Select Worked Problems on Relations

§3.1 #1d Let $A = \{(2,4), (3,1)\}$ and $B = \{(4,1), (2,3)\}$ find $A \times B$ and $B \times A$

$$A \times B = \{((2,4),(4,1)), ((2,4),(2,3)), ((3,1),(4,1)), ((3,1),(2,3))\}$$

$$B \times A = \{((4,1),(2,4)), ((2,3),(2,4)), ((4,1),(3,1)), ((2,3),(3,1))\}$$

§3.1 #3b Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof: Note $(x, y) \in A \times (B \cap C)$

iff $x \in A$ and $y \in (B \cap C)$

iff $x \in A$ and $y \in B$ and $y \in C$

iff $(x, y) \in A \times B$ and $(x, y) \in A \times C$

iff $(x, y) \in (A \times B) \cap (A \times C)$. Therefore, $A \times (B \cap C) = (A \times B) \cap (A \times C)$. //

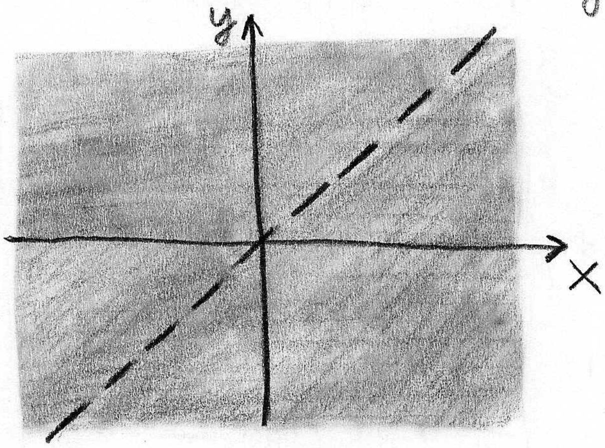
§3.1 #6h Find dom(R) and Rng(R) if $R = \{(x,y) \in \mathbb{R}^2 \mid x \neq y\}$

We have $x R y$ iff $x \neq y$.

$$\text{dom}(R) = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} \text{ such that } x \neq y\} = \mathbb{R}$$

$$\text{range}(R) = \{y \in \mathbb{R} \mid \exists x \in \mathbb{R} \text{ such that } x \neq y\} = \mathbb{R}$$

This example helps illustrate how relations are not the same as functions. (no function's graph nearly fills the plane, the vertical line test is very limiting)



§ 3.1 # 8f Find the inverse relation for $R_6 = \{ (x,y) \in \mathbb{R}^2 \mid y < x+1 \}$

By def^{n} $(a,b) \in R_6^{-1} \Leftrightarrow b \, R_6 \, a \Leftrightarrow a < b+1$ thus
$R_6^{-1} = \{ (a,b) \in \mathbb{R}^2 \mid a < b+1 \} = \{ (x,y) \in \mathbb{R}^2 \mid x < y+1 \}$

In other words, $x \, R_6^{-1} \, y$ iff $x < y+1$.

§ 3.1 # 9a Let $R = \{ (1,5), (2,2), (3,4), (5,2) \}$, $S = \{ (2,4), (3,4), (3,1), (5,5) \}$
Find $R \circ S$.

$x \, (R \circ S) \, y$ iff $\exists a$ such that $x \, S \, a$ and $a \, R \, y$.
iff $\exists a$ such that $(x,a) \in S$ and $(a,y) \in R$.

$\boxed{R \circ S = \{ (3,5), (5,2) \}}$

§ 3.2 # 1b Consider the relation $\leq$ on $\mathbb{N}$. Is this relation reflexive, symmetric and/or transitive?

$m \leq n \not\Rightarrow n \leq m$ $(3 \leq 4$ but $4 \not\leq 3)$ <u>not symmetric</u>

$m \leq m$ true since $m = m$, it is <u>reflexive</u>

$a \leq b$ and $b \leq c \Rightarrow a \leq c$

§ 3.2 # 2b Let $A = \{1, 2, 3\}$ find relation which is reflexive, not symmetric and not transitive

$$R = \{ (1,2), (1,1), (2,2), (2,3), (3,3) \}$$

Notice $1R1$, $2R2$, $3R3$ so it's reflexive.
However, $1R2$ yet $2 \not R 1$
Notice $1R2$ and $2R3$ yet $1 \not R 3$

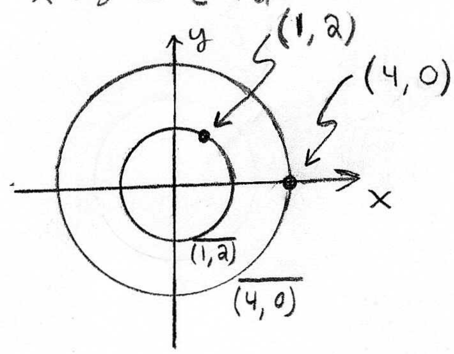
§3.2 #4e The relation T on $\mathbb{R}\times\mathbb{R}$ is defined by $(x,y)\,T\,(a,b)$ iff $x^2+y^2=a^2+b^2$. Show T is an equivalence relation and sketch the equivalence classes containing $(1,2)$ and $(4,0)$

T is reflexive since $x^2+y^2=x^2+y^2$.

T is symmetric since $x^2+y^2=a^2+b^2 \Rightarrow a^2+b^2=x^2+y^2$.

T is transitive since $(x,y)\,T\,(a,b)$ and $(a,b)\,T\,(c,d)$ yields $x^2+y^2=a^2+b^2$ and $a^2+b^2=c^2+d^2$ thus $x^2+y^2=c^2+d^2$ which shows $(x,y)\,T\,(c,d)$.

Thus T is an equivalence relation.



The equivalence classes form circles in the plane and the origin $\overline{(0,0)}=(0,0)$. Notice that these classes partition the plane, they cover each and every point and each circle is disjoint from the other circles.

(Remark, not part of sol$^{\underline{n}}$ to question, just an observation here)

§3.2 #4i Let S be set of all 2nd degree polynomials with real roots. Let H be the relation on S defined by fHg iff f and g have same zeros. What is the equivalence class of x^2-1 and x^2-2x+1

We wish to show H is an equivalence relation. Clearly fHf and $fHg \Rightarrow gHf$. since $f(r_1)=f(r_2)=0$ and $g(r_1)=g(r_2)=0$ is symmetric w.r.t. f and g. Also if fHg and gHh then f shares zeros with g and g shares zeroes with h thus f and h have same zeros.

$$\overline{x^2-1} = \{A(x^2-1) \mid A\in\mathbb{R},\, A\neq 0\}$$
$$= \{A(x+1)(x-1) \mid A\in\mathbb{R},\, A\neq 0\}$$
$$\overline{x^2-2x+1} = \{A(x-1)^2 \mid A\in\mathbb{R},\, A\neq 0\}$$

Remark: these also partition S but I cannot draw a picture of it.

$\boxed{\text{§3.2\# 12b}}$ R is symmetric iff $R = R^{-1}$

<u>Proof:</u> $\Rightarrow$] Suppose R is symmetric then $(a,b) \in R \Rightarrow (b,a) \in R$.
Let $(x,y) \in R$ then $(x,y) \in R^{-1}$ since $(x,y) \in R^{-1}$ iff $(y,x) \in R$
and we know $(y,x) \in R$ since R is symmetric. Thus $R \subseteq R^{-1}$.
Suppose $(x,y) \in R^{-1}$ then $\exists\ (y,x) \in R \Rightarrow (x,y) \in R$ since R symmetric.
Hence $R^{-1} \subseteq R$. Consequently, $R = R^{-1}$.

$\Leftarrow$] Assume $R = R^{-1}$. Let $(x,y) \in R$ then $(x,y) \in R^{-1}$ but
then $\exists\ (y,x) \in R$ by definition of R^{-1}. Thus R is symmetric.

We conclude that R is symmetric $\Leftrightarrow R = R^{-1}$. //

$\boxed{\text{§3.3\#3b}}$ Describe the partition for the equivalence relation R defined by $n,m \in \mathbb{Z}$, nRm iff $n+m$ is even.

the sum of even + even = even, odd + odd = even
but even + odd = odd.

$\bar{0} = \{0, \pm 2, \pm 4, \pm 6, \dots\}$

$\bar{1} = \{\pm 1, \pm 3, \pm 5, \dots\}$

$\boxed{\text{§3.3\#6c}}$ Given the partition $\{(-\infty,0), \{0\}, (0,\infty)\}$ of $\mathbb{R}$ describe the equivalence relation which corresponds to this partition

We want xRy if $x,y \in (-\infty,0)$ or $x=y=0$ or $x,y \in (0,\infty)$.

$\cancel{\boxed{xRy \text{ iff } xy \geq 0.}}$

(I left this to give you some idea of the sort of thought process I use)

$(-1)R(-3)$ since $(-1)(-3) = 3 > 0$

$2R5$ since $2(5) = 10 > 0$

$0R0$ since $0(0) = 0$.

oops, not quite, $0R3$, need to fix that ... $\boxed{xRy \text{ iff } xy > 0 \text{ or } x=y=0}$