§ 5.1#1 Show $\approx$ is reflexive, symmetric and transitive. that is show $\approx$ is an equivalence relation on all sets. ($\approx$ is defined on pg. 222, it says $A \approx B$ iff $\exists f : A \to B$ which is <u>one-one</u> and <u>onto</u>)

<u>Proof</u>: We show $A \approx A$. Let $i : A \to A$ be defined by $i(x) = x$. Clearly i is 1-1 and onto thus $A \approx A$.

Next, suppose $A \approx B$ then $\exists f : A \to B$ which is one-one and onto. Define $f^{-1}(y) = x$ such that $f(x) = y$. Since f is onto $\exists x \in A$ such that $f(x) = y$. We can show $f^{-1} : B \to A$ is one-one (i) and onto (ii)

i.) <u>one-one</u>: $f^{-1}(y) = f^{-1}(z) \Rightarrow f(f^{-1}(y)) = f(f^{-1}(z))$
$\Rightarrow y = z$.

ii.) <u>onto</u>: Let $a \in A$. Clearly $f(a)$ maps to a under f^{-1} since $f^{-1}(f(a)) = a$ by def$^{\underline{n}}$ of f^{-1}.

Thus $f^{-1} : B \to A$ is a <u>one-one</u> correspondence from B to A which shows $B \approx A$.

Next, suppose $A \approx B$ and $B \approx C$. Then $\exists f : A \to B$ and $\exists g : B \to C$ which are one-one and onto. I leave it to you to show $g \circ f : A \to C$ is a one-one and onto map thus $A \approx C$ and $\approx$ is hence transitive.

<u>Remark</u>: I'm reinventing the wheel here probably. If you see a property already proved in homework (that you've done) then feel free to reference those facts and make good use of them.
(See § 4.3 # 4, 5, 6 etc...)

§5.1 #2 Complete proof of Th^{m} 5.2(a) by showing if $h: A \to C$ and $g: B \to D$ are one-one correspondences then $f: A\times B \to C\times D$ given by $f(a,b) = (h(a), g(b))$ is a one-one correspondence

Proof: If $h: A \to C$ and $g: B \to D$ are one-one correspondences then $h(A) = C$ and $g(B) = D$ and h and g are one-one. We wish to show $f: A\times B \to C\times D$ given by $f(a,b) = (h(a), g(b))$ is a one-one correspondence. Suppose,

$$
\begin{aligned}
f(a,b) = f(x,y) &\implies (h(a), g(b)) = (h(x), g(y)) \\
&\implies h(a) = h(x) \text{ and } g(b) = g(y) \\
&\implies a = x \text{ and } b = y \quad \text{as } h, g \text{ are } 1\text{-}1. \\
&\implies (a,b) = (x,y) \\
&\implies f \text{ is } 1\text{-}1.
\end{aligned}
$$

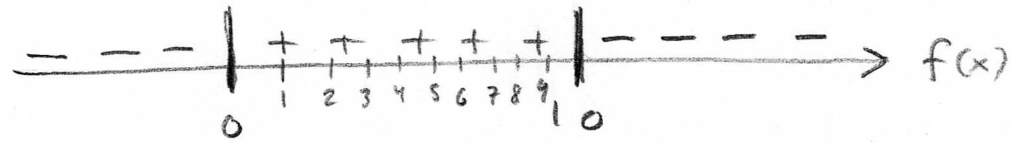
Let $(c,d) \in C\times D$ then $c \in C$ and $d \in D$ thus $\exists\, a \in A$ with $h(a) = c$ and $\exists\, b \in B$ with $g(b) = d$ as h and g are onto. Obsere that

$$f(a,b) = (h(a), g(b)) = (c,d)$$

Thus $f: A\times B \to C\times D$ is <u>surjective</u> (or <u>onto</u>)

§5.1 #5m Is the set $\{x \in \mathbb{N} \mid x(10-x) > 0\}$ finite or infinite?

Notice $x(10-x) = 0$ for $x = 0$ and $x = 10$ thus we can draw a sign chart and think of the extension of x to $\mathbb{R}$ where the Intermediate Value Th$^{\underline{m}}$ holds : $f(x) = x(10-x)$, $x \in \mathbb{R}$ is continuous



$f((0,10)) > 0$ yet $f|_{\mathbb{N}}(0,10) = \underbrace{\{f(1), \dots, f(9)\}}_{9 \text{ elements.}}$

Yes it is finite, it is $\approx$ to $\mathbb{N}_9$

§5.1 #5n the set of all complex numbers $a+ib$ such that $a^2+b^2 = 1$. Is this finite?

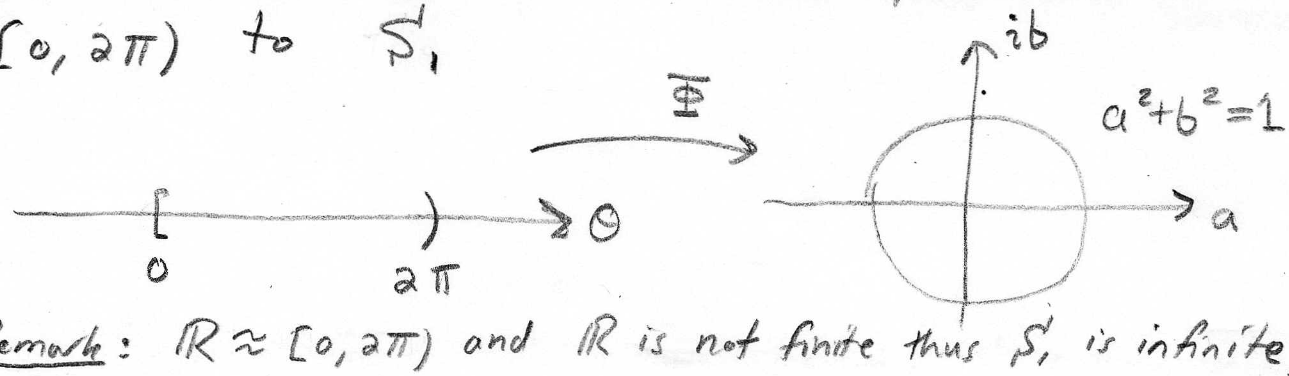
No. Notice
$$e^{i\theta}e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$$

So there is a natural mapping from $[0, 2\pi)$ to the set given here

$$\Phi : [0, 2\pi) \longrightarrow \{a+ib \mid a^2+b^2 = 1\} = S_1$$

$$\Phi(\theta) = \cos\theta + i\sin\theta = e^{i\theta}$$

You can show Φ is a 1-1 correspondence from $[0, 2\pi)$ to S_1



(Remark: $\mathbb{R} \approx [0, 2\pi)$ and $\mathbb{R}$ is not finite thus S_1 is infinite)

§5.1 #6a Let A and B be sets. Show that if A is finite then $A \cap B$ is finite

Suppose A is finite yet $A \cap B$ is not finite. If $A \cap B$ is not finite then $\nexists\ \Phi : A \cap B \to \mathbb{N}_k$.
However, as A is finite $\exists\ \psi : A \to \mathbb{N}_m$ which is a bijective map. The restriction of $\psi|_{A \cap B} : A \cap B \to \psi(A \cap B) \subseteq \mathbb{N}_m$ is one-one clearly $\psi(A \cap B) \subseteq \mathbb{N}_m$ thus $\psi(A \cap B)$ is a finite set. Thus $\exists\ k \leq m$ with $\psi|_{A \cap B} : A \cap B \to \mathbb{N}_k$ and thus $\exists\ \Phi = \psi|_{A \cap B}$ a clear <u>contradiction</u>.

Hence, using proof by contradiction, $A \cap B$ is finite. //

<u>Remark:</u> A direct proof is probably also natural here. <u>Sketch:</u>

$$\exists\ \psi : A \to \mathbb{N}_m \Rightarrow \psi|_{A \cap B} : A \cap B \to \psi(A \cap B) \subseteq \mathbb{N}_m$$

Thus $A \cap B \approx \psi(A \cap B) \subseteq \mathbb{N}_m \Rightarrow A \cap B \approx \mathbb{N}_k$ for some $k \leq m$. (Using Lemma 5.5 all subsets of $\mathbb{N}_k$ are finite)

§5.1 #13 Prove if $r > 1$ and $x \in \mathbb{N}_r$ then $\mathbb{N}_r - \{x\} \approx \mathbb{N}_{r-1}$

This seems obvious, but how do we prove it?
We can define the following map $\Psi : \mathbb{N}_r - \{x\} \to \mathbb{N}_{r-1}$

$$\Psi(a) = \begin{cases} a & \text{if } 1 \leq a \leq x-1 \\ a-1 & \text{if } x+1 \leq a \leq r \end{cases}$$

Notice $\Psi(1) = 1$ and $\Psi(r) = r-1$ and $\Psi(x+1) = x$ thus it is clear Ψ is onto $\mathbb{N}_{r-1}$. Thus (using Exercise 17 b) Ψ is 1-1 hence Ψ is a bijective correspondence. Therefore, $\mathbb{N}_r - \{x\} \approx \mathbb{N}_{r-1}$

<u>Remark:</u> this could also be seen as an cor. of Lemma 5.4.

Remark: §5.1 #14 "If S has cardnal number n and cardnal number m then $n=m$" To start I'd note $S \approx \mathbb{N}_n$ and $S \approx \mathbb{N}_m$. Past that I'd use Exercise §5.1# 1 to prove $\mathbb{N}_n = \mathbb{N}_m \Rightarrow n=m$. (this is actually pretty easy)

Remark: Outside of §5.1 I rarely see anyone be so careful about finite sets. Basically it's obvious that the finite union and/or intersection of finite sets is finite. It is still a good exercise to appreciate the detail which is hidden in such casual comments.

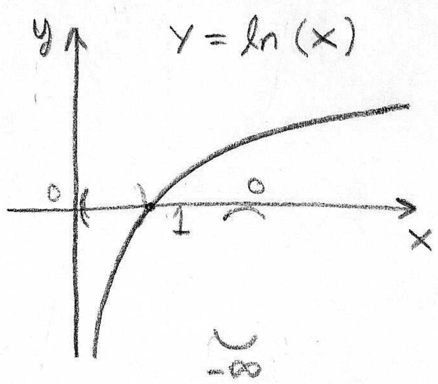
§5.2 #1c | Prove that $T^+ = 3\mathbb{Z}$ is denumerable

That is show $T^+ = \{3k \mid k \in \mathbb{N}\}$.

Define $\psi : T^+ \longrightarrow \mathbb{N}$ by $\psi(x) = \frac{1}{3}x$. Let $k \in \mathbb{N}$ then $3k \in T^+$ and $\psi(3k) = \frac{1}{3}(3k) = k$ therefore ψ is onto $\mathbb{N}$. Next suppose $\psi(x) = \psi(y)$ then $\frac{1}{3}x = \frac{1}{3}y \Rightarrow x = y \therefore \psi$ is one-one. Thus we find $T^+ \approx \mathbb{N}$ which means <u>T^+ is denumerable</u>

§5.2 #2b | Prove (a, ∞) has cardnality c

We would like to find a bijection from $(0,1) \longrightarrow (a,\infty)$.



$f(x) = a - \ln(x)$

(i.) $f(x) \longrightarrow \infty$ as $x \longrightarrow 0^+$

(ii.) $f(x) \longrightarrow a - \ln(1) = a$ as $x \longrightarrow 1^-$

$f : (0,1) \longrightarrow (a, \infty)$

and you can verify that f is a one-one and onto map.

§5.3 #5d) Does $\{2^x / x \in \mathbb{N}\}$ have cardinal number $\aleph_0$? Or cardinal number c?

Notice $\{2^x / x \in \mathbb{N}\} = \{2^1, 2^2, 2^3, \dots\} \approx \{1, 2, 3, \dots\}$ the correspondence is simply $2^x \longleftrightarrow x$. Thus the set is denumerable thus it has cardinality $\aleph_0$.

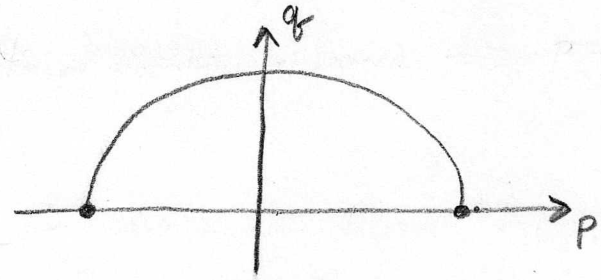
§5.2 #5f) What is the cardinality of $T = \{(p,q) \in \mathbb{R}^2 / q = \sqrt{1-p^2}\}$

Let's picture the set T in $\mathbb{R}^2$, Note $q = \sqrt{1-p^2}$

$\Rightarrow q^2 = 1 - p^2$

$\Rightarrow p^2 + q^2 = 1$

$\begin{pmatrix} q \geq 0 \\ -1 \leq p \leq 1 \end{pmatrix}$



It is intuitively obvious that $\overline{\overline{T}} = c$.
I'll give you a bonus point if you can prove it.

§5.2 #11 (HINT)) Given $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ has cardinality c Show $\mathbb{C}$ has cardinality c.

Try $\psi : \mathbb{R}^2 \longrightarrow \mathbb{C}$ defined by

$$\psi(a,b) = a + ib.$$

Prove ψ is one-one and onto.

§5.9 #9 (Hint))

Need to find $\psi : A \times B \to \mathbb{N}$ that is 1-1 and onto.
Need to use data! We are given $A \approx \mathbb{N}$ and $B \approx \mathbb{N}$ thus $\exists$ bijections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$.

§5.4 # 8a) Arrange the cardnal numbers in order from smallest to largest.

$$\overline{\overline{(0,1)}},\ \overline{\overline{[0,1]}},\ \overline{\overline{\{0,1\}}},\ \overline{\overline{\{0\}}},\ \overline{\overline{\mathcal{P}(\mathbb{R})}},\ \overline{\overline{\mathbb{Q}}},\ \overline{\overline{\varnothing}},\ \overline{\overline{\mathbb{R}-\mathbb{Q}}},\ \overline{\overline{\mathcal{P}(\mathcal{P}(\mathbb{R}))}},\ \overline{\overline{\mathbb{R}}}$$

Notice $\overline{\overline{\{0\}}} = 1$, $\overline{\overline{\{0,1\}}} = 2$ and $\overline{\overline{\varnothing}} = 0$. These are finite next the denumerable sets are $\mathbb{Q}$. The sets with cardnality $c = \aleph_1$ are $\mathbb{R} - \mathbb{Q}$ (irrationals) and $\mathbb{R}$, $[0,1]$, $(0,1)$.
$\aleph_2$ has $\mathcal{P}(\mathbb{R})$ by Cantor's Theorem, then $\overline{\overline{\mathcal{P}(\mathcal{P}(\mathbb{R})}} = \aleph_3$.

In ~~total~~,

$$\underbrace{\overline{\overline{\varnothing}},\ \overline{\overline{\{0\}}},\ \overline{\overline{\{0,1\}}}}_{\text{finite}},\ \underbrace{\overline{\overline{\mathbb{Q}}}}_{\substack{\text{denumerable}\\ \aleph_0}},\ \underbrace{\overline{\overline{(0,1)}},\ \overline{\overline{[0,1]}},\ \overline{\overline{\mathbb{R}}},\ \overline{\overline{\mathbb{R}-\mathbb{Q}}}}_{\aleph_1 = c},\ \underbrace{\overline{\overline{\mathcal{P}(\mathbb{R})}}}_{\aleph_2},\ \underbrace{\overline{\overline{\mathcal{P}(\mathcal{P}(\mathbb{R}))}}}_{\aleph_3}$$

Assuming Continuum Hypothesis which says c is the next infinity past $\aleph_0$ so it makes sense to lable it $\aleph_1$.

(See pg. 258 for some discussion after Cor. 5.37 concerning undecidability of the continuum hypothesis and Axiom of Choice)