Set Theory Homework, Select Solutions

§2.1#2b Is $\mathbb{Q} \subseteq \mathbb{Z}$?

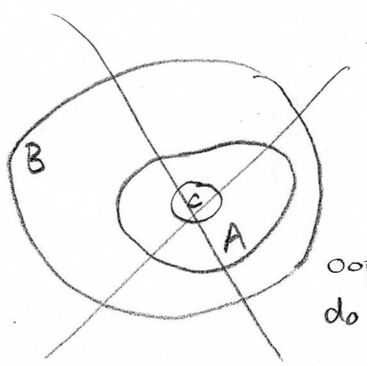
No, $\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z},\ q \neq 0 \}$ and $\frac{1}{2} \notin \mathbb{Z}$.

§2.1#4a,b a.) Is $\emptyset \in \{\emptyset, \{\emptyset\}\}$ b.) $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$. True or False ?

a.) Yes $\emptyset$ is an element of $\{\emptyset, \{\emptyset\}\}$, TRUE.

b.) Yes $\emptyset$ is a subset of $\{\emptyset, \{\emptyset\}\}$, THIS IS ALWAYS TRUE, IT HAS NOTHING TO DO WITH $\{\emptyset, \{\emptyset\}\}$. For example, let $\mathcal{B} = \{$kangaroos with one eye$\}$ then $\emptyset \subset \mathcal{B}$.

§2.1#5b Give example of A, B, C such that $A \subseteq B$, $B \subseteq C$, $C \subseteq A$



oops, can't do this $B \not\subseteq C$

$\boxed{A = B = C = \mathbb{R}.}$

(since $\subseteq$ we can choose $=$ option here)

§2.1#5c Give example of A, B, C such that $A \not\subseteq B$, $B \not\subseteq C$ and $A \subseteq C$

The constraint $A \subseteq C$ is most demanding here, there are many ways $A \not\subseteq B$ and $B \not\subseteq C$ can occur.

For example, $A = [0, 2]$, $B = [5, 6]$, $C = [0, 3]$.

(Many other sol^{n}'s possible for this question)

§2.1 #6e Let $X = \{1,2,3,4\}$. List out $\mathcal{P}(X)$

Remember: $\mathcal{P}(X) =$ set of all subsets of X, anticipate $2^4 = 16$

$$\mathcal{P}(X) = \{\phi, X, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$$

Remark: the fact that $\overline{\overline{\mathcal{P}(X)}} = 2^{\overline{\overline{X}}}$ is very useful. It helps me make sure I didn't miss anything

§2.1 #7a,c List proper subsets of a.) ϕ and c.) $\{1,2\}$

a.) There are no proper subsets of ϕ, (ϕ is a subset but $\phi = \phi$)

c.) The proper subsets of $\{1,2\}$ are ϕ, $\{1\}$ and $\{2\}$.

§2.1 #8a Give example of set A such that $\overline{\overline{\mathcal{P}(A)}} = 64$

By $\overline{\overline{\mathcal{P}(A)}} = 64$ we simply mean there are 64 elements in the power set of A. Note $64 = 2^6$ Thus $A = \{1, 2, 3, 4, 5, 6\}$ will do nicely.

§2.1 #8c Give example of A with $\mathcal{P}(A) = \phi$ if possible.

Not possible. Notice $A = \phi$ has $\mathcal{P}(A) = \{\phi\}$. There is a difference between ϕ and $\{\phi\}$

$\phi =$ the empty set

$\{\phi\} =$ the set containing the empty set.

Remark: You can construct $\mathbb{N}$ using sets like $\{\phi\}$...

§2.1 #12 Prove Th^m 2.2
Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Proof: Suppose $A \subseteq B$ and $B \subseteq C$. Let $x \in A$ then $x \in B$ by definition of $A \subseteq B$. If $x \in B$ then $x \in C$ since $B \subseteq C$. Thus, $x \in A \Rightarrow x \in C$. This proves $A \subseteq C$.

§2.1 #16 Prove that $X = Y$ where $X = \{x \in \mathbb{Z} \mid |x| \leq 3\}$ and $Y = \{-3, -2, -1, 0, 1, 2, 3\}$.

Frankly, this problem is so easy it's confusing.

$$x \in \mathbb{Z} \text{ and } |x| \leq 3 \iff x = -3, -2, -1, 0, 1, 2, 3$$
$$\iff x \in Y$$

Thus $x \in X \iff x \in Y$. Therefore, $X = Y$.

§2.2 #1a,b,c,e $A = \{1, 3, 5, 7, 9\}$, $B = \{0, 2, 4, 6, 8\}$ and $C = \{1, 2, 4, 5, 7, 8\}$ and $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
Find $A \cup B$, $A \cap B$, $A - B$, $(A - B) - C$

$$A \cup B = \{1, 3, 5, 7, 9, 0, 2, 4, 6, 8\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
$$A \cap B = \emptyset$$
$$A - B = A \quad \text{since } A \cap B = \emptyset.$$
$$(A - B) - C = A - C = \{1, 3, 5, 7, 9\} - \{1, 2, 4, 5, 7, 8\} = \{3, 9\}$$

Remark: this is not subtraction in the ordinary sense!

§2.2 # 2a, d, e | Let $U = \mathbb{Z}$ and suppose $E = \{2k \mid k \in \mathbb{Z}\}$, $D = \{2k+1 \mid k \in \mathbb{Z}\}$, $P = \{k \mid k \in \mathbb{Z}, k \geq 1\}$ and $N = \{k \mid k \leq -1, k \in \mathbb{Z}\}$. Find

a.) $E - P$, d.) $P - N$, e.) $\tilde{P}$

a.) $E - P = \{x \in E \mid x \notin P\}$

$= \{x \in \mathbb{Z} \mid x = 2k$ for some $k \in \mathbb{Z}$ and $x \ngeq 1\}$

$= \{\dots, -6, -4, -2, 0\}$

d.) $P - N = \{x \in P \mid x \notin N\}$

$= P$ since $P \cap N = \emptyset$.

e.) $\tilde{P} = U - P = \mathbb{Z} - P = \{x \in \mathbb{Z} \mid x \notin P\}$

$= \{x \in \mathbb{Z} \mid x \leq 0\}$

$= \underline{N \cup \{0\}}$.

§2.2 # 3l | Let $U = \mathbb{R}$ and $D = (5, \infty)$ find $\tilde{D}$

$\tilde{D} = \mathbb{R} - D = \{x \in \mathbb{R} \mid x \notin (5, \infty)\}$

$= \{x \in \mathbb{R} \mid x \leq 5\}$

$= \boxed{(-\infty, 5] = \tilde{D}}$

Notice: $\tilde{D} \cup D = (-\infty, 5] \cup (5, \infty) = \mathbb{R}$

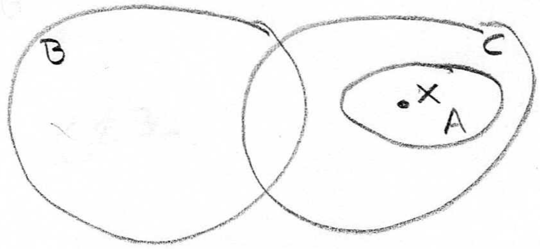
§2.2 # 4a, b, c hint | Let $U = \{1, 2, 3\}$ and $A = \{1, 2\}$

$\tilde{A} = U - A = \{1, 2, 3\} - \{1, 2\} = \{3\}$.

the rest I leave to you, it's not too bad. Check yours with a classmate.

§2.2 #10b Prove that if $A \subseteq B \cup C$ and $A \cap B = \emptyset$ then $A \subseteq C$.

Let's draw a model of the proof,



(we see $x \notin B$)

Proof: Let $A \subseteq B \cup C$ and $A \cap B = \emptyset$. Suppose $x \in A$, then since $A \subseteq B \cup C \Rightarrow x \in B$ or $x \in C$. Suppose that $x \in B$ then $x \in A$ and $x \in B$, but this is impossible since $A \cap B = \emptyset$. Thus $x \notin B$. This means $x \in C$ and consequently $A \subseteq C$.

§2.2 #10c Prove that: $C \subseteq A \cap B$ iff $C \subseteq A$ and $C \subseteq B$

Proof: We need to show $\Rightarrow$ and $\Leftarrow$.

$\Rightarrow$ | Assume $C \subseteq A \cap B$. Let $x \in C$ then $x \in A$ and $x \in B$ since $x \in A \cap B$. Thus $C \subseteq A$ and $C \subseteq B$. //

$\Leftarrow$ | Assume $C \subseteq A$ and $C \subseteq B$. Suppose $x \in C$, then $x \in A$ since $C \subseteq A$, and $x \in B$ since $C \subseteq B$. Therefore $x \in A$ and $x \in B \Rightarrow x \in A \cap B$. Thus, $C \subseteq A \cap B$ //.

$\boxed{\text{§2.2 \#13a} \mid \text{Prove that } \mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)}$

Proof: I remind the reader that

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}$$
$$\mathcal{P}(B) = \{Y \mid Y \subseteq B\}$$
$$\mathcal{P}(A \cap B) = \{Z \mid Z \subseteq A \cap B\}$$

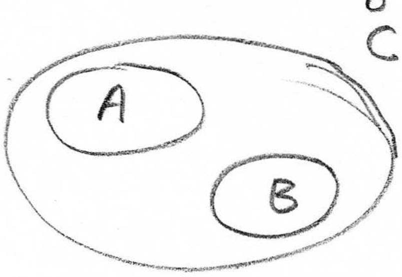
Suppose that $U \in \mathcal{P}(A \cap B)$ then $U \subseteq A \cap B$ thus by #10c $U \subseteq A$ and $U \subseteq B$. Thus we find $U \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$. Hence, $\underline{\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)}$. I

Next suppose that $V \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then we know $V \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$. Thus $V \subseteq A$ and $V \subseteq B$, and by #10c $V \subseteq A \cap B$. Thus, $V \in \mathcal{P}(A \cap B)$. So we find $\underline{\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)}$ II

Thus by I and II we conclude $\boxed{\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)}$.

$\boxed{\text{§2.2\#14} \mid \text{Provide a counterexample to the false claim "If } (A \cup C) \subseteq (B \cup C) \text{ then } A \subseteq B\text{"}}$

Let's see, why is this wrong? One possibility is that C is so big that it dominates the union.



$$A \cup C = C$$
$$B \cup C = C$$
But $A \not\subseteq B$

Let $C = \mathbb{R}$ and $A = [0,1]$ and $B = [2,3]$.

§2.2#16a,d) Let $U = \mathbb{R}$ and for $B \subseteq \mathbb{R}$ define $B^* = B \cup \{0\}$. Prove that $B \subseteq B^*$ and $(B^*)^* = B^*$

<u>Proof</u>: Let $x \in B$ then $x \in B \cup \{0\}$ thus $B \subseteq B^*$.

<u>Proof</u>: The proof of $(B^*)^* = B^*$ follows from the assertion that $A_1 \cup A_2 = (A_1 \cup A_2) \cup A_2$. We leave the proof of that assertion to the reader. Notice

$$(B^*)^* = B^* \cup \{0\} = (B \cup \{0\}) \cup \{0\} = B \cup \{0\} = B^*$$

Therefore $(B^*)^* = B^*$ //.

§2.3#1e) Find the union, intersection of the family of sets $\mathcal{A}$, where $\mathcal{A}$ is the set of all sets of integers that contain 10

$$\bigcup_{A \in \mathcal{A}} A = \mathbb{Z} \quad \text{and} \quad \bigcap_{A \in \mathcal{A}} A = \{10\}.$$

§2.3#7a) Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \Delta\}$ and $\mathcal{B} = \{B_\beta \mid \beta \in \Gamma\}$. rewrite $\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) \cap \left(\bigcup_{\beta \in \Gamma} B_\beta\right)$ as a union of intersections

you may use the following facts from §2.3#6 to help,

$$B \cap \bigcup_{\alpha \in \Delta} A_\alpha = \bigcup_{\alpha \in \Delta} (B \cap A_\alpha) \quad \& \quad B \cup \bigcap_{\alpha \in \Delta} A_\alpha = \bigcap_{\alpha \in \Delta} (B \cup A_\alpha)$$

$$\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) \cap \left(\bigcup_{\beta \in \Gamma} B_\beta\right) = \bigcup_{\alpha \in \Delta} \left[A_\alpha \cap \bigcup_{\beta \in \Gamma} B_\beta\right]$$

$$= \boxed{\bigcup_{\alpha \in \Delta} \bigcup_{\beta \in \Gamma} (A_\alpha \cap B_\beta)}$$

$$= \bigcup_{(\alpha, \beta) \in \Delta \times \Gamma} A_\alpha \cap B_\beta$$

(§3.1 comment) $\Delta \times \Gamma$ is the Cartesian product of Δ and Γ

§2.3 #6a hint | To prove $B \cap \bigcup_{\alpha \in \Delta} A_\alpha = \bigcup_{\alpha \in \Delta} (B \cap A_\alpha)$

You might try breaking it up into two parts

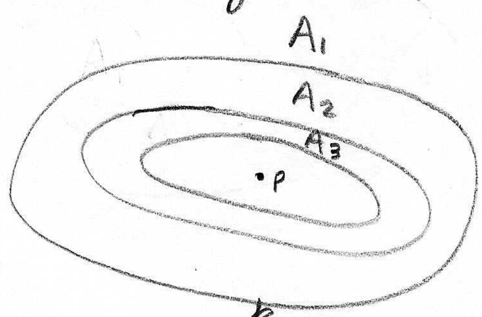
1.) Show $B \cap \bigcup_{\alpha \in \Delta} A_\alpha \subseteq \bigcup_{\alpha \in \Delta} (B \cap A_\alpha)$

2.) Show $\bigcup_{\alpha \in \Delta} (B \cap A_\alpha) \subseteq B \cap \left(\bigcup_{\alpha \in \Delta} A_\alpha \right)$

In both cases you'll use the standard prove $A \subseteq C$ technique. You assume $x \in A$ then work to show that $x \in C$ with the given data. We followed this logical pattern many times by now.

§2.3 #17 | Let $\mathcal{A} = \{ A_i \mid i \in \mathbb{N} \}$ be a nested family of sets. That is $\forall i, j \in \mathbb{N}$ if $i \leq j$ then $A_j \subseteq A_i$. Prove that for every $k \in \mathbb{N}$, $\bigcap_{i=1}^{k} A_i = A_k$

nested families of sets are important because they can be used to mathematically zoom in on some point.



$A_2 \subset A_1$

$A_3 \subset A_2 \subset A_1$

$A_4 \subset A_3 \subset A_2 \subset A_1$

$\vdots$

Proof: Let $x \in \bigcap_{i=1}^{k} A_i$ then $x \in A_i$ for each $i = 1, 2, \dots, k$. thus $x \in A_k$ and we find $\bigcap_{i=1}^{k} A_i \subset A_k$. Next suppose $x \in A_k$ then since $A_k \subseteq A_{k-1} \Rightarrow x \in A_{k-1}$ and so forth we find $x \in A_{k-2}, \dots, x \in A_2, x \in A_1$. Thus $x \in \bigcap_{i=1}^{k} A_i$ and $A_k \subseteq \bigcap_{i=1}^{k} A_i$. By def^{n} of set equality we conclude that $\bigcap_{i=1}^{k} A_i = A_k$ //

§2.4 #2c Use PMI to prove $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$

Proof: Let $P(n)$ be the proposition $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$.

Observe that $\sum_{i=1}^{1} 2^i = 2$ and $2^{1+1} - 2 = 2$ thus $P(1)$ is true.

Assume inductively that $P(n)$ is true. Consider,

$$\begin{aligned}
\sum_{i=1}^{n+1} 2^i &= 2^{n+1} + \sum_{i=1}^{n} 2^i \\
&= 2^{n+1} + 2^{n+1} - 2 \quad \text{by the induction hypothesis.} \\
&= 2(2^{n+1}) - 2 \\
&= 2^{(n+1)+1} - 2 \quad \therefore P(n) \Rightarrow P(n+1).
\end{aligned}$$

Thus by PMI we conclude $P(n)$ true $\forall n \in \mathbb{N}$. //

§2.4 #2g Prove $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Proof: Let $P(n)$ be the above proposition. Observe for $n=1$,

$$\frac{1}{2!} = 1 - \frac{1}{(1+1)!} = 1 - \frac{1}{2} = \frac{1}{2} \quad \therefore P(1) \text{ true.}$$

Assume inductively $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Add $\frac{n+1}{(n+1+1)!}$ to both sides of the induction hypothesis,

$$\begin{aligned}
\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} + \frac{n+1}{(n+1+1)!} &= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+1+1)!} \\
&= 1 - \frac{n+2}{(n+1)!(n+2)} + \frac{n+1}{(n+2)!} \\
&= 1 - \frac{n+2-n-1}{(n+2)!} \\
&= 1 - \frac{1}{((n+1)+1)!}.
\end{aligned}$$

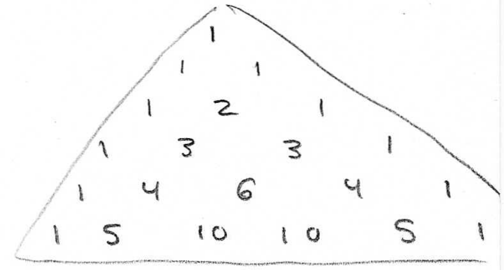
Thus $P(n) \Rightarrow P(n+1)$. We conclude $P(n)$ is true $\forall n \in \mathbb{N}$ by PMI.

§2.4 #8i) Let $P(n)$ be the proposition $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15}$ is an integer. Prove $P(n)$ is true $\forall n \in \mathbb{N}$.

Consider $P(1)$. Notice, $\frac{1}{3} + \frac{1}{5} + \frac{7}{15} = \frac{5+3+7}{15} = \frac{15}{15} = 1 \in \mathbb{Z}$.

Assume inductively that $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{Z}$. Consider,

$$\frac{(n+1)^3}{3} + \frac{(n+1)^5}{5} + \frac{7(n+1)}{15} =$$



$$= \frac{1}{3}(n^3 + 3n^2 + 3n + 1)$$
$$+ \frac{1}{5}(n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1)$$
$$+ \frac{7}{15}(n+1)$$

$$= \frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15}$$
$$+ n^2 + n + \frac{1}{3} + n^4 + 2n^3 + 2n^2 + n + \frac{1}{5} + \frac{7}{15}$$

$$= \underbrace{\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15}}_{\text{integer by induction hypothesis.}} + \underbrace{n^4 + 2n^3 + 3n^2 + 2n + 1}_{\text{an integer.}}$$

Thus $P(n) \Rightarrow P(n+1)$. Thus $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{Z}$ for all $n \in \mathbb{N}$ by the Principle of Mathematical Induction.

§2.4 #8j) Prove $4^n - 1$ is divisible by 3 $\forall n \in \mathbb{N}$

Proof: $n=1$ yields $4^1 - 1 = 3$ which is divisible by 3. Assume inductively $3 \mid (4^n - 1)$. Then $\exists k \in \mathbb{Z}$ such that $4^n - 1 = 3k$. Consider

$$4^{n+1} - 1 = 4\,4^n - 1 = 4(4^n - 1) + 3 = 4(3k) + 3$$

using induction hypothesis.

Thus, $4^{n+1} - 1 = 3(4k+1)$ which shows us $3 \mid (4^{n+1} - 1)$. Therefore, by PMI $4^n - 1$ is divisible by 3 $\forall n \in \mathbb{N}$. //

§2.4#8v) Prove by PMI: If $\overline{\overline{A}} = n$ then $\overline{\overline{\mathcal{P}(A)}} = 2^n$

Proof: Suppose $\overline{\overline{A}} = 1$ then w.l.o.g. we can denote $A = \{x\}$ thus $\mathcal{P}(A) = \{\phi, \{x\}\}$ and clearly $\overline{\overline{\mathcal{P}(A)}} = 2 = 2^1$.

Assume inductively that $\overline{\overline{A}} = n \Rightarrow \overline{\overline{\mathcal{P}(A)}} = 2^n$ for some $n > 1$. Let B be such that $\overline{\overline{B}} = n+1$. Since $n > 1$, $\exists x_o \in B$ and we can write $B = \{x_o\} \cup B_*$ where $B_* = B - \{x_o\}$. Clearly $\overline{\overline{B_*}} = (n+1) - 1 = n$ thus by the induction hypothesis $\overline{\overline{\mathcal{P}(B_*)}} = 2^n$. Let us study the power set of B,

$$\mathcal{P}(B) = \mathcal{P}(B_* \cup \{x_o\})$$

$$= \underbrace{\underbrace{\{\phi, U_1, U_2, \dots, U_{2^n-1}}_{\mathcal{P}(B_*) \text{ has } 2^n \text{ elements by induction hypothesis}}, U_1 \cup \{x_o\}, U_2 \cup \{x_o\}, \dots, U_{2^n-1} \cup \{x_o\}, \{x_o\}\}}_{\mathcal{P}(B) \text{ has } 2^n + 2^n \text{ elements again using induction hypothesis}}$$

Hence $\overline{\overline{\mathcal{P}(B)}} = 2^n + 2^n = 2^{n+1}$. Therefore, by PMI we find $\overline{\overline{A}} = n \Rightarrow \overline{\overline{\mathcal{P}(A)}} = 2^n$ for all $n \in \mathbb{N}$.

§2.4#6b) Define $\{n : n \in \mathbb{N}$ and $n > 10\}$ inductively (recursively)

Let Θ be the set of natural numbers defined by

(i.) $11 \in \Theta$

(ii.) If $n \in \Theta$ then $n+1 \in \Theta$.

You can verify $\Theta = \{n \mid n \in \mathbb{N} \text{ and } n > 10\}$. Frankly, I prefer a direct definition over a recursive one when possible.

§2.5 #1 Prove that every $n \in \mathbb{N}$ with $n > 3$ may be written as a linear combination of 2 and 5, that is $\exists x, y \in \mathbb{Z}$ such that $n = 2x + 5y$

Proof (Using PCI): Let $S = \{n \in \mathbb{N} \mid \exists x, y \in \mathbb{Z}$ such that $n = 2x + 5y, n > 3\}$

We desire to show $n-1 \in S \Rightarrow n \in S$. Suppose $n-1 \in S$, $n \geq 6$ then $\exists x, y \in \mathbb{Z}$ such that $n-1 = 2x + 5y$. Consider

$$
\begin{aligned}
n &= 1 + 2x + 5y \\
&= 2 - 1 + 2x + 5y \\
&= 2(x+1) + 5y - 1 \\
&= 2(x+1) + 5(y-1) + 4 \\
&= 2(x+3) + 5(y-1)
\end{aligned}
$$

Remark: I think I could use the same argument for a PMI proof

note $x+3, y-1 \in \mathbb{Z}$ thus n is linear combination of 2 and 5 over $\mathbb{Z}$, hence $n \in S$ for $n \geq 5$.

Also $4 = 2(2) + 5(0)$ thus $4 \in S$ and $5 = 2(0) + 5(1)$ $\therefore 4, 5 \in S$.
Thus $n-1 \in S \Rightarrow n \in S$ for $n \geq 5$. Thus the statement holds for $n \geq 4$ by PCI.

Remark: Notice your text seems to say PCI is for $S = \mathbb{N}$ yet in practice we can adjust the starting point for the induction set. I'm a bit annoyed with your text on this point. $S = \mathbb{N}$ in all the examples and I find no evidence that PCI is to be thought of as needed in the exercise above. There is not much to fix though,

$$S = \{n_o, n_o+1, n_o+2, \dots\} \longrightarrow \tilde{S} = \{1, 2, \dots, n_o-1, n_o, \dots\}$$

We can apply the text's PCI to $\tilde{S}$ where $\{1, 2, \dots, n_o-1\}$ are for free by def^{n} of $\tilde{S}$ then we'd have to check that $n_o \in S$ and $n-1 \in \tilde{S} \Rightarrow n \in \tilde{S}$ for $n > n_o$ to conclude $\tilde{S} = \mathbb{N}$ by PCI. This means PCI for $\mathbb{N}$ extends naturally to PCI for S above.

§2.5#6a Use PCI to prove the Fibonacci numbers $f_n \in \mathbb{N}$ for $n \in \mathbb{N}$

Proof: Let $S = \{m \mid f_m \in \mathbb{N}\}$. Recall $f_1 = 1$ and $f_2 = 1$ thus $1, 2 \in S$. Assume $m \in S$ for each $m \leq n-1$. That is assume inductively $f_1, f_2, \dots, f_{n-1} \in \mathbb{N}$. By definition,

$$f_n = f_{n-1} + f_{n-2}$$

By assumption $f_{n-1}, f_{n-2} \in \mathbb{N}$ and by properties of $\mathbb{N}$ $f_{n-1} + f_{n-2} \in \mathbb{N}$ hence $f_n \in \mathbb{N}$ which shows $n \in S$. Thus by PCI $f_n \in \mathbb{N}$ for $n \geq 1$.

Remark: I like this problem as an example for good use of PCI. Here we needed information about $(n-1)$ and $(n-2)$ in order to extract a conclusion about n. In "weak" induction (PMI) we only assume the hypothesis for n then try to extract information about $n+1$. #6a is natural to prove with PCI.

§2.5#7 (Assigned) Wish to use WOP to show $\sqrt{2}$ is irrational.

Hint: Let $A = \{b \in \mathbb{N} \mid \exists a \in \mathbb{N},\ a^2 = 2b^2\}$

$$\begin{aligned} A \neq \emptyset &\Leftrightarrow \exists a, b \in \mathbb{N} \text{ such that } a^2 = 2b^2 \\ &\Leftrightarrow \exists a, b \in \mathbb{N} \text{ such that } \left(\tfrac{a}{b}\right)^2 = 2 \\ &\Leftrightarrow \sqrt{2} \in \mathbb{Q} \end{aligned}$$

So we should suppose A is non-empty and work towards a contradiction.

(Notice $A \subset \mathbb{N}$ thus A has a smallest element by WOP)

Remark: Many WOP proofs use contradiction. Often the proof boils down to showing it is a contradiction for a least element to exist.