Inverse Operators, Green's Functions, Convolution, Transfer Functions

We solved $L[y] = f$ by finding the homogeneous sol$^{\underline{n}}$ $y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ where $L[y_j] = 0$ for $j = 1, 2, 3, \ldots, n$ and y_p (the particular sol$^{\underline{n}}$) with $L[y_p] = f$. Formally, you might like to find L^{-1} such that

$$L^{-1}[L[y]] = y \qquad \text{hence} \qquad y = L^{-1}[f].$$

Unfortunately L^{-1} cannot be single-valued due to the infinity of possible homogeneous sol$^{\underline{n}}$'s. But, if we insist on find a sol$^{\underline{n}}$ for $L[y] = f$ such that

$$y(0) = 0, \quad y'(0) = 0, \quad \ldots, \quad y^{(n-1)}(0) = 0$$

then a unique sol$^{\underline{n}}$ of $L[y] = f$ exists for each given forcing function f (we assume f is sufficiently nice to solve $L[y] = f$, certainly piecewise continuous will suffice, but I assume exponential order α to allow Laplace transform arguments to follow)

First Order Case:

Suppose $L = \frac{d}{dt} + P$. Consider $L[y] = Q$. We have

$$y' + Py = Q \quad \Rightarrow \quad e^{\int P\,dt}\frac{dy}{dt} + Pe^{\int P\,dt}\,y = Qe^{\int P\,dt}$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{\int P\,dt}\,y\right) = Qe^{\int P\,dt}$$

set $y(0) = 0$.

$$\Rightarrow \quad e^{\int P\,dt}\,y(t) = \int_0^t Q(t)\,e^{\int P\,dt}\,dt$$

$$\Rightarrow \quad y(t) = e^{-\int P\,dt}\int_0^t Q(u)\,e^{\int P(u)\,du}\,du$$

$$\Rightarrow \quad \boxed{y(t) = \int_0^t Q(u)\,K(u,t)\,du \quad \text{where we've defined } K(u,t) = \exp\left(\int P\,du - \int P\,dt\right)}$$

We define $\underline{L^{-1}(f)(t) = \int_0^t K(u,t)\,f(u)\,du}$. You can verify that $L[y] = Q$ has solution $y(t) = L^{-1}(Q)(t)$. The function $\underline{\underline{K(u,t)}}$ is called the <u>Green's function</u> for L.

Second Order Case

Consider $ay'' + by' + cy = f$ and define $L = aD^2 + bD + c$ where $D = d/dt$. Once more we seek L^{-1} such that $L[y] = f$ has sol$^{\underline{n}}$ $L^{-1}[L[y]] = L^{-1}[f] \Rightarrow y = L^{-1}[f]$.

I'll use variation of parameters. Suppose y_1, y_2 are sol$^{\underline{n}}$'s of $L[y] = 0$ such that $\{y_1, y_2\}$ is LI on the interval of interest $(0, \infty)$ (this argument can be made for other intervals, again I have Laplace tranforms in mind...)

We derived that

$$y = y_1 \int \frac{-y_2 f \, dt}{W[y_1, y_2]} + y_2 \int \frac{y_1 f \, dt}{W[y_1, y_2]}$$

Keep in mind $y(0) = 0$ and $y'(0) = 0$,

$$y(t) = y_1(t) \int_0^t \frac{-y_2(u) f(u) du}{y_1(u) y_2'(u) - y_1'(u) y_2(u)} + y_2(t) \int_0^t \frac{y_1(u) f(u) \, du}{y_1(u) y_2'(u) - y_1'(u) y_2(u)}$$

$$y(t) = \int_0^t \underbrace{\left[\frac{y_1(u) y_2(t) - y_1(t) y_2(u)}{y_1(u) y_2'(u) - y_1'(u) y_2(u)} \right]} f(u) \, du$$

$K(u, t)$ is Green's Function for L

$$\boxed{L^{-1}(f)(t) = \int_0^t K(u, t) f(u) du}$$

You can verify $L[y] = f$ has sol$^{\underline{n}}$ $y(t) = L^{-1}(f)(t)$.

Clearly $y(0) = \int_0^0 K(u,t) f(u) du = 0$.

<u>Comment</u>: it appears $K(u,t)$ may depend on our choice of fundamental sol$^{\underline{n}}$'s y_1 & y_2. However, this is <u>not</u> the case. Read Finney & Ostberg 163–177 of Differential Eq$^{\underline{n}}$'s with Linear Algebra.

<u>Remark</u>: I've constructed the Green's Function in ways which do not immediately connect L^{-1} with L. In contrast, the calculation ② shows how the coefficient functions of $L = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0$ give rise to the formula for L^{-1}.

n^{th} order, constant coefficient case

Consider $a_n y^{(n)}(t) + \dots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t)$ or $L[y] = f$.

Take the Laplace transform, use $y(0) = y'(0) = \dots = y^{(n-1)}(0) = 0$ to derive that

$$\underbrace{\left(a_n s^n + \dots + a_2 s^2 + a_1 s + a_0\right)}_{P(s)} Y(s) = \mathcal{L}\{f(t)\}(s) = F(s)$$

$P(s) \leftarrow$ characteristic polynomial for L evaluated at s.

Hence,

$$\underline{Y(s) = \frac{1}{P(s)} F(s)}.$$

Thus, $\underline{y(t) = \mathcal{L}^{-1}\left\{\frac{1}{P(s)} F(s)\right\}}$ where $\mathcal{L}^{-1}\{F(s)\}(t) = f(t)$.

Define $H(s) = \dfrac{Y(s)}{F(s)} = \dfrac{1}{P(s)}$ the "<u>transfer</u> <u>function</u>"

we let $\mathcal{L}^{-1}\{H\} = h$ as usual. To find the Green's function $K(u,t)$ we need to somehow write $y(t) = \mathcal{L}^{-1}\left\{\frac{1}{P(s)} F(s)\right\} = \int_0^t K(u,t) f(u)\,du$.

> **Def^{n}/** If $\mathcal{L}\{f\} = F$ and $\mathcal{L}\{g\} = G$ then $f * g$ is the function such that $\mathcal{L}\{f*g\} = FG$. In other words,
> $$\mathcal{L}^{-1}\{FG\} = \mathcal{L}^{-1}\{F\} * \mathcal{L}^{-1}\{G\}$$
> ($*$ is called the <u>convolution</u> product)

(we'll attempt a derivation of the formula for $f * g$ shortly but for now let's appreciate the connection to Green's functions...)

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{P(s)} \cdot F(s)\right\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{P(s)}\right\} * \mathcal{L}^{-1}\{F\} \\ &= (h * f)(t) \end{aligned}$$

(recall $H(s) = \frac{1}{P(s)}$ is the transfer function.)

Thus, finding the convolution product will allow us to find L^{-1} and the Green's function. Note that $\mathcal{L}^{-1}\left\{\frac{1}{P(s)}\right\}$ is a well-understood problem which we can handle by partial fractions for a given L.

Derivation of Convolution Product

We desire a formula for $f * g$ such that

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\}$$

$$\int_0^\infty e^{-st}(f*g)(t)\,dt = \int_0^\infty e^{-st} f(t)\,dt \int_0^\infty e^{-su} g(u)\,du$$

jump!

$$= \int_0^\infty\int_0^\infty e^{-s(t+u)} f(t)\, g(u)\, dt\, du$$

$$\boxed{\begin{aligned} w &= t+u \\ t &= w-u \end{aligned}}$$

$$= \int_0^\infty\int_0^\infty e^{-sw} f(w-u)\,g(u)\,dw\,du$$

$$= \int_0^\infty e^{-sw}\left(\int_0^\infty f(w-u)\,g(u)\,du\right) dw$$

$$= \int_0^\infty e^{-st}\left(\underbrace{\int_0^t f(t-u)\,g(u)\,du}\right) dt$$

<u>note</u>: $f(x) = 0$ for $x < 0$

We <u>define</u>

$$\boxed{(f*g)(t) = \int_0^t f(t-u)\,g(u)\,du}$$

Remark: the text proves this product is well-defined and satisfies $\star$. This calculation is <u>not</u> a proof. However, I do hope it helps you see where the mysterious $*$ comes from.

(tornado)

Observe that it is clear that

$$0 * g = 0 \quad \text{and} \quad (cf * g) = c(f*g) = f*(cg)$$

and

$$(f_1 + f_2) * g = f_1 * g + f_2 * g$$

Consider

$$\boxed{\begin{aligned} t-u &= w \\ u &= t-w \end{aligned}}$$

$$\boxed{\begin{aligned} w(0) &= t-0 = t. \\ w(t) &= t-t = 0. \\ dw &= -du \end{aligned}}$$

$$\boxed{\text{(flip bounds)} \quad \int_t^0 -g(w)f(t-w)\,dw}$$

$$(g*f)(t) = \int_0^t g(t-u)f(u)\,du$$

$$= \int_0^t g(w)\,f(t-w)\,dw$$

$$= \int_0^t f(t-w)\,g(w)\,dw$$

$$= (f*g)(t).$$

You can also show $f*(g*h) = (f*g)*h$.

Continuing the n^{th} order constant coefficient $L = P(D)$ discussion.

Recall, $L[y] = f$

$$P(D)[y] = f \Rightarrow \mathcal{L}\{P(D)[y]\} = \mathcal{L}\{f\}$$
$$\Rightarrow P(s)\,Y(s) = F(s)$$
$$\Rightarrow Y(s) = \left(\frac{1}{P(s)}\right)F(s)$$
$$\Rightarrow \underbrace{Y(s)}_{\text{output}} = \underbrace{H(s)}_{\text{transfer function}}\ \underbrace{F(s)}_{\text{input}}$$

(How engineers think on these things)

Use convolution to take inverse Laplace transform,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{H(s)F(s)\} \\ &= \mathcal{L}^{-1}\{H(s)\} * \mathcal{L}^{-1}\{F(s)\} \\ &= (h * f)(t) \\ &= \int_0^t h(t-u)\, f(u)\, du \end{aligned}$$

Identify that $K(u,t) = h(t-u)$, the Green's function is the inverse tranform of the transform function evaluated at $t-u$.

$$\boxed{L^{-1}(f)(t) = \int_0^t h(t-u)\, f(u)\, du}$$

The formula above shows how the force f applied to $L[y] = 0$ system gives a response dictated by the transfer function (or Green's function if you prefer) of the system. The general sol$^{\underline{n}}$ has the form

$$\boxed{y(t) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + \int_0^t h(t-u)\, f(u)\, du}$$

We constructed $y_p(t) = \int_0^t h(t-u) f(u)\, du$ to have initial conditions $y_p(0) = 0,\ y_p'(0) = 0, \dots, y_p^{(n-1)}(0) = 0$. You might read Ritger & Rose for further physical comments on the transfer function, or Finney and Ostberg for more methematical details on Green's function. There are many interesting generalities to comment upon.

Example II. find "h" for the general distinct root $n=2$ problem. we have $P(s) = (s-r_1)(s-r_2)$ for $r_1 \neq r_2$ if the coefficient of y'' is 1. Then the transfer function

$$H(s) = \frac{1}{P(s)} = \frac{1}{(s-r_1)(s-r_2)}$$

We can simplify by partial fractions,

$$\frac{1}{(s-r_1)(s-r_2)} = \frac{A}{s-r_1} + \frac{B}{s-r_2}$$

$$1 = A(s-r_2) + B(s-r_1)$$

$$\left.\begin{array}{lll} \underline{s=r_1} & 1 = A(r_1-r_2) & \Rightarrow A = \frac{1}{r_1-r_2} \\ \underline{s=r_2} & 1 = B(r_2-r_1) & \Rightarrow B = \frac{-1}{r_1-r_2} \end{array}\right\} \quad \underline{H(s) = \frac{1}{r_1-r_2}\left(\frac{1}{s-r_1} - \frac{1}{s-r_2}\right)}.$$

Therefore, $h(t) = \mathcal{L}^{-1}\{H(s)\}(t) = \underline{\frac{1}{r_1-r_2}\left(e^{r_1 t} - e^{r_2 t}\right)}.$

The general sol^{n} to $(D-r_1)(D-r_2)[y] = f$ with $y(0) = y'(0) = 0$ is

$$\boxed{y(t) = \int_0^t \frac{1}{r_1-r_2}\left(e^{r_1(t-v)} - e^{r_2(t-v)}\right) f(v)\,dv}$$

This gives an integral sol^{n} for any distinct-root, constant coeff, nonhomogeneous problem. Let us apply it to a particular problem,

$$y'' + 3y' + 2y = e^{-t}$$

$$P(s) = s^2 + 3s + 2 = (s+1)(s+2) \Rightarrow r_1 = -1,\ r_2 = -2$$

Thus, $h(t) = \frac{1}{-1+2}\left(e^{-t} - e^{-2t}\right) = e^{-t} - e^{-2t}$. Therefore, the sol^{n} with $y(0) = y'(0) = 0$ is,

$$\begin{aligned} y(t) &= (h * f)(t) \\ &= \int_0^t h(t-v)\, f(v)\,dv \\ &= \int_0^t \left[e^{-(t-v)} - e^{-2(t-v)}\right] e^{-v}\,dv \\ &= \int_0^t \left[e^{-t} - e^{v-2t}\right] dv \\ &= \left(ve^{-t} - e^{v-2t}\right)\Big|_0^t \\ &= \underline{te^{-t} - e^{-t} + e^{-2t}}. \end{aligned}$$

To create a general sol^{n} we simply add $c_1 e^{-t} + c_2 e^{-2t}$, these terms are called <u>transient</u> as they vanish as $t \to \infty$. Actually the same is true for $y(t)$ here, but if $f = \cos(t)$ etc... often $y(t) \not\to 0$.

Example III: find transfer function and h for the arbitrary complex root case with $a=1$; $P(s) = (s-\alpha+i\beta)(s-\alpha-i\beta)$.
We should work out the partial fractal decomp for

$$\frac{1}{P(s)} = \frac{1}{(s-\alpha)^2+\beta^2} = \frac{A\beta}{(s-\alpha)^2+\beta^2} + \frac{B(s-\alpha)}{(s-\alpha)^2+\beta^2}$$

$$1 = A\beta + B(s-\alpha)$$
$$1 = A\beta - \alpha B + Bs \implies B = 0 \text{ \& } A = \frac{1}{\beta}$$

(well duh.)

$$H(s) = \frac{1}{\beta}\left[\frac{\beta}{(s-\alpha)^2+\beta^2}\right]$$

$$\Rightarrow \underline{h(t) = \frac{1}{\beta} e^{\alpha t} \sin \beta t}.$$

Therefore, to solve $[(D-\alpha)^2+\beta^2](y) = f$ we may calculate

$$\boxed{y(t) = \int_0^t \frac{1}{\beta} e^{\alpha(t-v)} \sin[\beta(t-v)]\, f(v)\, dv}$$

For example, to solve $y''+y = f = \sin t$ solve, for $h(t) = \sin t$:

$$\begin{aligned}
y(t) &= \int_0^t \sin(t-v)\sin(v)\,dv \\
&= \int_0^t (\sin t \cos v - \sin v \cos t)\sin v\, dv \\
&= \sin t \int_0^t \cos v \sin v\, dv - \cos t \int_0^t \sin^2 v\, dv \\
&= \sin t \left(\tfrac{1}{2}\sin^2(v)\right)\Big|_0^t - \cos t\left(\tfrac{v}{2} - \tfrac{1}{4}\sin(2v)\right)\Big|_0^t \\
&= \tfrac{1}{2}\sin^3 t - \tfrac{1}{2} t\cos t + \tfrac{1}{4}\cos(t)\sin(2t) \\
&= \tfrac{1}{2}\sin t\,[\sin^2 t + \cos^2 t] - \tfrac{1}{2} t \cos t \\
&= \underline{\tfrac{1}{2}\sin t - \tfrac{1}{2} t \cos t}.
\end{aligned}$$

(You can check, $y(0) = 0$
$y'(0) = 0$.)

Example III: Examine the repeated root case.

$$y'' - 2ry' + r^2 y = f$$

$$P(s) = s^2 - 2sr + r^2 = (s-r)^2 \Rightarrow H(s) = \frac{1}{(s-r)^2}$$

I'll use convolution to calculate $\mathcal{L}^{-1}\{H\}$.

$$\begin{aligned}
h(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s-r)^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{1}{s-r}\right)\left(\frac{1}{s-r}\right)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s-r}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-r}\right\} \\
&= e^{rt} * e^{rt} \\
&= \int_0^t e^{r(t-v)} e^{rv}\, dv \\
&= \int_0^t e^{rt}\, dv \\
&= e^{rt} \int_0^t dv \\
&= \underline{te^{rt}}.
\end{aligned}$$

$$\Rightarrow \boxed{y(t) = \int_0^t (t-v)\, e^{r(t-v)} f(v)\, dv}$$