No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. This test has 107 points, 7 are bonus points. Make sure to at least attempt each part.

1) [42pts] Let $\vec{A} = <1, 0, 3>$ and $\vec{B} = <0, 3, 4>$ find the following quantities,

    a.) $\vec{A} + \vec{B} + \hat{i} = <1, 0, 3> + <0, 3, 4> + <1, 0, 0>$

$$= \boxed{<1, 3, 7>}.$$

    b.) $\vec{A} \cdot \vec{B} = <1, 0, 3> \cdot <0, 3, 4> = 1 \cdot 0 + 0 \cdot 3 + 3 \cdot 4 = \boxed{12}.$

    c.) A nonzero vector which is perpendicular to both $\vec{A}$ and $\vec{B}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 3 \\ 0 & 3 & 4 \end{vmatrix} = <-9, -4, 3>$$

Note $\vec{A} \cdot (\vec{A} \times \vec{B}) = -9 + 9 = 0$ ¢

$\vec{B} \cdot (\vec{A} \times \vec{B}) = -12 + 12 = 0$ $\therefore$ $\underline{\vec{A} \times \vec{B} \perp \text{ to } \vec{A} \, \& \, \vec{B}}$

    d.) $\hat{A} = \dfrac{1}{|\vec{A}|} \vec{A} = \dfrac{1}{\sqrt{1+9}} \vec{A} = \boxed{\dfrac{1}{\sqrt{10}} <1, 0, 3> = \hat{A}}$

e.) $proj_{\vec{A}}(\vec{B}) = \left(\vec{B}\cdot\hat{A}\right)\hat{A} = \frac{1}{A^2}\left(\vec{B}\cdot\vec{A}\right)\vec{A} = \frac{1}{10}(12)\vec{A} = \frac{6}{5}\vec{A}$

$$\boxed{\frac{6}{5}\langle 1, 0, 3\rangle}$$

f.) Angle between $\vec{B}$ and $\vec{C} = \sqrt{2}\,\hat{i} + \sqrt{7}\,\hat{k}$

$|\vec{B}| = \sqrt{9+16} = \sqrt{25} = 5 = B$

$|\vec{C}| = \sqrt{2+7} = \sqrt{9} = 3 = C$

$\vec{B}\cdot\vec{C} = \langle 0, 3, 4\rangle \cdot \langle \sqrt{2}, 0, \sqrt{7}\rangle$

$BC\cos\theta = 4\sqrt{7} \longrightarrow \boxed{\theta = \cos^{-1}\left(\frac{4\sqrt{7}}{15}\right)}$

g.) Find the equation of the plane which contains $\vec{A}, \vec{B}$ and the line $\vec{r}(t) = < 1-t, 3, 2-3t >$.
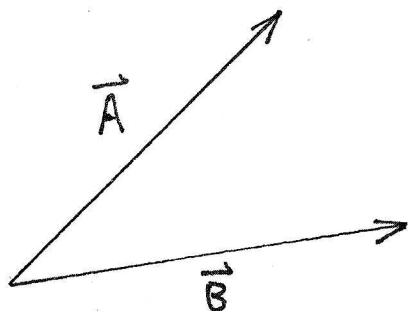
$\vec{A}\times\vec{B}$ is normal to plane. $\langle a, b, c\rangle = \langle -9, -4, 3\rangle$

$\vec{r}(0) = \langle 1, 3, 2\rangle$ point on plane

Thus, the eq$^{=}$ of the

$$\boxed{-9(x-1) - 4(y-3) + 3(z-2) = 0}$$

2) [3pts] Consider the vectors illustrated below, does $\vec{A} \times \vec{B}$ go into the page or out of the page ?
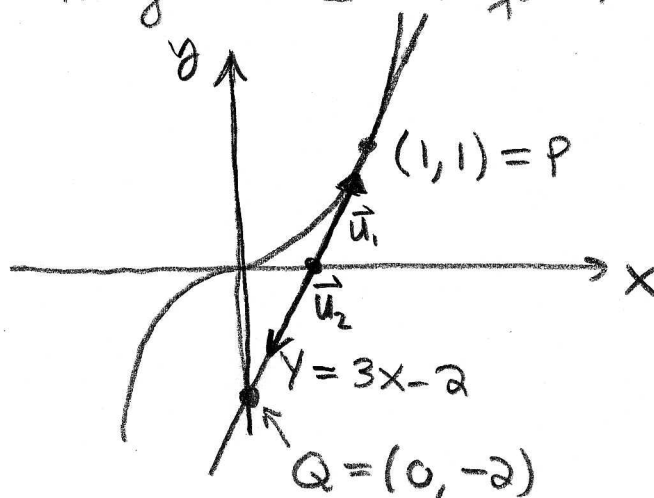
$\vec{A}$

$\vec{B}$

into the page
by right hand
rule

3) [10pts] Let $y = f(x) = x^3$. Find two distinct unit vectors that point along the tangent line to the function at (1,1). Include a good sketch of the curve, tangent line and both vectors.

$$f'(x) = 3x^2 \longrightarrow f'(1) = 3$$

Thus $y = f(1) + f'(1)(x-1) = 1 + 3(x-1) = 3x-2$
is the tangent line in question,

$(1,1) = P$

$\vec{u_1}$

$\vec{u_2}$

$y = 3x-2$

$Q = (0,-2)$

$$\vec{V} = P - Q = \langle 1, 3 \rangle$$

$$\vec{u_1} = \frac{1}{|\vec{V}|} \langle 1, 3 \rangle = \boxed{\frac{1}{\sqrt{10}} \langle 1, 3 \rangle}$$

and $\boxed{\frac{-1}{\sqrt{10}} \langle 1, 3 \rangle}$

unit vectors along the tangent line.

4) [20pts] Consider the vector-valued function of a real variable $\vec{r}(t) = <\cos(t), \sin(t), t>$.

a) Give the name of the curve $\vec{r}(t) = <\cos(t), \sin(t), t>$.

This is a helix.

b) Find the equation of the tangent line to the given curve at the point $(1,0,1)$. $(1,0,0)$

$$\vec{r}'(t) = <-\sin t, \cos t, 1>$$

$$\vec{r}(t) = (1,0,0) = <\cos t, \sin t, t> \rightarrow \underline{t=0}$$

Tangent line is

$$\vec{r_2}(s) = (1,0,0) + s\vec{r}'(0)$$
$$= \boxed{<1,0,0> + s<0,1,1> = \vec{r_2}(s)}$$

c) If $\vec{r}(t)$ represents the position of Hector at time t then find the velocity $\vec{v}(t)$.

$$\vec{v}(t) = \vec{r}'(t) = <-\sin t, \cos t, 1>$$

d) If $\vec{r}(t)$ represents the position of Hector time t then find the acceleration $\vec{a}(t)$.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}<-\sin t, \cos t, 1> = \underline{<-\cos t, -\sin t, 0>}$$

e) If $\vec{r_2}(t) = <1, 0, t-2>$ is the position of Dwight, will Dwight and Hector collide? Will their paths intersect?

Collision? $\vec{r}(t) = \vec{r_2}(t)$

$$<\cos t, \sin t, t> = <1, 0, t-2>$$

$$\left.\begin{array}{l}\cos t = 1 \\ \sin t = 0\end{array}\right\} \rightarrow t = 0, 2\pi n \quad n \in \mathbb{Z}$$

$$t = t-2 \rightarrow \underline{0 = -2} \ ! \quad \boxed{nope.}$$

Intersect?

$$\vec{r}(t) = \vec{r_2}(s) \begin{array}{l}\left.\begin{array}{l}\cos t = 1 \\ \sin t = 0\end{array}\right\} \rightarrow t = 0 \\ t = s-2 \rightarrow s = t+2\end{array}$$

$$\boxed{\text{Yes the paths intersect for } t=0, s=2}$$

5) [9pts] Suppose $f(x,y) = x^2y + y$. Calculate the partial derivatives indicated below

a) $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left[x^2y + y\right] = \boxed{2xy = f_x}$

b) $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[x^2y + y\right] = \boxed{x^2 + 1 = f_y}$

c) $f_{xx}(1,3)$

$$f_{xx} = \dfrac{\partial}{\partial x}(f_x) = \dfrac{\partial}{\partial x}\left[2xy\right] = 2y$$

$$f_{xx}(1,3) = 2(3) = \boxed{6 = f_{xx}(1,3)}$$

6) [4pts] Let $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of two variables; $f = f(x, y)$. State the definition of the partial derivative with respect to x at the point (x_o, y_o).

$$\frac{\partial f}{\partial x}(x_o, y_o) \equiv \lim_{h \to 0} \left(\frac{f(x_o + h, y_o) - f(x_o, y_o)}{h} \right)$$

Could also use $\frac{d}{dx}$ to define it as in notes.

7) [10pts] Let $z = x^y$. Suppose that $x = st$ and $y = \sin(s^2 t)$. Calculate $\frac{\partial z}{\partial s}$ using the chain rule for functions of several variables.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$= \frac{\partial}{\partial x}(x^y) \frac{\partial}{\partial s}[st] + \frac{\partial}{\partial y}(x^y) \frac{\partial}{\partial s}\left[\sin(s^2 t)\right]$$

$$= (y x^{y-1}) t + \ln(x) x^y \cos(s^2 t) \frac{\partial}{\partial s}(s^2 t)$$

$$= \boxed{t y x^{y-1} + 2st \, x^y \ln(x) \cos(s^2 t)}$$

8) **[5pts]** Find the distance from the tip of the vector $\vec{W} = <1, 2, 2>$ to the plane
$x + y - 3(z - 7) = 0$. Assume the vector's tail is placed on the plane. Recall that we define distance between two objects to be the shortest distance between any pair of points on the respective objects.

$<1, 1, -3> = \vec{n}$ $\vec{W} = <1, 2, 2>$

normal
vector

$d = ?$
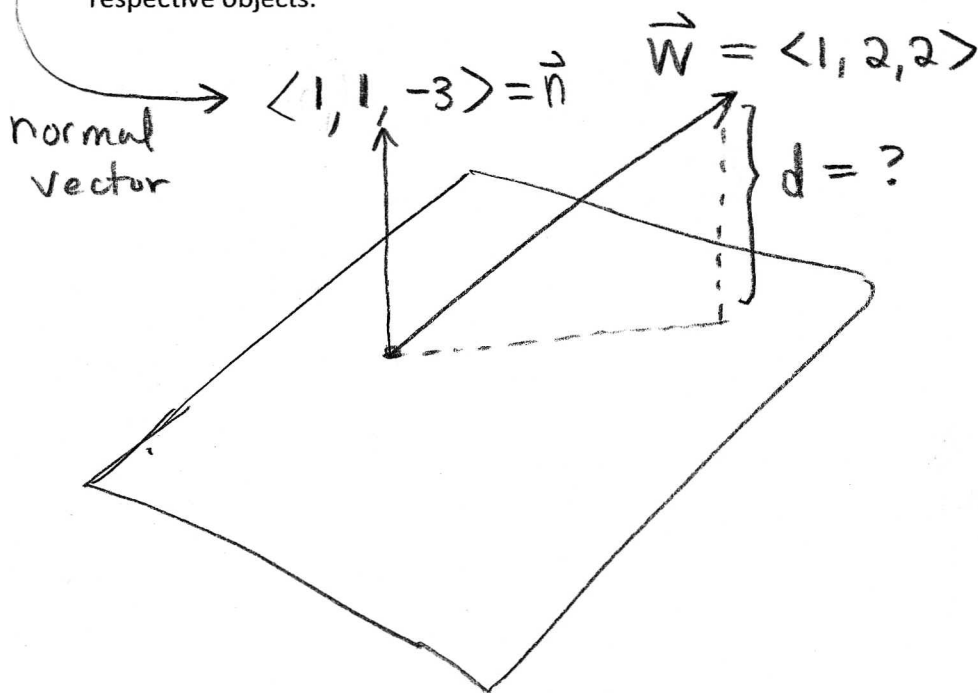
The distance is the length of $\vec{W}$
in the $\vec{n}$ direction, take absolute
value to get
rid of direction
indicating sign.
(we want length)
length ≥ 0.

$$d = |comp_{\hat{n}}(\vec{W})|$$

$$= |\vec{W} \cdot \hat{n}|$$

$$= |\frac{1}{n}(\vec{W} \cdot \vec{n})|$$

$$= |\frac{1}{\sqrt{1+1+9}} <1, 2, 2> \cdot <1, 1, -3>|$$

$$= |\frac{1}{\sqrt{11}}(1 + 2 - 6)|$$

$$= \boxed{3/\sqrt{11}}$$

9) [4pts] Let $\vec{r}: \mathbb{R} \to \mathbb{R}^2$ be a vector valued function of a real variable. Assume that the component functions are differentiable everywhere. Show that the **unit** tangent vector

$$\vec{T} = \frac{1}{|\,d\vec{r}/dt\,|}\frac{d\vec{r}}{dt} \qquad \text{(this is definition of } \vec{T})$$

is perpendicular its derivative $\frac{d\vec{T}}{dt}$. You should use properties proven in the homework project.

$$\vec{T} \cdot \vec{T} = 1$$

Now differentiate

$$\left(\text{since } |\vec{T}| = \left| \frac{1}{|\vec{F}'(t)|}\vec{F}'(t) \right| \right.$$
$$= |\vec{F}'(t)| / |\vec{F}'(t)|$$
$$\left. = 1 \right)$$

$$\frac{d}{dt}\left(\vec{T} \cdot \vec{T} \right) = \frac{d\vec{T}}{dt} \cdot \vec{T} + \vec{T} \cdot \frac{d\vec{T}}{dt} = \frac{d}{dt}(1) = 0$$

$$\Rightarrow \quad \frac{d\vec{T}}{dt} \cdot \vec{T} + \frac{d\vec{T}}{dt} \cdot \vec{T} = 0$$

$$\Rightarrow \quad 2\frac{d\vec{T}}{dt} \cdot \vec{T} = 0$$

$$\text{So} \quad \frac{d\vec{T}}{dt} \cdot \vec{T} = 0$$

$$\text{Thus} \quad \vec{T} \perp \text{ to } \frac{d\vec{T}}{dt}.$$