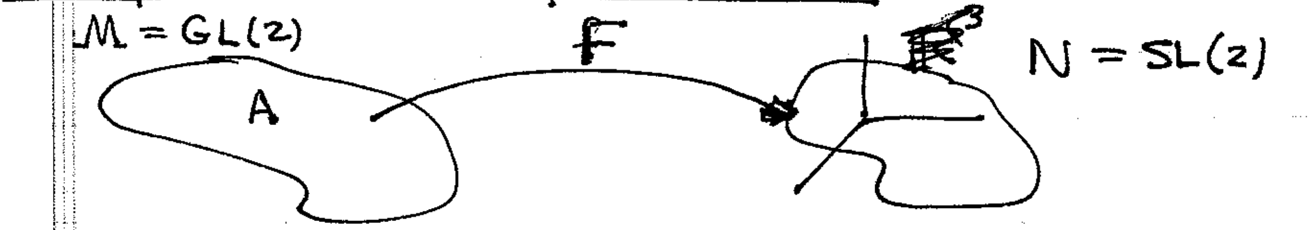
Example of abstract push-forward



$$F(A) = \frac{1}{\sqrt{|\det(A)|}} A^2 \qquad \text{clearly } \det(F(A)) = \frac{\det A}{\det A} =$$

$$F(A) = \frac{A^2}{\det(A)} \qquad \det(F(A)) = \frac{\det(A^2)}{(\det(A))^2} = 1.$$

$$\chi_M(A) = (x^{11}(A), x^{12}(A), x^{21}(A), x^{22}(A)) = (A_{11}, A_{12}, A_{21}, A_{22})$$

$$N = \{ B \in \mathbb{R}^{2\times 2} \mid \det(B) = 1 \}$$

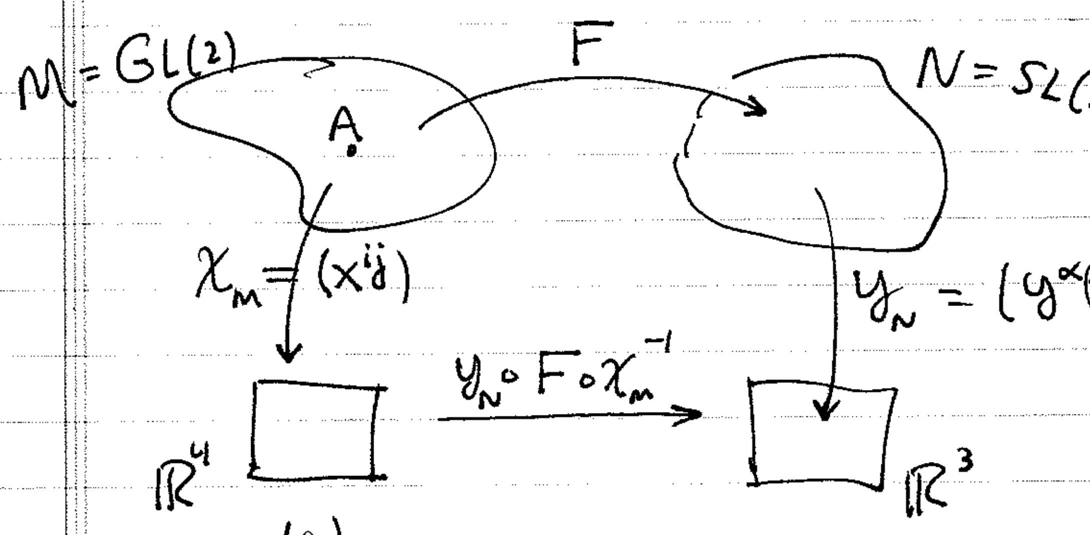
$$B_{11} B_{22} - B_{12} B_{21} = 1 \hookrightarrow B_{22} = \frac{1 + B_{12} B_{21}}{B_{11}}$$

Consider, $\chi_N(B) = (B_{11}, B_{12}, B_{21})$

for B with $B_{11} \neq 0$

$$\chi_N^{-1}(B_{11}, B_{12}, B_{21}) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & \frac{1 + B_{12} B_{21}}{B_{11}} \end{bmatrix}$$

$$\det(\chi^{-1}(\vec{B})) = B_{11}\left(\frac{1 + \cancel{B_{12} B_{21}}}{B_{11}}\right) - \cancel{B_{12} B_{21}} = 1.$$



(Assume A has $(A^2)_{11} \neq 0$ so y_N is right chart.)

$$y_N \circ F \circ \chi_M^{-1}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = y_N\left(F\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= y_N\left(\frac{1}{(ad-bc)^2}\begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & cb+d^2 \end{bmatrix}\right) = \frac{(a^2+bc, ab+bd, ac+dc)}{(ad-bc)^2}$$

$$F(x^{11}, x^{12}, x^{21}, x^{22}) = \left(\frac{x_{11}^2 + x_{12}x_{21}}{(x_{11}x_{22} - x_{12}x_{21})^2}, -, - \right)$$

$$d_A F\left(\frac{\partial}{\partial x^{11}}\right)(y^{11}) = \frac{\partial}{\partial x^{11}}\left[y^{11} \circ F \right]$$

$$= \frac{\partial}{\partial x^{11}}\left[\frac{x_{11}^2 + x_{12}x_{21}}{(x_{11}x_{22} - x_{12}x_{21})^2} \right]$$

$$= \frac{2x_{11}(x_{11}x_{22} - x_{12}x_{21})^2 + (x_{11}^2 + x_{12}x_{21})2x_{22}(\ )^{2}}{(x_{11}x_{22} - x_{12}x_{21})^4}$$

$$= \frac{2[x_{11}(x_{11}x_{22} - x_{12}x_{21})^2 + (x_{11}^2 + x_{12}x_{21})x_{22}]}{(x_{11}x_{22} - x_{12}x_{21})^4} \quad (A)$$

$$d_A F\left(\frac{\partial}{\partial x^{11}}\right)(y^{12}) = \frac{\partial}{\partial x^{11}}\left[y^{12} \circ F \right] \qquad F(x) = \left(-, \frac{x_{11}x_{12} + x_{12}x_{22}}{(\det x)^2} \right)$$

$$= \frac{\partial}{\partial x^{11}}\left[\frac{x_{11}x_{12} + x_{12}x_{22}}{(x_{11}x_{22} - x_{12}x_{21})^2} \right]$$

$$= \frac{x_{12}(x_{11}x_{22} - x_{12}x_{21})^2 + (x_{11}x_{12} + x_{12}x_{22})2x_{22}\det x}{\det(x)^4} \quad (B)$$

$$d_A F\left(\frac{\partial}{\partial x^{11}}\right)(y^{21}) = \frac{\partial}{\partial x^{11}}\left[\frac{x_{11}x_{21} + x_{21}x_{22}}{\det(x)^2} \right] \qquad \det x = x_{11}x_{22} - x_{12}x_{21}$$

$$= \frac{x_{21}(\det(x))^2 - (x_{11}x_{21} + x_{21}x_{22})2x_{22}\det(x)}{\det(x)^4}$$

$$= \frac{x_{21}\det(x) - 2x_{22}(x_{11}x_{21} + x_{21}x_{22})}{\det(x)^3}$$

$$= \frac{x_{21}(x_{11}x_{22} - x_{12}x_{21}) - 2x_{22}(x_{11}x_{21} + x_{21}x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^3}$$

$$= \frac{-x_{12}x_{21}^2 - x_{11}x_{22}x_{21} - 2x_{21}x_{22}^2}{(x_{11}x_{22} - x_{12}x_{21})^3} \quad (C)$$

$$\therefore \quad d_A F\left(\frac{\partial}{\partial x^{11}}\right) = A\frac{\partial}{\partial y^{11}} + B\frac{\partial}{\partial y^{12}} + C\frac{\partial}{\partial y^{21}}$$

$\underline{\text{Example}}$ $\quad F : GL(2) \longrightarrow \mathbb{R}^2$

$$F(A) = (\det(A),\ \operatorname{trace}(A))$$

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc,\ a + d)$$

$$F\begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix} = (x^{11}x^{22} - x^{12}x^{21},\ x^{11} + x^{22})$$

$$\begin{tikzcd} GL(2) \arrow[r, "F"] \arrow[d, "\chi = (x^{ij})"'] & \mathbb{R}^2 \arrow[d, "Id\ \ y = (y^1, y^2)"] \\ \mathbb{R}^4 \arrow[r, "F \circ \chi^{-1}"] & \mathbb{R}^2 \end{tikzcd}$$

$$d_A F(X_p)(g) = X_p(g \circ F)$$

(I) $d_A F\left(\frac{\partial}{\partial x^{11}}\right)(y^1) = \frac{\partial}{\partial x^{11}}\left[y^1 \circ F\begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix}\right]$

$= \frac{\partial}{\partial x^{11}}\left[y^1 (x^{11}x^{22} - x^{12}x^{21},\ x^{11} + x^{22})\right]$

$= \frac{\partial}{\partial x^{11}}\left[x^{11}x^{22} - x^{12}x^{21}\right]$

$= x^{22}$

(II) $d_A F\left(\frac{\partial}{\partial x^{11}}\right)(y^2) = \frac{\partial}{\partial x^{11}}\left(x^{11} + x^{22}\right) = 1.$

(III) $d_A F\left(\frac{\partial}{\partial x^{12}}\right)(y^1) = \frac{\partial}{\partial x^{12}}\left(x^{11}x^{22} - x^{12}x^{21}\right) = -x^{21}$ $\quad$ (V) $d_A F\left(\frac{\partial}{\partial x^{21}}\right)(y^1) = -x^{12}$

(IV) $d_A F\left(\frac{\partial}{\partial x^{12}}\right)(y^2) = \frac{\partial}{\partial x^{12}}\left(x^{11} + x^{22}\right) = 0$ $\quad$ (VI) $d_A F\left(\frac{\partial}{\partial x^{21}}\right)(y^2) = 0$

(VII) $d_A F\left(\frac{\partial}{\partial x^{22}}\right)(y^1) = \frac{\partial}{\partial x^{22}}\left[x^{11}x^{22} - x^{12}x^{21}\right] = x^{11}$, $\quad d_A F\left(\frac{\partial}{\partial x^{22}}\right)(y^2) = 1$ (VIII)

Collecting I, II, ..., VIII

I & II say $d_A F\left(\frac{\partial}{\partial x^{11}}\right) = X^{22}\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}$

III & IV say $d_A F\left(\frac{\partial}{\partial x^{12}}\right) = -X^{21}\frac{\partial}{\partial y^1}$

V & VI say $d_A F\left(\frac{\partial}{\partial x^{21}}\right) = -X^{12}\frac{\partial}{\partial y^1}$

VII & VIII say $d_A F\left(\frac{\partial}{\partial x^{22}}\right) = X^{11}\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}$

Thus in terms of the x, y coordinate bases on $T_A GL(2)$ and $T_{F(A)}\mathbb{R}^2$ we have

$$X_A = V_A = V^{11}\partial_{11} + V^{12}\partial_{12} + V^{21}\partial_{21} + V^{22}\partial_{22} \in T_A GL(2)$$

$$\begin{aligned} d_A F(V_A) &= d_A F\left(\sum_{i,j=1}^{2} V^{ij}(A)\, \partial_{ij}\big|_A\right) \\ &= \sum_{i,j=1}^{2} V^{ij}(A)\, d_A F\left(\partial_{ij}\big|_A\right) \\ &= V^{11}(A)\left[X^{22}(A)\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}\right] + V^{12}(A)\left[X^{21}\cancel{(A)}\frac{\partial}{\partial y^1}\right] \\ &\quad + V^{21}(A)\left[-X^{12}(A)\frac{\partial}{\partial y^1}\right] + V^{22}(A)\left[X^{11}(A)\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}\right] \\ &= \left(V^{11}A^{22} - V^{12}A^{21} - V^{21}A^{12} + V^{22}A^{11}\right)\partial/\partial y^1 \\ &\quad \left(V^{11} + V^{22}\right)\partial/\partial y^2 \end{aligned}$$

$$[d_A F]\begin{bmatrix} V^{11} \\ V^{12} \\ V^{21} \\ V^{22} \end{bmatrix} = \underbrace{\begin{bmatrix} A^{22} & -A^{21} & -A^{12} & A^{22} \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{2\times 4}\begin{bmatrix} V^{11} \\ V^{12} \\ V^{21} \\ V^{22} \end{bmatrix} = \begin{bmatrix} V^{11}A^{22} - V^{12}A^{21} - V^{21}A^{12} + V^{22}A^{11} \\ V^{11} + V^{22} \end{bmatrix}$$

$$[d_A F] = \left[\frac{\partial (y\circ F)}{\partial x}\right]$$