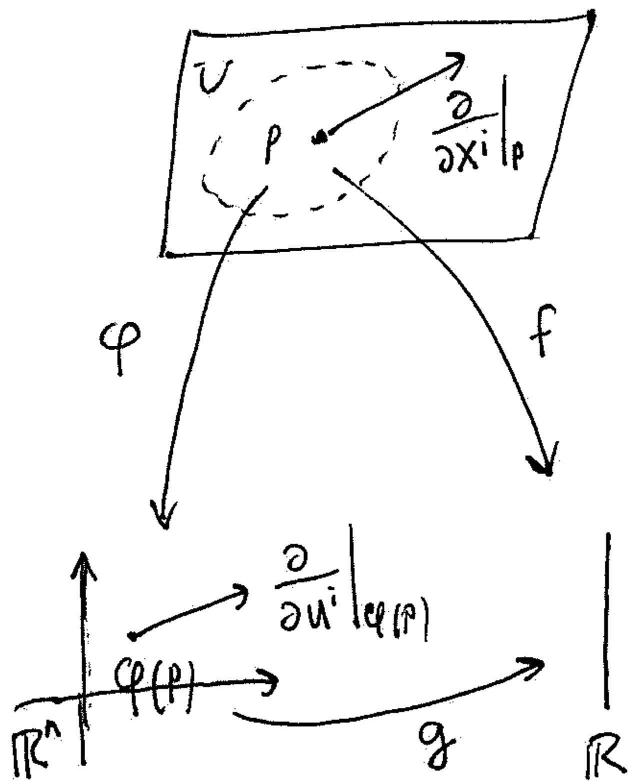


$$\boxed{\underline{\text{Claim}}:\ d\varphi_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial u^i}\Big|_{\varphi(p)}}$$

Let $g \in C^\infty(\mathbb{R}^n)$ and consider,

$$
\begin{aligned}
d\varphi_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) g &= \frac{\partial}{\partial x^i}\Big|_p (g\circ\varphi) \\
&= \frac{\partial f}{\partial x^i}(p) \quad \text{where } \underbrace{f = g\circ\varphi \in C^\infty M}_{\text{usual extension to M story}} \\
&= \frac{\partial}{\partial u^i}\Big|_{\varphi(p)}\left[(f\circ\varphi^{-1})(u)\right] \\
&= \frac{\partial}{\partial u^i}\Big|_{\varphi(p)}(g\circ\varphi\circ\varphi^{-1})(u) \\
&= \frac{\partial g}{\partial u^i}(\varphi(p)) \\
&= \left(\frac{\partial}{\partial u^i}\Big|_{\varphi(p)}\right)(g)
\end{aligned}
$$

$$\therefore\ \boxed{d\varphi_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial u^i}\Big|_{\varphi(p)}}$$

a.k.a. $dX_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial u^i}\Big|_{X(p)}$

$$T_pM = \{ v : C^\infty(M) \longrightarrow \mathbb{R} \mid \underbrace{v[f+cg] = v[f] + cv(g)}_{\text{linearity}} \ \& \ \underbrace{v[fg] = f(p)v[g] + g(p)v[f]}_{\text{product rule}} \}$$

$$\mathcal{C}_p = \{ [\gamma] \mid \gamma : J \subseteq \mathbb{R} \longrightarrow M, \gamma(0) = p \} = V_pM$$ (pg. 72 John Lee text) Smooth Manifolds

$$\gamma_1 \sim \gamma_2 \iff \gamma_1(0) = \gamma_2(0) \text{ and } \gamma_1'(0) = \gamma_2'(0)$$

$$[\gamma_1] = \{ \gamma : J \subseteq \mathbb{R} \to M \mid \gamma(0) = \gamma_1(0) \text{ and } \gamma_1'(0) = \gamma'(0) \}$$

($\sim$ forms equivalence relation on set of all curves through $p \in M$ of the form $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ where $0 \in J$ and $\gamma(0) = p$)

$$\gamma'(t_0)f = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right)f = \left.\frac{d}{dt}\right|_{t_0}(f \circ \gamma) = (f \circ \gamma)'(t_0)$$

$$\gamma'(t_0) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(t_0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)}$$

Isomorphism of T_pM and $\mathcal{V}_pM$

$$\Psi : \mathcal{V}_pM \longrightarrow T_pM$$

$$\Psi([\gamma]) = \gamma'(0)$$

Note if $[\gamma_1] = [\gamma_2]$ then $\gamma_1 \sim \gamma_2$ and $\gamma_1'(0) = \gamma_2'(0)$ by def$^{\underline{n}}$ of $\sim$

Thus Ψ is well-defined. To prove Ψ is an isomorphism of vector spaces we need to explain how $\mathcal{V}_pM$ is a vector space.

$$\Psi(c[\gamma_1] + [\gamma_2]) = \Psi([\alpha]) \quad \text{where } \alpha'(0) = c\gamma_1'(0) + \gamma_2'(0)$$

$$= \alpha'(0)$$

$$= c\gamma_1'(0) + \gamma_2'(0)$$

$$= c\,\Psi([\gamma_1]) + \Psi([\gamma_2])$$

It remains to show Ψ is a bijection.

If $v \in T_pM$ then, using chart x at p, $v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p$ where $v^i = v(x^i)$.

construct $\gamma(t) = x^{-1}(x(p) + t\langle v^1, v^2, \dots, v^n\rangle)$ thus $(x \circ \gamma)(t) = x(p) + t\langle v^1, \dots, v^n\rangle$

Hence $\quad \Psi([\gamma]) = \gamma'(0) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(0) \frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p = v.$

So Ψ is surjective.

Why is $\Psi : \mathcal{V}_p M \longrightarrow T_p M$ injective?

$$\Psi([\gamma]) = \gamma'(0)$$

Suppose $\Psi([\gamma_1]) = \Psi([\gamma_2])$

$$\gamma_1'(0) = \gamma_2'(0)$$

$$\Rightarrow [\gamma_1] = [\gamma_2]$$

$$\Rightarrow \Psi \text{ one-to-one}$$

$$\Psi^{-1}(V) = [\gamma_V] \quad \text{where} \quad \underbrace{\gamma_V(t) = x^{-1}(x(p) + t\langle V^1, \dots, V^n \rangle)}$$

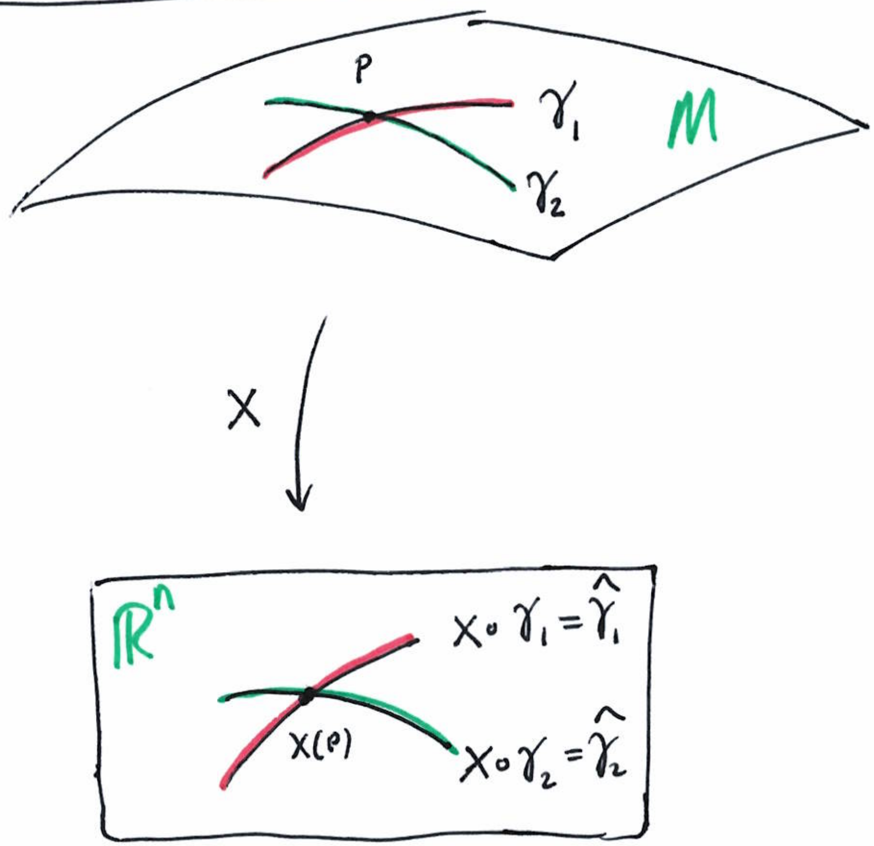
$$(x \circ \gamma_V)(t) = x(p) + t\vec{V}$$

$$\gamma_V'(0) = V$$

$$\boxed{\text{Def}^{n}/ \quad c[\gamma_V] + [\gamma_W] = [\gamma_{cV+W}]}$$

this notation makes $\mathcal{V}_p M$ vector space structure easier to sort through

Vector Space $\mathcal{V}_p M$ (I)



$$\hat{\gamma}_1(0) = \hat{\gamma}_2(0) = x(p)$$

$$\hat{\gamma}_1'(0) = \sum \frac{d(u^i \circ \hat{\gamma}_1)}{dt}(0) \frac{\partial}{\partial u^i}\Big|_{x(p)} = \sum_{i=1}^{n} \frac{d\hat{\gamma}_1^i}{dt}(0) \frac{\partial}{\partial u^i}\Big|_{x(p)}$$

$$\hat{\gamma}_2'(0) = \sum_{i=1}^{n} \frac{d\hat{\gamma}_2^i}{dt}(0) \frac{\partial}{\partial u^i}\Big|_{x(p)}$$

$$\hat{\alpha}(t) = x(p) + \left(c \left\langle \frac{d\hat{\gamma}_1^1}{dt}(0), \dots, \frac{d\hat{\gamma}_1^n}{dt}(0) \right\rangle + \left\langle \frac{d\hat{\gamma}_2^1}{dt}(0), \dots, \frac{d\hat{\gamma}_2^n}{dt}(0) \right\rangle \right) t$$

Lift the line $\hat{\alpha}$ back up to M,

$$\alpha = x^{-1} \circ \hat{\alpha}$$

Has $\alpha(0) = x^{-1}(\hat{\alpha}(0)) = x^{-1}(x(p)) = p$.

Also, by construction,

$$\alpha'(0) = \sum_{i=1}^{n} \frac{d(x^i \circ \alpha)}{dt}(0) \frac{\partial}{\partial x^i}\Big|_p$$

$$= \sum_{i=1}^{n} \frac{d}{dt}\Big|_0 \left(x^i \circ x^{-1} \circ \hat{\alpha} \right) \frac{\partial}{\partial x^i}\Big|_p$$

$$= \sum_{i=1}^{n} \frac{d\hat{\alpha}^i}{dt} \frac{\partial}{\partial x^i}\Big|_p \qquad (x^i \circ x^{-1})(u) = u^i$$

$$= \sum_{i=1}^{n} \left(c \frac{d\hat{\gamma}_1^i}{dt}(0) + \frac{d\hat{\gamma}_2^i}{dt}(0) \right) \frac{\partial}{\partial x^i}\Big|_p$$

$$= c\,\gamma_1'(0) + \gamma_2'(0)$$

(II) $\mathcal{V}_p M = \{ [\gamma] \mid \gamma : \text{dom}\,\gamma \subseteq \mathbb{R} \longrightarrow M,\ \gamma(0) = p \}$

If we consider $[\gamma_1], [\gamma_2] \in \mathcal{V}_p M$ then construct

$\hat{\alpha}(t) = x(p) + t(c\vec{v}_1 + \vec{v}_2)$ where $(\vec{v}_1)^i = \gamma_1'(0)x^i$ and $(\vec{v}_2)^i = \gamma_2'(0)x^i$

where we suppose $(U, x) \in \mathcal{A}_M$ and $p \in U$. Then

$$\boxed{\text{Def}^{\underline{n}}/\ c[\gamma_1] + [\gamma_2] = [\alpha] \text{ where } \alpha = x^{-1} \circ \hat{\alpha}}$$

To see this is well-defined notice $\alpha'(0) = c\gamma_1'(0) + \gamma_2'(0)$ hence if $[\gamma_1] = [\gamma_3]$ and $[\gamma_2] = [\gamma_4]$ we'd construct α_2 for which $\alpha_2'(0) = c\gamma_3'(0) + \gamma_4'(0) = c\gamma_1'(0) + \gamma_2'(0)$ thus $[\alpha] = [\alpha_2]$ as $\alpha(0) = p$ and $\alpha_2(0) = p$ as constructed previous page.

I invite the reader to check on the vector space axioms for $\mathcal{V}_p M$.

$$T_p^{physics} M = \{ [(p, \vec{v}, x)] \mid p \in M, \vec{v} \in \mathbb{R}^n, x \text{ chart at } p \}$$

$$\underbrace{(p, v^i, x) \sim (\bar{p}, \bar{v}^i, \bar{x})}_{\text{equivalent iff } p = \bar{p} \text{ and } \bar{v}^i = \sum_{j=1}^n \frac{\partial \bar{x}^i}{\partial x^j} v^j} \text{ if } \cancel{\bar{v}^i = \sum \frac{\partial x^j}{\partial \bar{x}^i}}$$

$$\left(\text{recall, } \quad v = v^i \frac{\partial}{\partial x^i}\Big|_p = \bar{v}^i \frac{\partial}{\partial \bar{x}^i}\Big|_p \Rightarrow \bar{v}^j = v^i \frac{\partial \bar{x}^j}{\partial x^i}. \right)$$

$$\boxed{\begin{array}{l} \text{Def}^{\underline{n}}/ \; c[(p, \vec{v}, x)] + [(p, \vec{w}, x)] = [(p, c\vec{v} + \vec{w}, x)] \\ \text{for } [(p, \vec{v}, x)], [(p, \vec{w}, x)] \in T_p^{physics} M \text{ where } c \in \mathbb{R}. \end{array}}$$

If $(p, v^i, x) \sim (\bar{p}, \bar{v}^i, \bar{x})$ and $(p, w^i, x) \sim (\bar{p}, \bar{w}^i, \bar{x})$

then $c[(\bar{p}, \bar{v}^i, \bar{x})] + [(\bar{p}, \bar{w}^i, \bar{x})] = [(p, c\bar{v}^i + \bar{w}^i, \bar{x})]$

Notice $$c\bar{v}^i + \bar{w}^i = \sum_j c \frac{\partial \bar{x}^i}{\partial x^j} v^j + \sum_j \frac{\partial \bar{x}^i}{\partial x^j} w^j = \sum_j \frac{\partial \bar{x}^i}{\partial x^j}(cv^j + w^j)$$

thus $(\bar{p}, c\bar{v}^i + \bar{w}^i, \bar{x}) \sim (p, cv^i + w^i, x)$ hence $T_p^{PHYSICS} M$ has well-defined addition & scalar multiplication. We can prove $T_p^{PHYSICS} M$ is a vector space over $\mathbb{R}$.

Isomorphisms with $T_p^{PHYSICS} M$ and T_pM

$$\Phi : T_pM \longrightarrow T_p^{PHYSICS} M$$

$$\Phi(v) = \Phi\left(v^i \frac{\partial}{\partial x^i}\Big|_p\right) = [(p, v^i, x)]$$

If $v = \bar{v}^i \frac{\partial}{\partial \bar{x}^i}\Big|_p$ then notice $\Phi\left(\bar{v}^i \frac{\partial}{\partial \bar{x}^i}\Big|_p\right) = [(p, \bar{v}^i, \bar{x})]$

But, $(p, v^i, x) \sim (p, \bar{v}^i, \bar{x})$ as $\bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^j} v^j$ as before.

Thus Φ is well-defined.

$$\Phi(cv+w) = \Phi\left((cv^i + w^i)\frac{\partial}{\partial x^i}\Big|_p\right)$$
$$= [(p, cv^i + w^i, x)]$$
$$= c[(p, v^i, x)] + [(p, w^i, x)]$$
$$= c\,\Phi(v) + \Phi(w).$$

$$\frac{\partial \bar{x}^i}{\partial x^k} v^k \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial}{\partial x^l}\Big|_p = \frac{\partial x^l}{\partial x^k}\frac{\partial}{\partial x^l}\Big|_p v^k$$
$$= v^k \frac{\partial}{\partial x^k}\Big|_p$$

$$\Phi^{-1}([(p, v^i, x)]) = v^i \frac{\partial}{\partial x^i}\Big|_p$$

If $[(p, v^i, x)] = [(p, \bar{v}^i, \bar{x})]$ note $\Phi^{-1}([(p, \bar{v}^i, \bar{x})]) = \bar{v}^i \frac{\partial}{\partial \bar{x}^i}\Big|_p = v^i \frac{\partial}{\partial x^i}\Big|_p$