origin.

Example 16.2.8

$\boxed{E8}$ $x^2 - y^2 = k \Rightarrow 2x - 2y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$

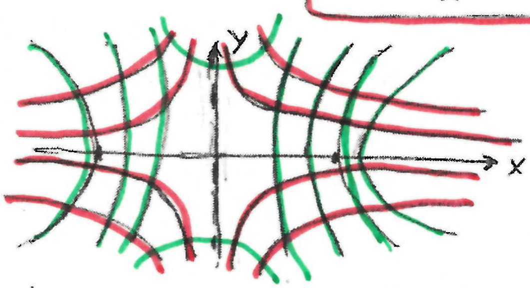
Hence the O.T. will have $\frac{dy}{dx} = \frac{-1}{x/y} = \frac{-y}{x}$

$$\therefore \frac{dy}{-y} = \frac{dx}{x} \Rightarrow -\ln|y| = \ln|x| + \tilde{c}$$

$$\Rightarrow \ln\left(\frac{1}{|y|}\right) = \ln(\bar{c}x)$$

$$\Rightarrow \frac{1}{|y|} = \bar{c}|x|$$

$$\Rightarrow \boxed{y = \frac{\pm c}{x} \text{ is the O.T.}}$$



$x^2 - y^2 = k$ is a hyperbola

$y = 0 \quad x = \pm k$

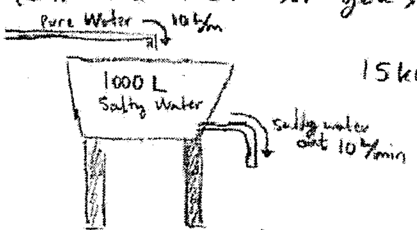
$x = 0 \quad y$ d.n.e

Example 16.2.9

E9 Mixing Problems

Consider some tank of fixed volume with some substance entering/exiting the tank, let $Y(t)$ be the amount of the substance at time t in the tank.

(I'll work #35 for you)



15 kg of salt at $t = 0$

Let $Y(t)$ = kg of salt in tank at time t

$$\frac{dY}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= 0 - \left(10 \frac{L}{min}\right) \cdot \left(\frac{Y(t)}{1000 L}\right)$$

$$= -\frac{1}{100} Y(t) / min \leftarrow kg/min \quad \text{makes sense}$$

$$= -Y/100$$

Thus $\frac{dY}{Y} = -\frac{dt}{100}$ $\therefore$ $\ln(Y) = -\frac{t}{100} + \tilde{c}$ $\therefore$ $Y(t) = Y_0 e^{-t/100}$

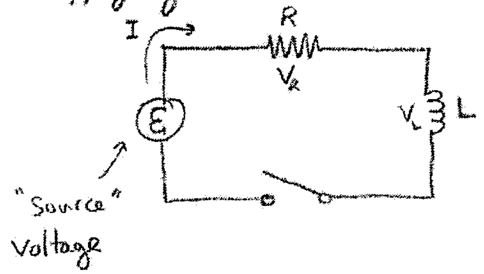
$Y(0) = 15 kg = Y_0$ $\therefore$ $\boxed{Y(t) = 15 e^{-t/100} kg}$

So after 20 minutes $Y(20) = 15 e^{-20/100} = \boxed{12.28 \text{ kg at } t = 20 \text{ min}}$

Example 16.2.10 (R is resistance, L is inductance)

E10 The RL - Circuit

Applying Kirchoff's Rules & a def^n or two we find



$$\mathcal{E} = V_R + V_L \quad \text{(Kirchoff)}$$

$$V_L = L\frac{dI}{dt} \quad \text{(Resists change in current)}$$

$$V_R = IR \quad \text{(Ohm's Law)}$$

$$\mathcal{E} = IR + L\frac{dI}{dt}$$

Now if $\mathcal{E}$ = constant and $I(0) = 0$ find $I(t)$,

$$\frac{\mathcal{E} - IR}{L} = \frac{dI}{dt} \quad \therefore \int \frac{dI}{\mathcal{E} - IR} = \int \frac{dt}{L}$$

$$\int \frac{dI}{\mathcal{E} - IR} = \frac{-1}{R}\ln(\mathcal{E} - IR)$$

$$\int \frac{dt}{L} = \frac{t}{L} + c$$

Thus $\frac{-1}{R}\ln(\mathcal{E} - IR) = \frac{t}{L} + \bar{c} \Rightarrow \mathcal{E} - IR = \bar{c}e^{-\frac{R}{L}t}$ that is

$$I = \frac{\mathcal{E}}{R}\left(1 + ce^{-\frac{R}{L}t}\right)$$

$$I(0) = \frac{\mathcal{E}}{R}(1 + c) = 0 \quad \therefore \quad c = -1$$

$$\therefore \boxed{I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$

Remark: the limiting current as $t \to \infty$ is $\mathcal{E}/R$. That is physically speaking the inductor is a short circuit for "long" times ($\tau = L/R$ then $5\tau \approx \infty$ pragmatically speaking.)

16.3. EXPONENTIAL GROWTH AND DECAY

We begin by studying several basic growth and decay examples. Then the logistic equation is studied in some depth.

If the growth of a population P is proportional to it's size then

$$\frac{dP}{dt} = kP$$

Likewise if the rate of change of Y is proportional to y

$$\frac{dy}{dt} = kY$$

As discussed in $\boxed{E1}$ of §7.3

$$\boxed{Y(t) = Y_o e^{kt} \quad \text{where} \quad Y_o = Y(o).}$$

Likewise $P(t) = P_o e^{kt}$ where $P_o = P(o)$. Both follow simply from sep. of variables. By the way:

$$\frac{dP}{dt} = kP \iff \frac{1}{P}\frac{dP}{dt} = k \iff \text{relative growth rate constant}.$$

So $k \equiv$ the relative growth rate.

Example 16.3.1

$\boxed{E1}$ If the population doubles every 10 yrs what is k?

$$P(0) = P_o \quad \text{and} \quad P(10) = 2P_o = P_o e^{10k}$$

$$\Rightarrow \ln(2) = 10k \Rightarrow k = \frac{\ln(2)}{10\,yr.} = 0.0693\,\frac{1}{yr.}$$

The relative growth rate is 6.93%.

Exponential growth is hard to maintain for large periods of time. If the growth is exponential then the population will double for a fixed period of time. This means if we let the population grow through 10 doubling periods then the population is increased 1,024 times over. After 20 doubling periods the population will be increased 1,048,576 times over. After 40 doubling periods the population is increased 1,099,511,627,776 times. What does this mean? Exponential population growth is not a perfect model. In practice we can only assume it works over a finite period of time. The logistic equation is a more sophisticated population growth model, it assumes exponential growth for a while but then as the population approaches the so-called carrying capacity the growth slows to zero.

Comment: Over the past several centuries various carrying capacities have been proposed for the human population of earth. Again and again these have been proved incorrect. God has always allowed us to find new technologies which circumvent the doomsday scenario which was supposed to be inevitable. A model is only as good as its assumptions. The trouble with all the models of human population is they fail to acknowledge the fact that the unexpected is to be expected.

Radioactive Decay Models

Radioactive decay is a probabilistic process. The weak force allows certain particles to morph into other particles by the release or absorption of a W or Z boson. Details aside, these radioactive particles are unstable and the number of particles that decay at any time is proportional to the number of unstable particles at that time. This leads to the same mathematics as population growth, however, the "growth rate" is negative.

$\boxed{E2}$ Let $y(t) = m(t)$ be the mass of some radioactive substance then as the mass destabilizes via radiation we have

$$\frac{dm}{dt} = km \qquad (k < 0 \text{ since } m \text{ is decreasing})$$

$$\Rightarrow m(t) = m_o e^{kt}$$

If balonium has a half-life of 1yr. then what percentage of the balonium is still in the fridge after $\frac{1}{10}$ yr. ?

$$m(1) = \frac{m_o}{2} = m_o e^{k} \quad \therefore \quad k = \ln(1/2) = \underline{-0.693 = k}$$

$$\therefore \quad m\left(\tfrac{1}{10}\right) = m_o e^{-0.693\left(\frac{1}{10}\right)} = (0.933)m_o \implies \boxed{93.3\% \text{ remains}}$$

Comment: Sometimes scientists try to work this backwards. If you know the amount of a radioactive material and you know the initial abundance of the substance then you can extrapolate backwards and see how old something is. One problem, this assumes we know the initial abundance. How do we "know" such a thing? Personally I'm skeptical of what we "know" about the unrepeatable past. I have much more trust for thoroughly testable physical laws.

Comment Continued:

Please don't misunderstand, radioactive decay is a real observed phenomenon. I think where science may get into trouble is where it tries to extend past what can be tested. The same is true for creationists. We must be careful to not overstate our case. We have no reason to fear science so long as it truly seeks reality. We know the source of reality and we have a meaning for our existence. God is glorified whether or not we can "prove" Him. Of course the proof of God surrounds us every day. The question is do you accept the proofs He offers? Would it be enough that he sent His Son to appear in human form and alter the course of human history? He did that, yet the world still denies the existence of God.

I do think we are called to give a defense for the things we believe. However, I'm afraid sometimes (myself included) we are tricked into being on the defense about historical science which seems to contradict the history in Scripture. Sometimes it may be a better argument to simply say that we don't find the world's creation myth convincing. I don't believe that the universe created itself replete with physical law, logic and the plethora of beautiful mathematics which just happens to mirror nature in unexpected ways.

Truth be told, most of them don't really find their story convincing either. Why would they, it keeps changing. I don't mean to say our story of creation has not changed at all. Certainly as time goes on we may gain a better understanding of the nuts and bolts of the creation process, or perhaps not, I am not convinced one way or the other.

What I do know is that God will be at the center of our story no matter how much more information we gather about the universe. There is still much flexibility in the creationist's viewpoint. I would much rather be burdened with the supposedly troubling problem of assuming the existence of God.

Here is the difference; when we reinterpret our creation story the nature of God does not change. He is always good, just, loving, merciful and He keeps his promises without exception. When the world undertakes a scientific revolution the pure naturalist learns that his god was in fact a false god, nature is something different. I suppose he can put his faith in the ideal of perfectly modeled nature, but does that exist? What are you really trusting in when you put your trust in a changing universe? I choose to trust the unchanging God.

More on Population Models:

The Logistic Eqn

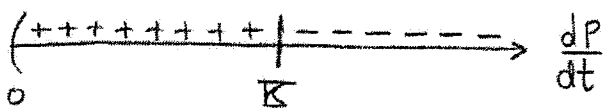
This is another model for population growth, the basic idea is that when the population P is small then $\frac{dP}{dt} = kP$ but as P gets big the resources are all used up and the population is unable to continue growing past some limiting population $K \equiv$ the carrying capacity. The simplest eqn encoorporating the above features is

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \quad \text{The Logistic Eq}^{n}}$$

Notice that as $P \longrightarrow K$ we have $\frac{dP}{dt} \longrightarrow 0$. As we desired the growth slows to zero as we approach the carrying capacity. Additionally when $P \ll K$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \cong kP$$

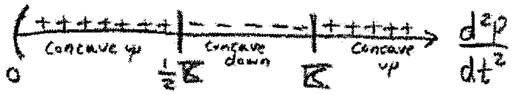
So for small population this model is like exponential growth. Now lets figure out what general features the sol^{n}'s to the Logistic Eq^{n} must have, (time for some calc. I)



P increases when $P < K$
P decreases when $P > K$

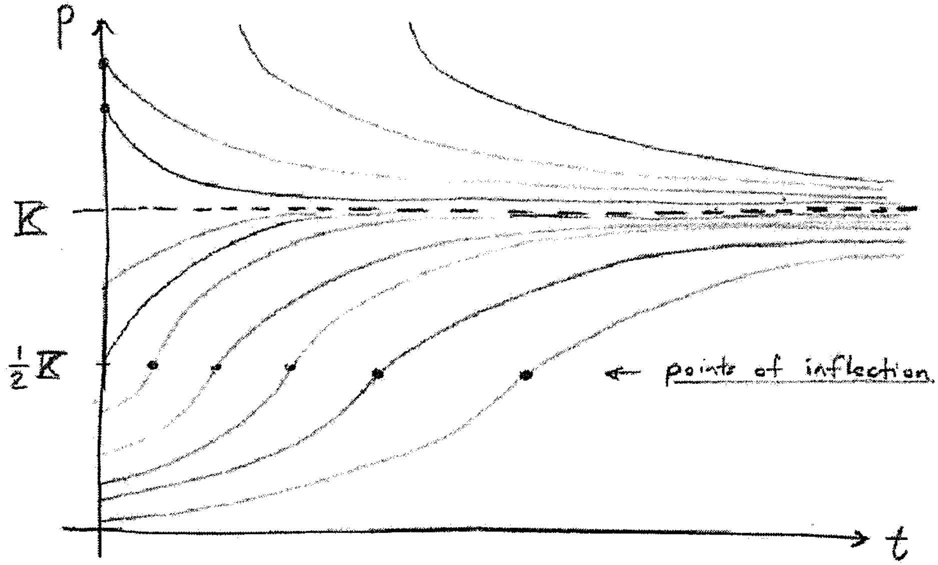
What about concavity? Lets differentiate,

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt}\left(1 - \frac{P}{K}\right) - \frac{k}{K}P\frac{dP}{dt}$$
$$= k\left(1 - \frac{2P}{K}\right)\frac{dP}{dt}$$
$$= k^2\left(1 - \frac{2P}{K}\right)\left(1 - \frac{P}{K}\right)$$



Notice $\frac{dP}{dt}$ is maximized at $P = \frac{1}{2}K$.

Graph of Sol^{n}'s to Logistic Eq^{n}



Inevitably as $t \to \infty$ the sol^{n} goes to K no matter what the intial condition was.

Remark: We have yet to find a sol^{n}. Next we'll explicitly solve the Log. Eq^{n}. I think its interesting we can see so much just from studying the DEq^{n} directly.

Logistic Eq$^{\underline{n}}$ Analytic Sol$^{\underline{n}}$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \Rightarrow \frac{dP}{P(1-P/K)} = k\,dt$$

Now to integrate in P we'll use partial fractions,

$$\frac{1}{P(1-P/K)} = \frac{A}{P} + \frac{B}{1-P/K}$$

$$1 = A(1-P/K) + BP \quad \xrightarrow{P=0} A = 1 \qquad \xrightarrow[P=K]{} 1 = BK \therefore B = 1/K$$

Thus $\dfrac{1}{P(1-P/K)} = \dfrac{1}{P} + \dfrac{1}{K-P}$

$$\int \frac{dP}{P(1-P/K)} = \int\left(\frac{1}{P} + \frac{1}{K-P}\right)dP = \ln|P| - \ln|K-P| = \ln\left|\frac{P}{K-P}\right|$$

$$\int k\,dt = kt + c$$

Hence $\ln\left|\dfrac{P}{K-P}\right| = kt + c \Rightarrow \left|\dfrac{P}{K-P}\right| = e^{c}e^{kt} \Rightarrow \dfrac{P}{K-P} = Ae^{kt}$

$A = \pm e^{c}$

Now solve for P

$$P = (K-P)Ae^{kt}$$

$$P(1 + Ae^{kt}) = AKe^{kt} \Rightarrow P = \frac{AKe^{kt}}{1+Ae^{kt}} = \boxed{\frac{K}{1+Ae^{-kt}} = P(t)}$$

Exercise: Verify for yourself that the conclusions we reached for inc/dec concave up/down etc... are duplicated by this sol$^{\underline{n}}$.

Remark: What ever the initial population is the final population is K

$$\lim_{t\to\infty}\left(\frac{K}{1+\underbrace{Ae^{-kt}}_{\to 0}}\right) = K$$

$\boxed{E1}$ Suppose that $\frac{dP}{dt} = 0.05P - 0.0005P^2$

Then what is the carrying capacity K ? and k ?

$$\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{100}\right) = kP\left(1 - \frac{P}{K}\right)$$

Comparing we identify $\boxed{K = 100 \text{ and } k = 0.05}$

$\boxed{E2}$ Suppose the carrying capacity of the US is 1000 (million). Additionally in 1990 $P = 250$ and in 2000 $P = 275$ million. Find $P(t)$ then predict the pop. in 2010 and 2100.

$$P(t) = \frac{1000}{1 + Ae^{-kt}}$$

Let 1990 be $t = 0$, then $P(0) = \frac{1000}{1+A} = 250 \Rightarrow \boxed{A = 3}$

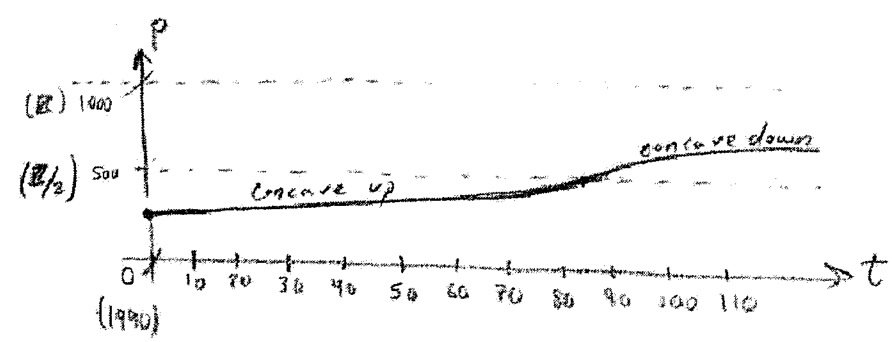
Additionally: $P(10) = \frac{1000}{1+3e^{-10k}} = 275 \Rightarrow 725 = 275(3e^{-10k})$

$$\Rightarrow \frac{725}{3 \cdot 275} = e^{-10k} = \frac{29}{33}$$

$$\Rightarrow \boxed{k = \frac{\ln(29/33)}{-10} = 0.01292}$$

$$P(20) = \frac{1000}{1 + e^{-0.13(20)}} = 301 \text{ million in } 2010$$

$$P(110) = \frac{1000}{1 + e^{-0.13(110)}} = 580 \text{ million in } 2100$$



$$P(t) = \frac{K}{2} = \frac{K}{1+3e^{-kt}} \Rightarrow 2 = 1 + 3e^{-kt}$$

$$\frac{1}{3} = e^{-kt} \Rightarrow \boxed{t = \frac{\ln(3)}{k} = \frac{\ln(3)}{0.01292} = 85 \Rightarrow P(85) = K/2}$$