# Select Worked Problems on Functions

**§4.1#1c** Let $R_3 = \{(1,2),(2,1)\}$ be a relation. Is this a function?

YES. $f : \{1,2\} \longrightarrow \{1,2\}$ with $f(1) = 2$ and $f(2) = 1$.

**§4.1#1d** Let $R_4 = \{(x,y) \in \mathbb{R}^2 \mid x = \sin(y)\}$. Is this a function?

Notice that $(0,\pi), (0,0)$ are both in $\mathbb{R}^4$ thus $R_4$ is not a function. However, $R_4^{-1} = \{(x,y) \in \mathbb{R}^2 \mid y = \sin(x)\}$ is a function. (Just saying.)

**§4.1#3x** Identify domain, range and possible codomain for the function $\{(x,y) \in \mathbb{Z}^2 \mid y = \frac{x^2-4}{x-2}\}$

This is just fancy notation for $f : \mathbb{Z} - \{2\} \longrightarrow \mathbb{R}$ where $f(x) = \frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = x+2$ for $x \neq 2$.

Thus $\text{dom}(f) = \mathbb{Z} - \{2\}$ and $\text{range}(f) = \mathbb{Z} - \{4\}$.

The codomain$(f)$ can be any set containing range$(f)$ for example $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ etc...

**§4.1#4c** Let $f(x) = x^2 - 1$ find all pre-images of 24

Def$^{n}$/ $x$ is the preimage of $y$ with respect to $f$ iff $f(x) = y$.

Pre-images of $24 = y$ have $f(x) = 24$ $\Rightarrow x^2 - 1 = 24$

$$\Rightarrow x^2 = 25$$

$$\Rightarrow \boxed{x = \pm 5}$$

$\boxed{\S 4.2 \# 1a}$ Let $f(x) = 2x + 5$ and $g(x) = 6 - 7x$. Find $f \circ g$ and $g \circ f$

$$(f \circ g)(x) = f(g(x)) = f(6-7x) = 2(6-7x)+5 = 17 - 14x$$

$$(g \circ f)(x) = g(f(x)) = g(2x+5) = 6 - 7(2x+5) = -29 - 14x$$

In this case $\text{dom}(f \circ g) = \text{dom}(g \circ f) = \mathbb{R}$. Notice that the slope of the composite of linear functions is the product of the slopes. This is where the chain-rule

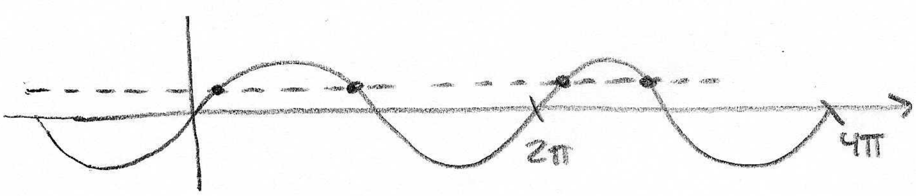
$$\frac{d}{dx}(f \circ u)(x) = \underbrace{\frac{df}{dx}(u(x))}_{\text{slope of } f} \underbrace{\frac{du}{dx}(x)}_{\text{slope of } u}$$

comes from. Look up my proof in the calc I notes you'll see it.

$\boxed{\S 4.2 \# 2d}$ $f(x) = \sin(x)$ is $f^{-1}(x)$ a function?

Well it depends on what we understand to be dom $(f)$. The maximal domain in $\mathbb{R}$ is $\mathbb{R}$. However, $f$ is not injective on $\mathbb{R}$ (sorry, but this is best discussed with section 4.3 concepts and I welcome you to do likewise)

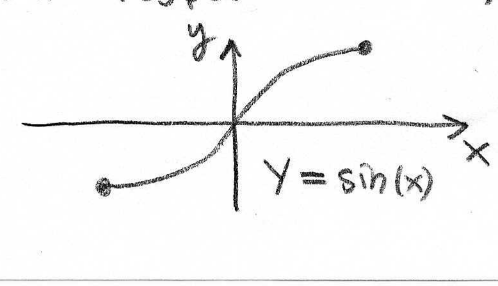
Note the graph $y = \sin(x)$




$\sin(\pi/6) = \sin(5\pi/6) = \frac{1}{2}$ thus $f(x) = \sin(x)$ is not one-one. What would we make $f^{-1}(1/2)$? Should it be $\pi/6$ or $5\pi/6$ we must choose one value per-input into $f^{-1}$.

**BUT: WHAT ABOUT $\sin^{-1}(x)$ ??**

It is customary to restrict $f$ to $f|_{[-\pi/2, \pi/2]}$ where $f$ is one-one with respect to $[-\pi/2, \pi/2]$. Then




$$\left(f_{[-\pi/2, \pi/2]}\right)^{-1}(x) = \sin^{-1}(x)$$

$\boxed{\S 4.2 \# 3f}$ Let $f(x) = \dfrac{1-x}{-x}$. Find $f^{-1}(x)$ if possible

Clearly $\text{dom}(f) = \mathbb{R} - \{0\}$. We want $f(f^{-1}(x)) = x$ let $y = f^{-1}(x)$, we want $f(y) = x$ that is,

$$x = \frac{1-y}{-y} \Rightarrow -xy = 1-y$$

$$\Rightarrow y(1-x) = 1$$

$$\Rightarrow y = \boxed{f^{-1}(x) = \frac{1}{1-x}}$$

Notice $f(x) = 1 - \frac{1}{x}$ has $\text{range}(f) = \mathbb{R} - \{1\}$ since it's just $y = \frac{1}{x}$ flipped and shifted one unit up. This is good since $\text{dom}(f^{-1}) = \mathbb{R} - \{1\}$ and $\text{range}(f^{-1}) = \mathbb{R} - \{0\}$. We ought to always find this pattern,

$$\begin{tikzcd} \text{dom}(f) \arrow[r, bend left, "f"] & \text{range}(f) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

$\text{dom}(f) = \text{range}(f^{-1})$ $\qquad$ $\text{range}(f) = \text{dom}(f^{-1})$.

$\boxed{\S 4.2 \# 7b}$ Describe two extensions of $f$ with domain $\mathbb{R}$ for the function $f = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid y = 3\}$

We want $y = 3$ for $x \in \mathbb{N}$ but otherwise it can be anything,

1.) $g(x) = 3\sin^2\left[\frac{\pi}{2}(2x+1)\right]$

Notice $f(1) = 3\sin^2\left(\frac{3\pi}{2}\right) = 3$. In fact, $\sin\left(\left(\frac{2n+1}{2}\right)\pi\right) = \pm 1$ then $f(n) = 3(\pm 1)^2 = 3$ for each $n \in \mathbb{N}$. Thus $g|_{\mathbb{N}} = f$ as required. That is $g$ is an extension of $f$ to $\mathbb{R}$.

2.) $h(x) = 3$ for $x \in \mathbb{Q}$. Clearly $h(n) = 3$ for $n \in \mathbb{N}$ thus $h|_{\mathbb{N}} = f$. This is a very lazy extension, I likes it.

§ 4.2 #9 Prove Th$^{m}$(4.6). Let $h$ and $g$ be functions with $\mathrm{dom}(h) = A$ and $\mathrm{dom}(g) = B$. If $A \cap B = \emptyset$ then $h \cup g$ is a function with domain $A \cup B$. Furthermore,

$$(h \cup g)(x) = \begin{cases} h(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Proof: We view functions as relations here. That is to say

$$h = \{(x,y) \mid x \in A,\ y = h(x)\}$$

$$g = \{(x,y) \mid x \in A,\ y = g(x)\}$$

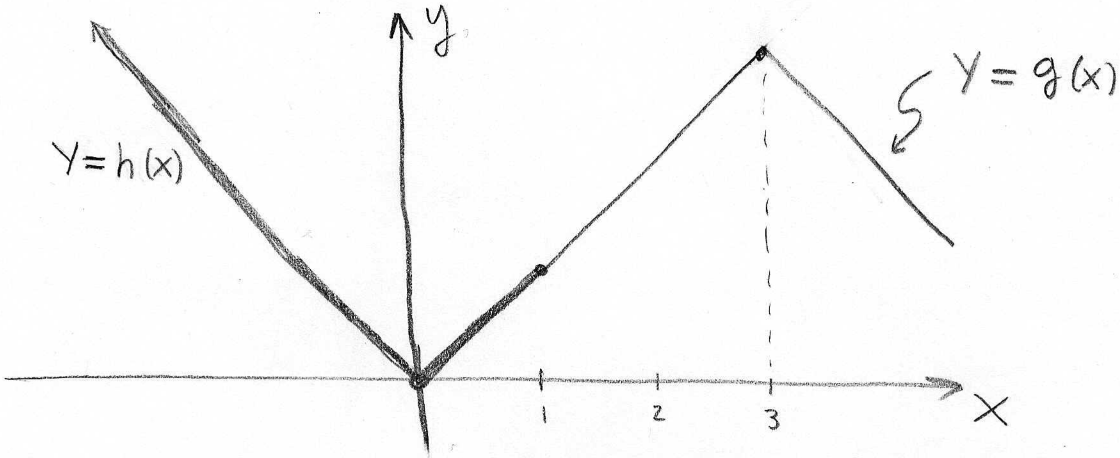
Notice $h \cup g = \{(x,y) \mid (x,y) \in h \text{ or } (x,y) \in g\}$. Clearly $h \cup g$ is a relation since it's a subset of $(A \times \mathrm{Rng}(f)) \cup (B \times \mathrm{Rng}(g))$. If $(x,y) \in h \cup g$ then either $x \in A$ or $x \in B$. We cannot have $x \in A$ and $x \in B$ since $A \cap B = \emptyset$. Let $(x,y) \in h \cup g$ such that $x \in A$ then $y = h(x)$, thus $(h \cup g)(x) = h(x)$. Likewise let $(x,y) \in h \cup g$ such that $x \in B$ then $y = g(x)$ thus $(h \cup g)(x) = g(x)$. Hence, $f \cup g$ is a function as described in a casewise fashion in the Th$^{m}$.

§ 4.2 #10 Let $f$ be a function with $\mathrm{dom}(f) = D$, and let $g$ be an extension of $f$ with $\mathrm{dom}(g) = A$. By definition, $f = g|_D$ and $D \subseteq A$. Let $i$ be the <u>inclusion</u> <u>mapping</u> from $D$ to $A$ given by $i(x) = x$ for each $x \in D$. Prove $f = g \circ i$

Proof: Let $x \in D$ then $f(x) = g(i(x))$.
Notice $\mathrm{dom}(i) = D$ and $\mathrm{range}(i) = D \subseteq A = \mathrm{dom}(g)$ thus the composition is well-defined, moreover since $f(x) = g(i(x))$ for each $x \in D$ it follows $f = g \circ i$ //

§ 4.2 # 12c | Let $h : (-\infty, 1] \to \mathbb{R}$ with $h(x) = |x|$, and suppose $g : [0, \infty) \to \mathbb{R}$ with $g(x) = 3 - |x-3|$. Is $h \cup g$ a function?

No, $h \cup g$ is not a function since $(0, g(0))$ and $(0, h(0))$ are both in $h \cup g$. Th$^{m}$ 4.6 does not apply since $\text{dom}(h) \cap \text{dom}(g) \neq \phi$.
Let me graph the situation,




Blast! I was tricked. $g(0) = h(0)$. This is a function.

$$(h \cup g)(x) = \begin{cases} h(x) & \text{if } x \in (-\infty, 0] \\ g(x) & \text{if } x \in [0, \infty) \end{cases}$$

§ 4.3 # 1e | Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sqrt{x^2 + 5}$ is this map onto its codomain? Is it injective?

Clearly $f$ is not onto since $\sqrt{x^2+5} \geq \sqrt{5}$.
Moreover, $f$ is not injective since

$$\begin{aligned} f(a) = f(b) &\Rightarrow \sqrt{a^2+5} = \sqrt{b^2+5} \\ &\Rightarrow a^2 + 5 = b^2 + 5 \\ &\Rightarrow a^2 = b^2 \\ &\Rightarrow a = \pm b \end{aligned}$$

thus $a \neq b$ in general.
$f(1) = f(-1)$ for example.

**§4.3 #6** If $f: A \to B$ and $g: B \to C$ such that $g \circ f: A \to C$ is one-one then $f: A \to B$ is one-one.

<u>Proof</u>: Suppose $f: A \to B$ and $g: B \to C$ such that $g \circ f: A \to C$ is 1-1. Let $x, y \in A$ such that $f(x) = f(y)$. We seek to show $x = y$. The proof is simple, act by $g$,

$$g(f(x)) = g(f(y)) \Rightarrow (g \circ f)(x) = (g \circ f)(y)$$

$$(\text{Using } g \circ f \text{ injective}) \Rightarrow x = y \quad \therefore f \text{ is one-one.} //$$

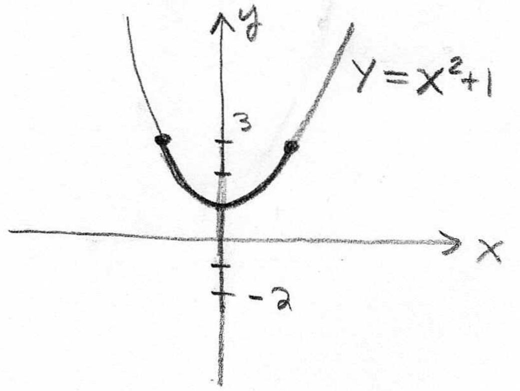
**§4.3 #10a,b** Let $A, B$ be sets, and $S \subset A \times B$. Let $\pi_1 : S \to A$ and $\pi_2 : S \to B$ be the projection maps onto $A$ and $B$ respective. Show $\pi_1$ need not be one-one.

a.) Let $A = B = \mathbb{R}$. Then $\pi_1(1,2) = 1$ and $\pi_1(1,3) = 1$ yet $(1,2) \neq (1,3)$ thus $\pi_1$ is not injective. Notice this is independent of our choice of $S \subset A \times B$.

b.) Let $S = \{(x,y) \mid x^2 + y^2 = 1\}$ then $\pi_1$ cannot map to onto $A$ since $2 \in A = \mathbb{R}$ yet $(x, 2) \notin \text{dom}(\pi_1) = S$ since $x^2 + 4 = 1$ has no real sol$^{n}$'s. We mean to point out $2 \notin \text{Range}(\pi_1)$.

$\boxed{\S 4.4\#2d}$ Let $f(x) = x^2+1$. Find $f^{-1}([-2,3])$

That is find $f^{-1}([-2,3]) = \{x \mid f(x) \in [-2,3]\}$




$3 = x^2+1$
$\Rightarrow x^2 = 2$
$\Rightarrow x = \pm\sqrt{2}$
$\Rightarrow \boxed{f^{-1}([2,3]) = [-\sqrt{2}, \sqrt{2}]}$

Notice, $f^{-1}([1,3]) = [-\sqrt{2}, \sqrt{2}]$ just the same.

$\boxed{\S 4.4\#3a,e}$ Find $f(A)$ for $f(x) = 1-2x$ in the cases $A = \{-1,0,1,2,3\}$ and $A = (1,4]$.

$$f(\{-1,0,1,2,3\}) = \{f(-1), f(0), f(1), f(2), f(3)\}$$
$$= \{1+2, 1, 1-2, 1-4, 1-6\}$$
$$= \underline{\{3, 1, -1, -3, -5\}}.$$

$$f((1,4]) = \{f(x) \mid x \in (1,4]\}$$
$$= \{1-2x \mid 1 < x \leq 4\}$$
$$= \underline{[-7, -1)}.$$

§4.4#11a) Let $\mathcal{A} = \{D_\alpha \mid \alpha \in \Delta\}$ and $\mathcal{B} = \{E_\beta \mid \beta \in \Gamma\}$ be families of subsets of $A$ and $B$ respectively. Prove that $f : A \to B$ (a function) has

$$f\left(\bigcap_{\alpha \in \Delta} D_\alpha\right) \subseteq \bigcap_{\alpha \in \Delta} f(D_\alpha)$$

Proof:

Suppose $y \in f\left(\bigcap_{\alpha \in \Delta} D_\alpha\right)$ then $\exists x \in \bigcap_{\alpha \in \Delta} D_\alpha$ such that $f(x) = y$. Since $x \in \bigcap_{\alpha \in \Delta} D_\alpha \Rightarrow x \in D_\alpha \ \forall \alpha \in \Delta$.

Notice that $x \in D_\alpha$ with $f(x) = y \Rightarrow y \in f(D_\alpha)$.

Moreover, $y \in f(D_\alpha)$ for each $\alpha \in \Delta \Rightarrow y \in \bigcap_{\alpha \in \Delta} f(D_\alpha)$.

Therefore, $f\left(\bigcap_{\alpha \in \Delta} D_\alpha\right) \subseteq \bigcap_{\alpha \in \Delta} f(D_\alpha)$.

Remark: why do you think $\subseteq$ was placed here instead of $=$. Why is $\bigcap_{\alpha \in \Delta} f(D_\alpha) \not\subseteq f\left(\bigcap_{\alpha \in \Delta} D_\alpha\right)$ in general? Can you illustrate $f(A) \cap f(B) \not\subseteq f(A \cap B)$ ?

§4.4#17) Let $f : A \to B$. Prove that if $X \subseteq A$ and $f$ is one-one then $f(A - X) = f(A) - f(X)$

Proof: let $f : A \to B$ with $X \subseteq A$ and $f$ one-one. Let $y \in f(A - X)$ then $\exists a \in (A - X)$ such that $f(a) = y$. Clearly $a \in A$ thus $f(a) = y \in f(A)$. Is it possible for $y \in f(X)$? Suppose it were the case that $y \in f(X)$ then $\exists x \in X$ with $f(x) = y$ yet $f(a) = y$ thus by 1-1 prop. of $f$ $x = a$ but that is a contradiction since $X \cap (A - X)$ is empty. Thus $y \in f(A)$ yet $y \notin f(X) \Rightarrow y \in f(A) - f(X)$. Hence, $\underline{f(A - X) \subseteq f(A) - f(X)}$.

(I leave the proof of $f(A) - f(X) \subseteq f(A - X)$ to the reader.)