Worked Problems for Math 200, Spring 2009

These are from "A TRANSITION TO ADVANCED MATHEMATICS" 6th Ed. by D. Smith, M. Eggen and R. St. Andre.

§1.1#2e The truth table for $P \wedge \sim Q$

P	Q	$\sim Q$	$P \wedge \sim Q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

$\longrightarrow$

P	Q	$P \wedge \sim Q$
T	T	F
T	F	T
F	T	F
F	F	F

§1.1#3b If P, Q and R are true while S and T are false then is $Q \vee (R \wedge S)$ true?

YES, since Q is true $Q \vee (\text{anything}) = \text{true}$.

§1.1#3i If P, Q and R are true while S and T are false then is $(P \vee S) \wedge (P \vee T)$ true?

1.) $P \vee S$ is true since P is true

2.) $P \vee T$ is true since P is true

Thus $(P \vee S) \wedge (P \vee T)$ is true as both input propositions for the $\wedge$ are true.

§1.1#4a Show $P \wedge P$ is equivalent to P

P	$P \wedge P$
T	T
F	F

Same truth values. proves equivalence.

§1.1#8c) Is the proposition $(P \wedge Q) \vee [(\sim P) \wedge (\sim Q)]$ a contradiction, tautology or neither?

We can find the answer easily from a truth table,

P	Q	$P \wedge Q$	$(\sim P) \wedge (\sim Q)$	$(P \wedge Q) \vee [(\sim P) \wedge (\sim Q)]$
0	0	0	1	1
0	1	0	0	0
1	0	0	0	0
1	1	1	0	1

(Let $0 = F$ and $1 = T$ in the table above.)

Thus $P \wedge Q \vee [(\sim P) \wedge (\sim Q)]$ is <u>not</u> a contradiction and it's not a tautology.

§1.1#10f) Give a denial of $P \equiv \{x < y \text{ or } m^2 < 1 \mid \underline{\text{fixed}}\ x, y, m \in \mathbb{R}\}$.

$$\sim P = \{x \geq y \text{ and } m^2 \geq 1 \mid \text{for fixed } x, y, m \in \mathbb{R}\}.$$

§1.2#4d) Is the following conditional sentence true?
If $5 < 2$ then $10 < 7$

Yes. This is a true sentence, the antecedent is false and the consequent is false

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

we find this problem fits into this case.

§1.2 #7a I'll prove part a. of Thm 1.2 here,

Claim: $\{P \Rightarrow Q\} \approxeq \{(\sim P) \vee Q\}$

Proof: by exhaustion.

P	Q	$\sim P$	$\sim P \vee Q$	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same truth values for all cases $\therefore$ statements equivalent.

Remark: the proof of the Thm is easy it's just tedious.

§1.2 #9a Show $\{(P \vee Q) \Rightarrow R\} \approxeq \{\sim R \Rightarrow (\sim P \wedge \sim Q)\}$

I find the notation T = 1, F = 0 less cumbersome for such problems.

Proof:

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$
0	0	0	0	1
0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

Remark: I was tempted to squeeze this into a single table but it's neater with two tables

remember, by definition of $(\Rightarrow)$ this is false = 0 whenever the antecendent $(P \vee Q)$ is true yet the consequent R is false.

§1.2 #9a Continued

P	Q	R	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$\sim R$	$\sim R \Rightarrow (\sim P \wedge \sim Q)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	0	1
0	1	0	1	0	0	1	0
0	1	1	1	0	0	0	1
1	0	0	0	1	0	1	0
1	0	1	0	1	0	0	1
1	1	0	0	0	0	1	0
1	1	1	0	0	1	0	1

Comparing the rightmost column of both truth tables reveals the given statements of $\{(P \vee Q) \Rightarrow R\}$ and $\{\sim R \Rightarrow (\sim P \wedge \sim Q)\}$ are equivalent.

Alternate Sol^{n}: Use Th^{m} 1.1a and Th^{m} 1.2d

$$\{(P \vee Q) \Rightarrow R\} \iff \{\sim R \Rightarrow \sim(P \vee Q)\} \quad : \text{by Th}^{m}\ 1.1a$$
$$\iff \{\sim R \Rightarrow (\sim P) \vee (\sim Q)\} \quad : \text{by Th}^{m}\ 1.2d$$

Remark: the alternate sol^{n} is more clever and thus easier. The truth table method is more brute-force. Either is correct.

§1.2 #10d Given an example of a true conditional sentence for which the contrapositive is true

If $|x| < 1$ then $|x|^2 < 1$. (true conditional sentence for some fixed $x \in \mathbb{R}$)

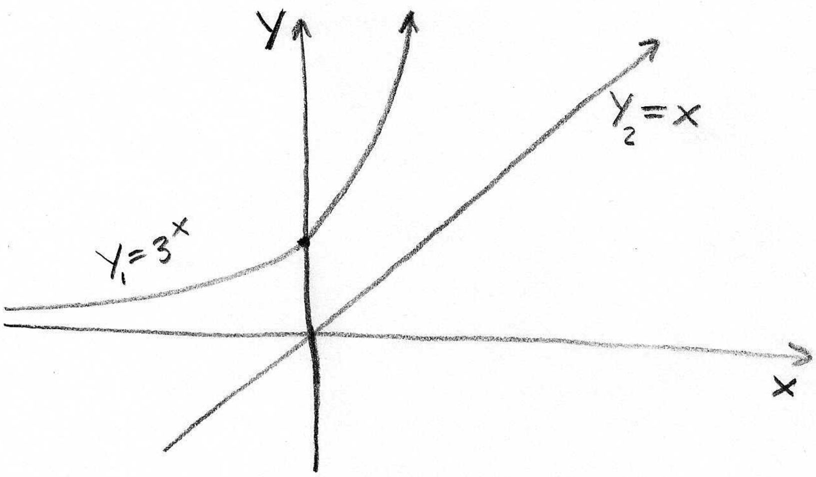
If $\underbrace{|x|^2 \geq 1}_{\text{negation of } |x|^2 < 1}$ then $\underbrace{|x| \geq 1}_{\text{negation of } |x| < 1}$.

§1.3#5b Is $(\forall x)(x+x \geq x)$ true for $x \in \mathbb{N}$?

Yes. Since $x \in \mathbb{N} \Rightarrow x > 0 \Rightarrow x+x > 0$. Hence $x+x \geq x$ for all $x \in \mathbb{N}$. [Note the x I chose from $\mathbb{N}$ was completely unconstrained except that I insisted $x \in \mathcal{U} = \mathbb{N}$. This is needed to conclude the "$\forall$" holds.]

§1.3#5e Is $(\exists x)(3^x = x)$ true for $\mathcal{U} = \mathbb{R}$?

No. Consider the graphs.



no intersection thus no sol$^{\underline{n}}$ to $3^x = x$.

Alternatively: $3^x = x$ is $\approx$ to $f(x) = 3^x - x = 0$

Consider $f(x) = 3^x - x$.

Notice $f'(x) = \ln(3)3^x - 1$ and $f''(x) = (\ln(3))^2 3^x$. Clearly $f''(x) \neq 0$ for all $x \in \mathbb{R}$. Note, $f'(0) = \ln(3) - 1 > 0$. It follows that $f'(x) > 0$ for all $x \in \mathbb{R}$.

Observe $f(0) = 3^0 - 1 = 1 > 0$.

Since f increases it follows $f(x) > 1$ for $x > 0$. Similar arguments follow for $x < 0$.

Remark: neither of these arguments are complete. Your text's answer key just says "No."

$\boxed{\S 1.3 \# 5i}$ Is the following sentence true?
$(\exists x)(x^2 + x + 41 \text{ is prime})$ (universe $U = \mathbb{N}$)

Let $x = 1$ then $1 + 1 + 41 = 43$. You can check that 43 is prime, thus the sentence is true, the truth set is nonempty for ($x^2 + x + 41$ is prime).

Remark: part j is a very different statement. Think about it.

$\boxed{\S 1.3 \# 6b}$ Translate: $(\exists! x)(x \geq 0 \wedge x \leq 0)$. (Real Numbers)

Translation: There exists a unique real number x such that $x \geq 0$ and $x \leq 0$.

$\boxed{\S 1.3 \# 8e}$ Find a useful denial for $(\exists! x) A(x)$.

$$\sim\{(\exists! x) A(x)\} \Longleftrightarrow \sim\{(\exists x) A(x) \wedge (\forall y)(\forall z)[A(y) \wedge A(z) \Rightarrow y = z]\}$$

$$\Longleftrightarrow \{\sim((\exists x) A(x)) \vee (\sim((\forall y)(\forall z)[A(y) \wedge A(z) \Rightarrow y = z]))\}$$

$$\Longleftrightarrow \{(\forall x)(\sim A(x)) \vee ((\exists y)(\exists z) \sim [A(y) \wedge A(z) \Rightarrow y = z])\}$$

$$\Longleftrightarrow \{(\forall x)(\sim A(x)) \vee (\exists y)(\exists z)\, \underbrace{[A(y) \wedge A(z) \wedge (y \neq z)]}_{\text{used Th}^{m}\ 1.2\ \text{part e.}}\}$$

Remark: $\{\sim (\exists x) A(x)\} \Leftrightarrow \{(\forall x)(\sim A(x))\}$ is much easier. That little ! makes a big conceptual difference.

§1.4 #1 Verify the tautologies on pg. 29

1.) Excluded Middle : $P \vee (\sim P)$

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

2.) $P \Rightarrow Q \Leftrightarrow (\sim Q \Rightarrow \sim P)$: Contrapositive.

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Thus $(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$ independent of the values of P & Q. In useful terms this means we may exchange a proof of $P \Rightarrow Q$ for a proof of the contrapositive $\sim Q \Rightarrow \sim P$. Often, when a direct proof fails to be clear it can be surmounted by an indirect proof which relies on a tautology.

6.) De Morgan's Laws :

$\sim(P \vee Q) \Leftrightarrow (\sim P) \wedge (\sim Q)$

$\sim(P \wedge Q \Leftrightarrow (\sim P) \vee (\sim Q)$

P	Q	$P \vee Q$	$P \wedge Q$	$\sim(P \vee Q)$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$\sim P \vee \sim Q$
0	0	0	0	1	1	1	1	1	1
0	1	1	0	0	1	1	0	0	1
1	0	1	0	0	1	0	1	0	1
1	1	1	1	0	0	0	0	0	0

observe $\sim(P \vee Q) \Leftrightarrow (\sim P) \wedge (\sim Q)$ by comparing columns 5 & 9
likewise $\sim(P \wedge Q) \Leftrightarrow (\sim P) \vee (\sim Q)$ by comparing columns 6 & 10.

§1.4#1

7.) $P \Leftrightarrow (\sim P \Rightarrow (Q \wedge \sim Q))$: Contradiction

P	Q	$\sim P$	$\sim Q$	$Q \wedge \sim Q$	$\sim P \Rightarrow (Q \wedge \sim Q)$	$P \Leftrightarrow [\sim P \Rightarrow (Q \wedge \sim Q)]$
0	0	1	1	0	0	1
0	1	1	0	0	0	1
1	0	0	1	0	1	1
1	1	0	0	0	1	1

Compare the 1st and 6th columns to verify the tautology $P \Leftrightarrow (\sim P \Rightarrow (Q \wedge \sim Q))$.

8.) $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$: Modus Ponens

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

§1.4#5c Prove that if $x, y \in \mathbb{Z}$ are odd then $x+y$ is even.

Proof: Suppose $x, y \in \mathbb{Z}$ are odd, then $\exists\, m, n \in \mathbb{Z}$ such that $x = 2m+1$ and $y = 2n+1$. Observe,

$$x + y = (2m+1) + (2n+1) = 2\underbrace{(m+n+1)}_{k}$$

Hence $x+y$ is even since $x+y = 2k$ for $k = m+n+1 \in \mathbb{Z}$.

§1.4#5d Prove that if $x, y \in \mathbb{Z}$ are even then xy is divisible by 4.

Proof: $x, y \in \mathbb{Z}$ even $\Rightarrow \exists\, m, n \in \mathbb{Z}$ such that $x = 2m$ and $y = 2n$. Observe that $xy = (2m)(2n) = 4\underbrace{mn}_{k} = 4k$. Thus, $xy = 4k$ for $k = mn \in \mathbb{Z}$ thus 4 divides xy with respect to $\mathbb{Z}$.

§1.4 # 6b) Prove that $|a-b| = |b-a|$

Proof: Notice that $|x| = \sqrt{x^2}$. Consider then,

$$\begin{aligned} |a-b| &= \sqrt{(a-b)^2} \\ &= \sqrt{[-(b-a)]^2} \\ &= \sqrt{(-1)^2(b-a)^2} \\ &= \sqrt{(b-a)^2} \\ &= |b-a|. \end{aligned}$$

Alternatively: Proceed case wise. Suppose $a, b \in \mathbb{R}$,

1.) $a = b$ then $|a-b| = |0| = |b-a|$

2.) $a < b$ then $a-b < 0$ thus $|a-b| = -(a-b) = b-a$ and $b-a > 0$ thus $|b-a| = b-a$ $\therefore$ $|a-b| = |b-a|$.

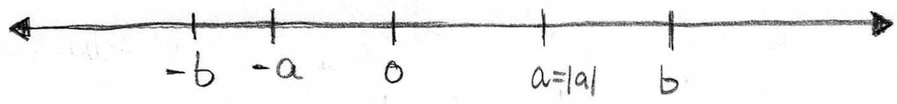
3.) $b < a$ then $b-a < 0$ thus $|b-a| = -(b-a) = a-b$ and $a-b > 0$ thus $|a-b| = a-b$ $\therefore$ $|a-b| = |b-a|$.

Thus, $|a-b| = |b-a|$ in all cases thus it is true.

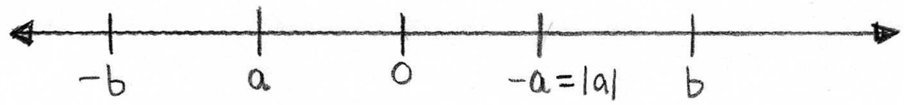
Remark: In practice step 3. was unecessary. There is a clear symmetry in the role played by a and b. I would give full credit for stating 1.) and 2.) with a qualification "without loss of generality we can assume $a < b$".

§1.4 #6e Prove $|a| \le b$ iff $-b \le a \le b$

Proof: If $a = b = 0$ then the statement is trivially satisfied. Suppose $a \neq 0$ then either $a < 0$ or $a > 0$. We may assume $b > 0$ in both cases since both of the conditionals are impossible for $b < 0$. Consider,

I.) 

: Case with $a > 0$ and $|a| = a \le b$.

II.) 

: case with $a < 0$ and $|a| = -a \le b$

These pictures help me guide my arguments.

I.) $(a>0)$ Suppose $|a| \le b \Rightarrow a \le b$. Since $b \ge 0$, $-b \le 0 < a$ thus $-b < a$. Hence, $-b \le a \le b$. Conversely, suppose $-b \le a \le b \Rightarrow a \le b \Rightarrow |a| \le b$.

II.) $(a<0)$ Suppose $|a| \le b \Rightarrow -a \le b \Rightarrow -b \le a$. Notice that $a < 0 < b \Rightarrow a \le b$. Consequently $-b \le a \le b$. Conversely suppose $-b \le a \le b \Rightarrow -a \le b \Rightarrow |a| \le b$.

§1.4 #7d Let $a \in \mathbb{Z}$ prove $a(a+1)$ is even

Proof: Either a is odd or a is even. Proceed casewise:

1.) a even $\Rightarrow \exists m \in \mathbb{Z}$ such that $a = 2m$. Consider that $a(a+1) = \underbrace{2m(2m+1)}_{k} = 2k$. Therefore, $a(a+1)$ is even since $a(a+1) = 2k$ for some $k \in \mathbb{Z}$.

2.) a odd $\Rightarrow \exists n \in \mathbb{Z}$ such that $a = 2n+1$. Consider that $a(a+1) = (2n+1)(2n+1+1) = (2n+1)(2n+2) = 2\underbrace{(2n+1)(n+1)}_{k}$. Therefore, $a(a+1)$ is even since $a(a+1) = 2k$ for some $k \in \mathbb{Z}$.

$\boxed{\text{§1.4 \# 7j}}$ If a/b and c/d then ac/bd. Prove this.

Proof: We suppose $a, b, c, d \in \mathbb{Z}$. If a/b and c/d then $\exists\, m, n \in \mathbb{Z}$ such that

$$b = ma \quad \text{and} \quad d = nc.$$

Consider that,

$$bd = (ma)(nc) = \underbrace{(mn)}_{k}ac = kac$$

Thus, $bd = k(ac)$ for some $k \in \mathbb{Z}$ hence ac / bd.

Remark: a/b is shorthand for "a divides b".
$a \nmid b$ is shorthand for "a does not divide b"

$\boxed{\text{§1.4 \# 9d}}$ Work backwards to set-up proof for $x^3 + 2x^2 < 0 \Rightarrow 2x + 5 < 11$

$$x^3 + 2x^2 < 0 \iff x^2(x+2) < 0 \iff x + 2 < 0$$
$$\iff x < -2$$
$$\iff 2x < -4$$
$$\iff 2x + 5 < 1 < 11.$$

Proof: Suppose $x \in \mathbb{R}$ such that $x^3 + 2x^2 < 0$. Observe that $x^3 + 2x^2 = x^2(x+2)$. Note $x \neq 0$ since otherwise $x^3 + 2x^2 = 0 \not< 0$. Hence, divide by $x^2 > 0$,

$$x^3 + 2x^2 = x^2(x+2) < 0 \Rightarrow x + 2 < 0$$

Multiply by 2,

$$2x + 4 < 0 \Rightarrow 2x + 5 < 1 < 11$$
$$\Rightarrow 2x + 5 < 11.$$

Remark: this question is weird.

§1.5 # 2e Outline proof of A and B are invertible iff the product AB is invertible. (use "two-part" proof)

Step 1: Show A, B invertible $\Rightarrow$ AB is invertible.
Step 2: Show AB invertible $\Rightarrow$ A, B invertible.
Conclude using $(P \Rightarrow Q) \wedge (Q \Rightarrow P) \Leftrightarrow (P \Leftrightarrow Q)$.

Proof: Suppose A, B are square matrices. We say A is invertible if $\exists\, A^{-1}$ a square matrix with $AA^{-1} = I = A^{-1}A$.
Suppose A, B are invertible matrices. Thus $\exists\, A^{-1}, B^{-1}$ such that $AA^{-1} = I = A^{-1}A$ and $BB^{-1} = I = B^{-1}B$. I claim that $(AB)^{-1} = B^{-1}A^{-1}$. Note

$$(AB)^{-1}(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$
$$AB(AB)^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = A^{-1}A = I$$

So my claim is true and AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$.
Next suppose AB is a product of matrices such that $(AB)^{-1}$ exists.
I wish to show A^{-1} and B^{-1} exist given $(AB)^{-1}AB = I = AB(AB)^{-1}$.
Claim $A^{-1} = B(AB)^{-1}$ and $B^{-1} = (AB)^{-1}A$,

$$A(B(AB)^{-1}) = AB(AB)^{-1} = I, \quad ((AB)^{-1}A)B = I \; : \; AA^{-1} = I = B^{-1}B$$

Now I'll be lazy and quote a Thm from linear algebra which states all left (or right) inverses are also right (or left) inverses.
Thus $A^{-1} = B(AB)^{-1}$ and $B^{-1} = (AB)^{-1}A$ are invertible. Notice that $A = (A^{-1})^{-1}$ and $B = (B^{-1})^{-1}$ thus $AB = (A^{-1})^{-1}(B^{-1})^{-1}$. //

Remark: you are not responsible for the proof given here, I just thought it would be good to see a proof here or there actually written out. The exercise §1.5 # 2e is sort of like chewing without food.

§1.5 #3e | Prove by contraposition: If $x+y$ is even, then either x and y are odd or x and y are even

Proof: If x, y are both either not even or not odd then one of x, y are even and the other is odd. Without loss of generality, assume $\exists\, m, n \in \mathbb{Z}$ such that

$$\underbrace{x = 2m}_{\text{even}} \quad \text{and} \quad \underbrace{y = 2n+1}_{\text{odd}}$$

Notice that $x+y = 2m + 2n + 1 = 2(m+n)+1 \therefore x+y$ is odd. Thus $x+y$ is not even. Thus by proof by contraposition we are done. $[(P \Rightarrow Q) \Longleftrightarrow (\sim Q \Rightarrow \sim P)]$.

§1.5 #3g | Prove by contraposition: If $\underbrace{8 \nmid (x^2-1)}_{P}$ then $\underbrace{x \text{ is even}}_{Q}$

$\sim Q = x$ is odd

$\sim P = 8 \mid x^2-1$

Proof: Assume x is odd. Then $\exists\, m \in \mathbb{Z}$ such that $x = 2m+1$. Consider that,

$$\begin{aligned} x^2 - 1 &= (2m+1)^2 - 1 \\ &= 4m^2 + 4m + 1 - 1 \\ &= 4(m^2 + m) \\ &= 4m(m+1) \\ &= 8k \end{aligned}$$

the product of any two consequtive integers is even thus $\exists\, k \in \mathbb{Z}$ such that $m(m+1) = 2k$.

Thus $8 \mid x^2-1$. //

Remark: I hope 3c and 3d are not so tricky. The big idea is simply that $(P \Rightarrow Q) \approxeq (\sim Q \Rightarrow \sim P)$.

§1.5#6b Prove by contradiction: Let $a, b \in \mathbb{Z}$ with $a, b > 0$. If ab is odd then both a and b are odd.

Proof: We have $P \Rightarrow Q$. The negation is $\sim(P \Rightarrow Q) \Leftrightarrow P \wedge \sim Q$.

So assume $P \wedge \sim Q$. That is let ab be odd and both a and b are not odd. Hence a, b are even $\Rightarrow \exists\, m, n \in \mathbb{Z}$ such that

$$a = 2m \quad , \quad b = 2n$$

Hence $ab = (2m)(2n) = 2(2mn) \Rightarrow ab$ is even.

This is a $\rightarrow\leftarrow$ (contradiction) since ab cannot be even & odd.

§1.5#7c: Let $a \in \mathbb{Z}$ with $a > 0$. Prove a is odd iff $a+1$ is even

Proof: $\Rightarrow$) Assume a is odd. Then $\exists\, m \in \mathbb{Z}$ such that $a = 2m+1$ hence $a+1 = 2m+1+1 = 2(m+1) \Rightarrow a+1$ is even.

$\Leftarrow$) Assume $a+1$ is even. Then $\exists\, n \in \mathbb{Z}$ such that $a+1 = 2n$ hence $a = 2n-1 = 2n-2+1 = 2(n-1)+1 \Rightarrow a$ is odd. //

§1.6#1b Prove $\exists\, m, n \in \mathbb{Z}$ such that $15m + 12n = 3$

Proof: $1, -1 \in \mathbb{Z}$ and $15(1) + 12(-1) = 3$.

§1.6#1d Prove $\nexists\, m, n \in \mathbb{Z}$ such that $12m + 15n = 1$.

Proof: (by contradiction) Suppose $\exists\, m, n \in \mathbb{Z}$ such that $12m + 15n = 1$ then 12 and 15 would be relatively prime. But both 12 and 15 have a common factor of 3. In other words, 3 divides the L.H.S. of $12m + 15n = 1$ BUT $3 \nmid 1$. $\rightarrow\leftarrow$ //

§1.6 #2c) Prove for all $a,b,c,d \in \mathbb{Z}$, if a/b then $\forall n \in \mathbb{N}$ a^n/b^n

Proof: $a/b \Rightarrow \exists m \in \mathbb{Z}$ such that $b = ma$. For each $n \in \mathbb{N}$,

$$b^n = (ma)^n = m^n a^n \quad \text{and} \quad m^n \in \mathbb{Z}.$$

thus a^n / b^n for an arbitrary $n \in \mathbb{N}$. Therefore a^n/b^n $\forall n \in \mathbb{N}$. //

§1.6 #5e) Prove or disprove. For $a,b,c,d \in \mathbb{Z}$, if $a|(b-c)$ and $a|(c-d)$, then $a|(b-d)$

Proof: Suppose $a|(b-c)$ and $a|(c-d)$ then $\exists m,n \in \mathbb{Z}$ such that $(b-c) = ma$ and $c-d = na$. Then we can solve for b and d as follows

$$b = c + ma \quad \& \quad d = c - na$$

Notice that

$$\begin{aligned} b-d &= (c+ma)-(c-na) \\ &= ma + na \\ &= (m+n)a \quad \therefore \quad a | b-d. // \end{aligned}$$

§1.6 #6b) Show that if $z \in \mathbb{C}$ then $\bar{z}z \in \mathbb{R}$

Proof: Suppose $z \in \mathbb{C}$ then $\exists a,b \in \mathbb{R}$ such that $z = a+ib$.
Consider that

$$\bar{z}z = (a-ib)(a+ib)$$

I leave the rest to you. Our definition of a real number in this context is that there are no "i"'s remaining.