Calculus III , In-Class-Part of Test 3, December 2, 2008 Name: _____.

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. This test has 50 points, none are bonus . Make sure to at least attempt each part. *(the take home test has 50 points +5 bonus so the total points possible are 105 out of a 100 point scale)*

PROBLEM 1: [10pts] Use the table for part a. *(you may leave the answer in terms of the spherical unit vectors e_ρ, e_ϕ, e_θ (our notation, the table uses "u" but they're the same objects.)).*

a. Let $f(\rho, \theta, \phi) = \theta\phi\rho$. Calculate ∇f.

b. Let $\vec{F}(x, y, z) =< xyz, y^2, z^2 >$ calculate $curl(\vec{F}) = \nabla \times \vec{F}$ and $div(\vec{F}) = \nabla \cdot \vec{F}$.

PROBLEM 2: [15pts] Let $\vec{F}(x, y) =< y, \ -x >$ and suppose C is the closed curve formed by following the circle $x^2 + y^2 = 4$ from (0,2) to (-2,0) and then the line segment from (-2,0) back to (2,0). Given all of that information, calculate the following line integral :

$$\int_C \vec{F} \cdot d\vec{r}$$

PROBLEM 3a: [10pts] Let $\vec{F}(x, y, z) =< g(x, y, z), 0, \cos(z) >$ with g is a continuous function. Suppose that S is the upwards-oriented rectangle with corners (0,0,0), (1,0,0), (1,1,0) and (0,1,0). Calculate:

$$\int_S \vec{F} \cdot d\vec{S}$$

PROBLEM 3b: [5pts] Let $\vec{F} =< 1, 0, 0 >$ and suppose that S is a sphere of radius one centered about the origin. Calculate $\int_S \vec{F} \cdot d\vec{S}$.

PROBLEM 4: [10pts] Let $\vec{F} =< 2x, \ 2y, \ 2z >$ and suppose C is a curve from $(0, 0, 0)$ to $(1, 2, 3)$. Calculate the following line integral, *(please use the conservative force idea here, you really have no other option since I didn't specify the details of the curve C)*

$$\int_C \vec{F} \cdot d\vec{r}$$

Takehome Questions

Instructions: You will have approximately 50.5 hours to complete the take-home. (I need them before I leave school Thursday at 5pm, exceptions are possible but must be requested soon). You may discuss your general idea with your classmates however you should not copy calculations line by line. I do expect that there will be a question on the final based on one of the take- home problems so it would be both unethical and short-sighted to simply copy someone else's work. I trust that you will honor God and your peers in this matter.

PROBLEM 5[10pts] (Green's Theorem) Let C be the boundary of a square in the xy-plane. Show that

$$\oint_C xy^2\,dx + (x^2y + 2x)dy$$

depends only on the area of the square and not on its location in the plane.

PROBLEM 6 [20pts] Let $\vec{F} =< y, xz, z^2 >$ and let C be the boundary of the triangle cut from the plane $x + y + z = 1$ in the first octant, oriented counterclockwise as view from above. Use Stoke's Theorem to calculate $\int_C \vec{F} \cdot d\vec{r}$ by calculating a certain surface integral. I repeat, please do not calculate the line integral directly. Thanks.

Bonus[5pts]: Let $\alpha = f\,dy$ where f is a smooth function. Let $\beta = g\,dx + h\,dz$ where g, h are smooth functions. Calculate $\gamma = \alpha \wedge \beta$. What vector fields do α, β and γ correspond to respectively?

•This Problem is Mostly About Flux and the Divergence Theorem, part e uses Stokes' Theorem and a simple (but interesting) idea we had not introduced for Stoke's Theorem up to this point.

PROBLEM 7 [20pts] In lecture we learned the rather disturbing truth that the most important vector field in all of electrostatics escapes the standard form of the divergence theorem. I discussed that the so-called "Coulomb Field" is the electric field which is generated by a "point charge". The charge density for a point charge is described by the mathematics of Dirac Delta functions Technically, this takes us into the mathematics of distributions. There is a rigorous theory of integration for distributions but that is beyond calculus III. So we find ourselves in a curious position, we need to use mathematics which skirts the edges of the standard vector calculus. This is the point of 422-423 in my notes.

There is another way around this dilemma. We can propose that electric charges are not really point charges. Rather they are spread out over a tiny sphere of radius A. If you make this assumption then you can show by Gauss' Law in electrostatics that the electric field due to this non-point charge is:

$$\vec{E}(x,y,z) = \begin{cases} \vec{E}_{in} & 0 \leq \rho \leq A \\ \vec{E}_{out} & \rho \geq A \end{cases}$$

With,

$$\vec{E}_{out} = \frac{Q}{4\pi\epsilon_o\,\rho^2} e_\rho \qquad \text{the external field}$$

$$\vec{E}_{in} = \frac{Q\rho}{4\pi\epsilon_o\,A^3} e_\rho \qquad \text{the internal field}$$

the external field is defined to be $\vec{E}_{out} = \frac{Q}{4\pi\epsilon_o\,\rho^2} e_\rho$ will match the standard Coulomb field due to a point charge Q placed at the origin. The internal field smoothly goes to zero as $\rho \to 0$. The point is that if we smear out the charge the infinity goes away.

Complete the following tasks:

a.) Find the curl of $\vec{E}$ inside and outside the little sphere.

b.) Find the divergence of $\vec{E}$ inside and outside the little sphere.

c.) Find the flux of $\vec{E}$ over the sphere $\rho = R$ for both cases $0 < R < a$ (inside) and $R > a$ (outside).

d.) Verify that the divergence theorem holds true everywhere. That is show that

$$\iint_{S_R} \vec{E} \cdot d\vec{S} = \iiint (\nabla \cdot \vec{E})\, dV$$

e.)[2pts]Why is it not possible for $\vec{E} = \nabla \times \vec{F}$ for some vector field $\vec{F}$? *Hint, apply Stoke's Theorem two times to a sphere. Cut the sphere into two pieces, the northern and southern hemispheres. The equator will be the boundary for both the upper and lower hemispheres. However, the equator will be given opposite orientations from the north verses the south hemispheres. This means the line integrals from Stokes' Theorem applied to the top and bottom will cancel yielding that the flux is zero.*

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Given Formulas:

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) **i**, **j**, and **k** are unit vectors in the directions of increasing x, y, and z. F_x, F_y, and F_z are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.	**Cylindrical** (r, θ, z) $\mathbf{u}_r$, $\mathbf{u}_\theta$, and **k** are unit vectors in the directions of increasing r, θ, and z. F_r, F_θ, and F_z are the scalar components of $\mathbf{F}(r, \theta, z)$ in these directions.	**Spherical** (ρ, ϕ, θ) $\mathbf{u}_\rho$, $\mathbf{u}_\phi$, and $\mathbf{u}_\theta$ are unit vectors in the directions of increasing ρ, ϕ, and θ. F_ρ, F_ϕ, and F_θ are the scalar components of $\mathbf{F}(\rho, \phi, \theta)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{u}_\theta + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{u}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{u}_\phi + \frac{1}{\rho \sin \phi}\frac{\partial f}{\partial \theta}\mathbf{u}_\theta$
Divergence	$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$	$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$	$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F_\rho) + \frac{1}{\rho \sin \phi}\frac{\partial}{\partial \phi}(F_\phi \sin \phi) + \frac{1}{\rho \sin \phi}\frac{\partial F_\theta}{\partial \theta}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$	$\nabla \times \mathbf{F} = \begin{vmatrix} \frac{1}{r}\mathbf{u}_r & \mathbf{u}_\theta & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & F_\theta & F_z \end{vmatrix}$	$\nabla \times \mathbf{F} = \begin{vmatrix} \frac{\mathbf{u}_\rho}{\rho^2 \sin \phi} & \frac{\mathbf{u}_\phi}{\rho \sin \phi} & \frac{\mathbf{u}_\theta}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_\rho & \rho F_\phi & \rho \sin \phi F_\theta \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin \phi}\frac{\partial}{\partial \phi}\left(\sin \phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2 \phi}\frac{\partial^2 f}{\partial \theta^2}$

For a sphere of radius R centered at origin

$$d\vec{S} = \left(R^2 \sin\phi \, d\theta \, d\phi\right) e_\rho$$

$$e_\rho = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$$

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a. Let $f(\rho, \theta, \phi) = \theta\phi\rho$. Calculate ∇f.

$$\nabla f = \frac{\partial f}{\partial \rho} u_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi} u_\phi + \frac{1}{\rho \sin\phi}\frac{\partial f}{\partial \theta} u_\theta$$

$$= \frac{\partial}{\partial \theta}(\theta\phi\rho)\, u_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\theta\phi\rho)\, u_\phi + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \theta}(\theta\phi\rho)\, u_\theta$$

$$= \boxed{\theta\phi u_\rho + \theta u_\phi + \frac{\phi}{\sin\phi} u_\theta}$$

b. Let $\vec{F}(x, y, z) = < xyz, y^2, z^2 >$ calculate $curl(\vec{F}) = \nabla \times \vec{F}$ and $div(\vec{F}) = \nabla \cdot \vec{F}$.

$$\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2), \frac{\partial}{\partial z}(xyz) - \frac{\partial}{\partial x}(z^2), \frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(xyz) \right\rangle$$

$$= \boxed{\langle 0, xy, -xz \rangle = \nabla \times \vec{F}}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= \boxed{yz + 2y + 2z = \nabla \cdot \vec{F}}$$

I mentioned in lecture I wanted just one complete circuit around the circle.

PROBLEM 2: [15pts] Let $\vec{F}(x,y) = < y, \ -x >$ and suppose C is the the circle $x^2 + y^2 = 4$ oriented in the counter-clockwise direction. Calculate,

$$\int_C \vec{F} \cdot d\vec{r}$$

Notice $x = 2\cos t$, $y = 2\sin t$ for $0 \le t \le 2\pi$ parametrizes the given circle; $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \right) dt$$

$$= \int_0^{2\pi} \langle 2\sin t, -2\cos t \rangle \cdot \langle -2\sin t, 2\cos t \rangle \, dt$$

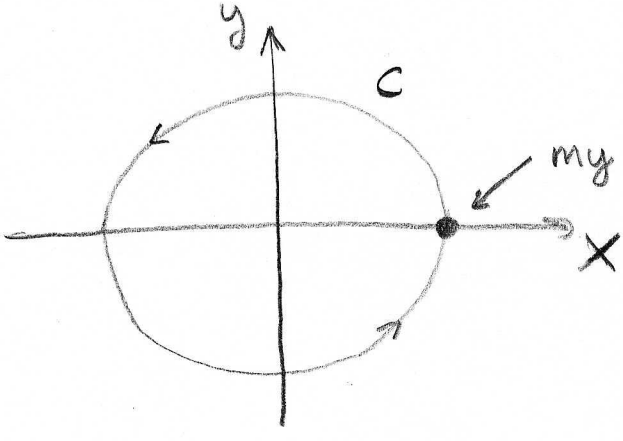
$$= \int_0^{2\pi} (-4\sin^2 t - 4\cos^2 t) \, dt$$

$$= -4 \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt$$

$$= -4 \int_0^{2\pi} dt$$

$$= -4t \Big|_0^{2\pi}$$

$$= \boxed{-8\pi}$$

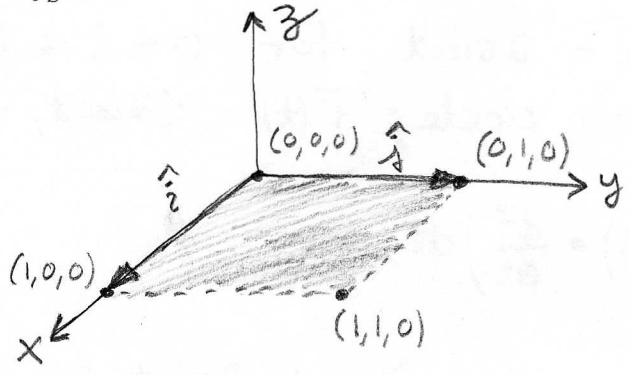


my parametrization begins and ends here, but all choices yield same answer by symmetry.

$x = 2\cos(t+\varphi)$, $y = 2\sin(t+\varphi)$ for $0 \le t \le 2\pi$ will yield same result for any $\varphi \in [0, 2\pi]$

PROBLEM 3a: [15pts] Let $\vec{F}(x,y,z) = < g(x,y,z), 0, \cos(z) >$ with g is a continuous function. Suppose that S is the upwards-oriented rectangle with corners (0,0,0), (1,0,0), (1,1,0) and (0,1,0). Calculate:

$$\int_S \vec{F} \cdot d\vec{S}$$



$$\Sigma(u,v) = u\hat{i} + v\hat{j}$$

for $0 \le u, v \le 1$.

$$\Sigma(u,v) = \langle u, v, 0 \rangle$$

$$\frac{\partial \Sigma}{\partial u} \times \frac{\partial \Sigma}{\partial v} = \hat{i} \times \hat{j} = \hat{k}$$

We find $x = u$, $y = v$, $z = 0$ and $d\vec{S} = (du\,dv)\hat{k}$ for the surface S thus,

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F}(u,v,0) \cdot \hat{k}\, du\, dv$$

$$= \int_0^1 \int_0^1 \langle g(u,v,0), 0, \cos(0) \rangle \cdot \langle 0, 0, 1 \rangle\, du\, dv$$

$$= \int_0^1 \int_0^1 du\, dv$$

$$= \boxed{1}$$

PROBLEM 3b: [15pts] Let $\vec{F}(x,y,z) = \langle \frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho} \rangle$ and suppose that S is an outward oriented sphere of radius one centered about the origin. Calculate $\int_S \vec{F} \cdot d\vec{S}$.

Notice $\vec{F}(x,y,z) = \langle \cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi \rangle = e_\rho$

and I gave $d\vec{S} = (R^2 \sin\phi \, d\theta d\phi)\, e_\rho$. Here

$R = 1$ and we want $\Sigma(\varphi, \theta) = \langle \cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi \rangle$

for $0 \le \theta \le 2\pi$ & $0 \le \varphi \le \pi$ — this usually need not match $\vec{F}$. (just happens here)

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \underbrace{e_\rho \cdot e_\rho}_{1} \sin\phi \, d\phi \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi \, d\phi\right)$$

$$= 2\pi \left(-\cos\phi \Big|_0^{\pi}\right)$$

$$= 2\pi \left(-\cos\pi + \cos(0)\right)$$

$$= \boxed{4\pi}$$

Remark: I wrote out $\Sigma(\phi, \theta)$ to emphasize where $d\vec{S}$ is coming from. Also for less trivial $\vec{F}$ we'd need to substitute in the eq^{n}'s $x = \cos\theta\sin\phi$, $y = \sin\theta\sin\phi$ & $z = \cos\phi$.

PROBLEM 4: [10pts] Let $\vec{F} =< 2x,\ 2y,\ 2z >$ and suppose C is a curve from $(0,0,0)$ to $(1,2,3)$. Calculate the following line integral, *(please use the conservative force idea here, you really have no other option since I didn't specify the details of the curve C)*

$$\int_C \vec{F} \cdot d\vec{r}$$

Observe that $f(x,y,z) = x^2+y^2+z^2$ is a potential function for $\vec{F}$ since $\nabla f = \langle 2x, 2y, 2z \rangle$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$$= f(1,2,3) - f(0,0,0) \quad \text{by FTC for line integrals.}$$

$$= 1+4+9-0$$

$$= \boxed{14}$$

Takehome Questions

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PROBLEM 5[10pts] (Green's Theorem) Let C be the boundary of a square in the xy-plane. Show that

$$\oint_C xy^2\,dx + (x^2y + 2x)dy$$

depends only on the area of the square and not on its location in the plane.

$$\oint_C xy^2\,dx + (x^2y+2x)dy = \pm \iint_{\substack{S \\ \partial S = C}} \left[\frac{\partial}{\partial x}(x^2y+2x) - \frac{\partial}{\partial y}(xy^2)\right] dA$$

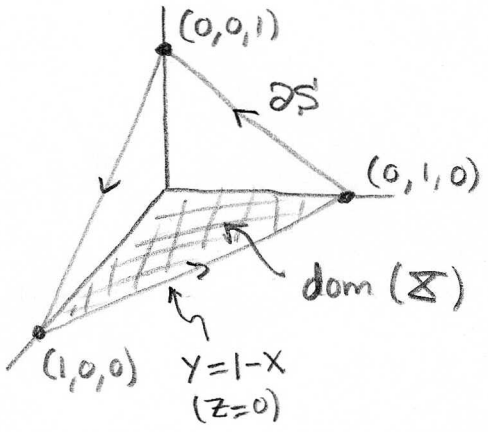
$$= \pm \iint_S (2xy + 2 - 2xy)\,dA$$

$$= \pm 2 \iint_S dA$$

$$= \pm 2(\text{Area of Square } S)$$

this is independent of the placement of the square in the xy-plane.

PROBLEM 6 [20pts] Let $\vec{F} = < y, xz, z^2 >$ and let C be the boundary of the triangle cut from the plane $x + y + z = 1$ in the first octant, oriented counterclockwise as view from above. Use Stoke's Theorem to calculate $\int_C \vec{F} \cdot d\vec{r}$ by calculating a certain surface integral. I repeat, please do not calculate the line integral directly. Thanks.



Since $z = 1-x-y$ we can use
Let $\Sigma(x,y) = \langle x, y, 1-x-y \rangle$
which gives $\Sigma_x = \langle 1, 0, -1 \rangle$
and $\Sigma_y = \langle 0, 1, -1 \rangle$ hence

$$d\vec{S} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} dx\,dy$$

$$\Rightarrow \quad d\vec{S} = \langle 1, 1, 1 \rangle\, dx\,dy$$

Wish to use $\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$. We have $d\vec{S}$

now calculate the curl,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & xz & z^2 \end{vmatrix} = \langle \partial_y(z^2) - \partial_z(xz), \partial_z(y) - \partial_x(z^2), \partial_x(xz) - \partial_y(y) \rangle$$

$$= \langle -x, 0, z-1 \rangle$$

Thus,

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^1 \int_0^{1-x} \langle -x, 0, -x-y \rangle\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (-2x - y)\, dy\, dx$$

$$= \int_0^1 \left[-2x(1-x) - \frac{1}{2}(1-x)^2 \right] dx$$

$$= \int_0^1 \left[-2x + 2x^2 - \frac{1}{2} + x - \frac{1}{2}x^2 \right] dx$$

$$= \int_0^1 \left(\frac{3}{2}x^2 - x - \frac{1}{2} \right) dx = \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = \boxed{\frac{-1}{2}}$$

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$$\nabla \times \vec{E} = \begin{vmatrix} \frac{u_\rho}{\rho^2 \sin\phi} & \frac{u_\phi}{\rho \sin\phi} & \frac{u_\theta}{\rho} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial\theta \\ E_\rho & \rho E_\phi & \rho \sin\phi\, E_\theta \end{vmatrix}$$

for both $\vec{E}_{in}$ and $\vec{E}_{out}$ we have $E_\phi = 0$ & $E_\theta = 0$.

$$\Rightarrow \nabla \times \vec{E} = \frac{-u_\phi}{\rho\sin\phi} \frac{\partial}{\partial\theta}(E_\rho) + \frac{u_\theta}{\rho} \frac{\partial}{\partial\phi}(E_\rho)$$

However, $E_\rho = \frac{Q}{4\pi\epsilon_o \rho^2}$ (outside) or $E_\rho = \frac{Q\rho}{4\pi\epsilon_o A^3}$ (inside)

are constant w.r.t. θ & ϕ $\therefore$ $\boxed{\nabla \times \vec{E} = 0 \text{ everywhere}}$

b.) Find the divergence of $\vec{E}$ inside and outside the little sphere.

Again we'll use the observation $E_\theta = 0$ & $E_\phi = 0$ both inside and out in the spherical formalism; $\vec{E} = E_\rho e_\rho + E_\theta e_\theta + E_\phi e_\phi$

Using my notes, page (± 382 ± 5 pages)

$$\nabla \cdot \vec{E} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left[\rho^2 E_\rho \right]$$

Inside,

$$\nabla \cdot \vec{E}_{in} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left[\rho^2 \frac{Q\rho}{4\pi\epsilon_0 A^3} \right]$$

$$= \frac{1}{\rho^2} \frac{3\rho^2 Q}{4\pi\epsilon_0 A^3}$$

$$= \boxed{\frac{3Q}{4\pi\epsilon_0 A^3} = \nabla \cdot \vec{E}_{in}}$$

Outside,

$$\nabla \cdot \vec{E}_{outo} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left[\rho^2 \frac{Q}{4\pi\epsilon_0 \rho^2} \right]$$

notice $\rho \neq 0$ in this case no ÷ by zero is ignored this calculation is legit.

$$= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left[\frac{Q}{4\pi\epsilon_0} \right]$$

$$= \boxed{0. = \nabla \cdot \vec{E}_{outo}}$$

c.) Find the flux of $\vec{E}$ over the sphere $\rho = R$ for both cases $0 < R < a$ (inside) and $R > a$ (outside).

Inside $R < A$

$$\iint_{S_R\ (\rho=R)} \vec{E}_{in} \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \left(\frac{QR}{4\pi\epsilon_0 A^3}\, e_\rho\right) \cdot \left(e_\rho R^2 \sin\phi \, d\phi \, d\theta\right)$$

$$= \frac{QR^3}{4\pi\epsilon_0 A^3} \int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{QR^3}{4\pi\epsilon_0 A^3} \left(\theta\Big|_0^{2\pi}\right)\left(-\cos\phi\Big|_0^{\pi}\right)$$

$$= \boxed{\frac{QR^3}{\epsilon_0 A^3} = \Phi_{\text{inside through sphere of radius } R<A}}$$

$$\iint_{S_R} \vec{E}_{outside} \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \left(\frac{Q}{4\pi\epsilon_0 R^2}\, e_\rho\right) \cdot \left(e_\rho R^2 \sin\phi \, d\phi \, d\theta\right)$$

$$= \frac{Q}{4\pi\epsilon_0} \int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{Q}{4\pi\epsilon_0} (4\pi)$$

$$= \boxed{\frac{Q}{\epsilon_0} = \Phi_{\text{sphere with } R>A}}$$

Φ is flux

d.) Verify that the divergence theorem holds true everywhere. That is show that

$$\iint_{S_R} \vec{E} \cdot d\vec{S} = \iiint (\nabla \cdot \vec{E}) dV$$

Inside:

$$\iiint_{B_R} (\nabla \cdot \vec{E}) dV = \iiint_{B_R} \frac{3Q}{4\pi\epsilon_0 A^3} dV$$

$$= \frac{3Q}{4\pi\epsilon_0 A^3} \iiint_{B_R} dV \quad \leftarrow \text{Volume of } B_R$$

$$= \left(\frac{3Q}{4\pi\epsilon_0 A^3}\right)\left(\frac{4\pi R^3}{3}\right)$$

$$= \frac{QR^3}{\epsilon_0 A^3} = \iint_{S_R = \partial B_R} \vec{E}_{in} \cdot d\vec{S}$$

from part c. we find div. Th^m

$$\iiint_{B_R} \nabla \cdot \vec{E}\, dV = \iint_{\partial B_R} E \cdot d\vec{S}$$

holds true.

Outside $R > A$

$$\iiint_{B_R} (\nabla \cdot \vec{E}) dV = \iiint_{B_R - B_A} (\nabla \cdot E_{out}) dV + \iiint_{B_A} \nabla \cdot \vec{E}_{in}\, dV$$

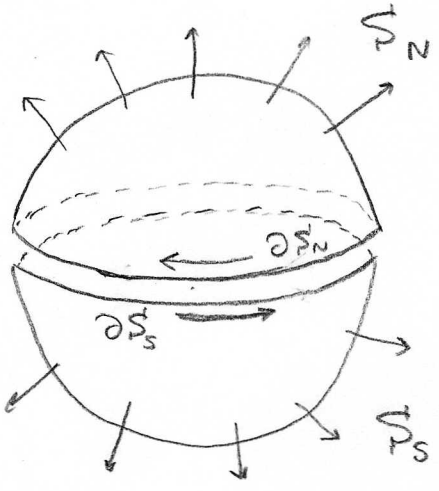
B_A is $\rho \leq A$

$$= 0 + \iiint_{B_A} \frac{3Q}{4\pi\epsilon_0 A^3} dV$$

$$= \frac{3Q}{4\pi\epsilon_0 A^3} \iiint_{B_A} dV \qquad \text{Volume of } B_A = \frac{4}{3}\pi A^3$$

$$= \frac{3Q}{4\pi\epsilon_0 A^3} \cdot \frac{4\pi A^3}{3} = \frac{Q}{\epsilon_0} = \iint_{B_R} \vec{E} \cdot d\vec{S}$$

e.)[2pts]Why is it not possible for $\vec{E} = \nabla \times \vec{F}$ for some vector field $\vec{F}$? *Hint, apply Stoke's Theorem two times to a sphere. Cut the sphere into two pieces, the northern and southern hemispheres. The equator will be the boundary for both the upper and lower hemispheres. However, the equator will be given opposite orientations from the north verses the south hemispheres. This means the line integrals from Stokes' Theorem applied to the top and bottom will cancel yielding that the flux is zero.*



$$S = S_N \cup S_S$$

$$\partial S_N = -\partial S_S$$

The equator obtains opposing orientations from S_N vs. S_S.

Consider that if $\vec{E} = \nabla \times \vec{F}$ then,

$$\Phi_E = \iint_S \vec{E} \cdot d\vec{S} = \iint_{S_N} \vec{E} \cdot d\vec{S} + \iint_{S_S} \vec{E} \cdot d\vec{S}$$

$$= \iint_{S_N} (\nabla \times \vec{F}) \cdot d\vec{S} + \iint_{S_S} (\nabla \times \vec{F}) \cdot d\vec{S}$$

applied Stoke's Th^{m} to S_S & S_N

$$= \int_{\partial S_N} \vec{F} \cdot d\vec{r} + \int_{\partial S_S} \vec{F} \cdot d\vec{r}$$

$$= \int_{-\partial S_S} \vec{F} \cdot d\vec{r} + \int_{\partial S_S} \vec{F} \cdot d\vec{r}$$

$$= -\int_{\partial S_S} \vec{F} \cdot d\vec{r} + \int_{\partial S_S} \vec{F} \cdot d\vec{r}$$

$$= 0$$

Yet we know from (c.) that $\Phi_E \neq 0$ for $R \neq 0$

therefore $\nexists \vec{F}$ such that $\nabla \times \vec{F} = \vec{E}$.

Bonus[5pts]: Let $\alpha = f\,dy$ where f is a smooth function. Let $\beta = g\,dx + h\,dz$ where g, h are smooth functions. Calculate $\gamma = \alpha \wedge \beta$. What vector fields do α, β and γ correspond to respectively?

$$\alpha = f\,dy = \omega_{\langle 0, f, 0\rangle}$$
$$\beta = g\,dx + h\,dz = \omega_{\langle g, 0, h\rangle}$$

} using the so-called work-form correspondance, $\omega_{\langle a,b,c\rangle} = a\,dx + b\,dy + c\,dz$

$$\alpha \wedge \beta = f\,dy \wedge (g\,dx + h\,dz)$$
$$= fg\,dy \wedge dx + fh\,dy \wedge dz$$
$$= -fg\,dx \wedge dy + fh\,dy \wedge dz = \Phi_{\langle fh, 0, -fg\rangle}$$

Where I'm using the flux-form correspondance,

$$\Phi_{\langle a,b,c\rangle} = a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy$$

Notice that

$$\langle 0, f, 0\rangle \times \langle g, 0, h\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & f & 0 \\ g & 0 & h \end{vmatrix}$$
$$= \langle fh, 0, -fg\rangle$$

Thus we see that

$$\alpha \wedge \beta = \gamma$$

gives a demo. of,

$$\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A} \times \vec{B}}$$

in the case $\vec{A} = \langle 0, f, 0\rangle$
$\vec{B} = \langle g, 0, h\rangle$