PROBLEM SET III SOLUTION : SPRING 2010

PROBLEM A27 Edwards 6.9 & 6.10 from pg. 128

6.9 Locate critical points of $f(x) = x^3(x-1)^4$ and use Th$^{\underline{m}}$ 6.3 of Edwards to classify the critical pts.

$$f'(x) = 3x^2(x-1)^4 + 4x^3(x-1)^3 = x^2(x-1)^3[3(x-1)+4x]$$

$$f'(x) = x^2(x-1)^3(7x-3).$$

$$\Rightarrow \quad x=0,\ x=1,\ x=3/7 \text{ critical pts.}$$

$$f''(x) = 2x(x-1)^3(7x-3) + 3x^2(x-1)^2\underset{(7x-3)}{\wedge} + x^2(x-1)^3(7)$$

$$= x(x-1)^2[2x(7x-3) + 3x(7x-3) + 7(x-1)]$$

$$= x(x-1)^2[14x^2 - 6x + 21x^2 - 9x + 7x - 7]$$

$$= x(x-1)^2[35x^2 - 8x - 7]$$

Note, $f''(0) = f''(1) = 0$ whereas $f''(3/7) = \frac{3}{7}\left(\frac{-4}{7}\right)^2\left[35\left(\frac{9}{49}\right) - \frac{24}{7} - 7\right]$

$$= \frac{48}{7^3}\left[\frac{315 - 7(24) - 49}{49}\right]$$

$$= \frac{48}{7^3}\left[\frac{315 - 7(31)}{49}\right]$$

$$= \frac{48}{7^3}\left[\frac{315 - 217}{49}\right] > 0$$

$\therefore$ $f(3/7)$ yields local minimum by 2^{nd} derivative test.

Continuing,

$$f'''(x) = (x-1)^2[35x^2 - 8x - 7] + 2x(x-1)[35x^2 - 8x - 7] + x(x-1)^2[70x - 8]$$

$$= (x-1)[(x-1) + 2x][35x^2 - 8x - 7] + (x-1)[x(x-1)[70x-8]]$$

$$= (x-1)[(3x-1)(35x^2 - 8x - 7) + (x^2 - x)(70x - 8)]$$

Clearly $f'''(1) = 0$ whereas $f'''(0) = -1[(-1)(-7) + 0] = -7 < 0$

Hence by Th$^{\underline{m}}$ 6.3 we find $f(0)$ is neither min nor max since the 1^{st} nonzero derivative is at odd power.

PROBLEM A27 Continued (6.9 continued from pg. 128)

$$f''''(x) = \left[(3x-1)(35x^2-8x-7)+(x^2-x)(70x-8)\right] + (x-1)\left[\cdots\cdots\right]$$

← since multiplied by $(x-1)$ I know these not important for ↙

$$\Rightarrow f''''(1) = \left[(3-1)(35-8-7)+(1-1)(70-8)\right] + \underline{0}$$

$$= 2(35-15)$$

$$= 2(20)$$

$$= 40 > 0 \quad \therefore \quad f(1) \text{ is local minimum}$$

by Th$^{\underline{m}}$ 6.3 (since even ~~power~~ order derivative is $1^{\underline{st}}$ nonzero for critical # 1.)

$\boxed{6.10}$ Let $f(x) = x\tan^{-1}(x) - \sin^2 x$.
Show $f(x) = \frac{7}{45}x^6 + R(x)$ where $R(x) \to 0$ as $x \to 0$.
Use that $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$ & $\sin^2(x) = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6$

$$f(x) = x\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right) - \left(x^2 - \frac{x^4}{3} + \frac{2}{45}x^6\right)$$

$$= \cancel{x^2} - \cancel{\frac{x^4}{3}} + \frac{x^6}{5} - \cancel{x^2} + \cancel{\frac{x^4}{3}} - \frac{2}{45}x^6$$

$$= \frac{9}{45}x^6 - \frac{2}{45}x^6$$

$$= \underline{\frac{7}{45}x^6}.$$

(the fact that $R(x) \to 0$ as $x \to 0$ follows from fact $\tan^{-1}(x)$ & $-\sin^2 x$ ~~only~~ have Taylor Series as given and all higher-order terms lead to terms order higher than 6.

Remark: I'm not sure what else Edwards was looking for here. Certainly there is more analysis we could do in order to explain why $\tan^{-1}(x) = x - x^3/3 + x^5/5$ etc... I assigned this because of the comment that

$$f(0) = f'(0) = f''(0) = f'''(0) = f''''(0) = f^{(5)}(0) = 0$$

I thought this was interesting; 5 derivatives with so little calculation.

PROBLEM A28 Continued

Edwards Problem 7.4 pg. 141) find 3^{rd} degree Taylor poly for the fnct. $f(x,y,z) = xy^2z^3$ at $(1,0,-1)$

$f = xy^2z^3$

$f_x = y^2z^3$

$f_y = 2xyz^3$

$f_z = 3xy^2z^2$

$f_{xx} = 0$

$f_{yy} = 2xz^3$

$f_{zz} = 6xy^2z$

$f_{xy} = 2yz^3$

$f_{xz} = 3y^2z^2$

$f_{yz} = 6xyz^2$

$f_{xxx} = f_{xxy} = f_{xxz} = 0$

$f_{yyy} = 0$

$f_{yyx} = 2z^3$

$f_{yyz} = 6xz^2$

$f_{xyz} = 6yz^2$

$f_{xzz} = 6y^2z$

$f_{yzz} = 12xyz$

$f_{zzz} = 6xy^2$

$f(1,0,-1) = 0$

$f_x(1,0,-1) = 0$

$f_y(1,0,-1) = 0$

$f_z(1,0,-1) = 0$

$f_{xx}(1,0,-1) = 0$

$f_{yy}(1,0,-1) = 2(1)(-1)^3 = -2$

$f_{zz}(1,0,-1) = 0$

$f_{xy}(1,0,-1) = 0$

$f_{xz}(1,0,-1) = 0$

$f_{yz}(1,0,-1) = 0$

$f_{xxx} = f_{xxy} = f_{xxz} = f_{yyy} = 0$ at $(1,0,-1)$.

$f_{yyx}(1,0,-1) = -2$

$f_{yyz}(1,0,-1) = 6$

$f_{xyz}(1,0,-1) = 6$

$f_{xzz}(1,0,-1) = 0$

$f_{yzz}(1,0,-1) = 0$

$f_{zzz}(1,0,-1) = 0$

$$f(x,y,z) = -y^2 - \frac{2\cdot 3}{6}(x-1)y^2 + \frac{6\cdot 3}{6}y^2(z+1) + \frac{6\cdot 3}{6}(x-1)y(z+1) + \cdots$$

$$\begin{aligned}
f(x,y,z) &= xy^2z^3 \\
&= [(x-1)+1][y^2][(z+1)-1]^3 \\
&= [y^2(x-1) + y^2][(z+1)^3 - 3(z+1)^2 + 3(z+1) - 1] \\
&= y^2(x-1)(z+1)^3 - 3(x-1)y^2(z+1)^2 + 3(x-1)y^2(z+1) - (x-1)y^2 + \\
&\quad + y^2(z+1)^3 - 3y^2(z+1)^2 + 3y^2(z+1) - y^2
\end{aligned}$$

Remark: sorry about this one.

$\boxed{\text{PROBLEM A28}}$ Edwards 7.3 & 7.4 from pg. 141

$\boxed{7.3}$ Let $f(x,y) = (x+y)^3$ find 3^{rd} degree Taylor poly. around $(0,0)$ & $(1,1)$

$$f(x,y) = \underbrace{x^3 + 3x^2y + 3xy^2 + y^3}_{\text{expansion around } (0,0).} \qquad \text{(just multiply it out)}$$

Next, calculate,

$$f_x = 3x^2 + 6xy + 3y^2 \qquad f_{xx} = 6x + 6y \qquad f_{xxx} = 6$$

$$f_y = 3x^2 + 6xy + 3y^2 \qquad f_{yy} = 6x + 6y \qquad f_{yyy} = 6$$

$$f_{xy} = 6x + 6y \qquad f_{xxy} = 6$$

$$f_{xyy} = 6$$

Evaluate at $(1,1)$ to obtain

$$f(1,1) = 8 \qquad f_{xx}(1,1) = 12$$

$$f_x(1,1) = 12 \qquad f_{yy}(1,1) = 12$$

$$f_y(1,1) = 12 \qquad f_{xy}(1,1) = 12$$

Hence, using $f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2} f_{xx}(a,b)(x-a)^2 + \dots$

$$\boxed{\begin{aligned} f(x,y) = 8 &+ 12(x-1) + 12(y-1) + 6(x-1)^2 + 12(x-1)(y-1) + 6(y-1)^2 + \\ &+ (x-1)^3 + 3(x-1)^2(y-1) + 3(x-1)(y-1)^2 + (y-1)^3 \end{aligned}}$$

Alternate Sol^{n}:

use the

1

1 1

1 2 1

1 3 3 1

of Pascal

$$(x+y)^3 = ((x-1) + (y-1) + 2)^3$$

$$= [(x-1)+(y-1)]^3 + 3[(x-1)+(y-1)]^2(2) + 3[(x-1)+(y-1)]4 + 8$$

$$= (x-1)^3 + 3(x-1)^2(y-1) + 3(x-1)(y-1)^2 + (y-1)^3$$

$$+ [3(x-1)^2 + 6(x-1)(y-1) + 3(y-1)^2](2) + 12(x-1) + 12(y-1) + 8$$

$$= 8 + 12(x-1) + 12(y-1) + 6(x-1)^2 + 12(x-1)(y-1) + 6(y-1)^2 +$$

$$+ 3(x-1)^2(y-1) + 3(x-1)(y-1)^2 + (x-1)^3 + (y-1)^3$$

PROBLEM 28 – II | Edwards # 7.10, 7.12 pg. 141

7.10 | Find and classify the critical points of $f(x,y) = (x^2+y^2)e^{x^2-y^2}$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$= \left\langle 2xe^{x^2-y^2} + (x^2+y^2)e^{x^2-y^2}(2x),\ 2ye^{x^2-y^2} + (x^2+y^2)e^{x^2-y^2}(-2y) \right\rangle$$

$$= \left\langle 2x(1+x^2+y^2)e^{x^2-y^2},\ 2y(1-x^2-y^2)e^{x^2-y^2} \right\rangle$$

Need $f_x = 0$ and $f_y = 0$ for critical point,

$$2x(1+x^2+y^2)e^{x^2-y^2} = 0 \quad \Rightarrow x = 0$$

$$2y(1-x^2-y^2)e^{x^2-y^2} = 0 \quad \Rightarrow \underline{y=0}. \text{ or } \underbrace{1-x^2-y^2=0}$$

note $x = 0$ from $f_x = 0$ thus
$1 - y^2 = 0$
$\Rightarrow (1-y)(1+y) = 0$
$\therefore \underline{y = \pm 1}$.

We must have $x = 0$ but y may take value $0, 1$ or -1.

$$\therefore \underbrace{(0,0),\ (0,1),\ (0,-1)}_{\text{critical points.}}$$

For $(0,0)$ | Notice we can expand exp using the known Maclaurin series expansion $e^u = 1 + u + \frac{1}{2}u^2 + \cdots$

$$f(x,y) = (x^2+y^2)\left(1 + x^2 - y^2 + \tfrac{1}{2}(x^2-y^2)^2 + \cdots\right)$$

$$= x^2 + x^4 - x^2y^2 + y^2 + y^2x^2 - y^4 + \cdots$$

$$= x^2 + y^2 + \text{higher than quadratic-type terms}$$

Thus $Q(x,y) = x^2 + y^2 = [x, y]\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$ and

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 = 0 \quad \therefore \lambda_1 = \lambda_2 = 1 \quad \therefore$$

$f(0,0)$ is local minimum of $z = f(x,y)$ since both e-values of the quadratic form of f at $(0,0)$ are positive.

For $(0,1)$ | Want to expand $f(x,y) = (x^2+y^2)e^{x^2-y^2}$ in powers of $(x-0) = x$ and $(y-1)$. ~~Ea~~ Easiest way I see to do this is as follows,

$$f(x,y) = (x^2+y^2)e^{x^2}e^{-y^2}$$

$$= \left(x^2 + (y-1+1)^2\right)e^{x^2}e^{-(y-1+1)^2}$$

$$= \left(x^2 + (y-1)^2 + 2(y-1) + 1\right)\left(1 + x^2 + \cdots\right)e^{-(y-1)^2-2(y-1)-1}$$

$$= \frac{1}{e}\left[x^2 + (y-1)^2 + 2(y-1) + 1\right]\left[1 + x^2 + \cdots\right]\left[1 - (y-1)^2 - 2(y-1) + \frac{1}{2}\left[(y-1)^2 + 2(y-1)\right]^2 + \cdots\right]$$

$$= \frac{1}{e}\left[x^2 + (y-1)^2 + 2(y-1) + 1 + x^2\right]\left[1 - (y-1)^2 - 2(y-1) + \frac{1}{2}\left[4(y-1)^2\right] + \cdots\right]$$

$$= \frac{1}{e}\left[1 + 2x^2 + 2(y-1) + (y-1)^2\right]\left[1 + (y-1)^2 - 2(y-1) + \cdots\right]$$

$$= \frac{1}{e}\left[1 + (y-1)^2 - \cancel{2(y-1)} + 2x^2 + \cancel{2(y-1)} - 4(y-1)^2 + (y-1)^2 + \cdots\right]$$

$$= \frac{1}{e}\left[1 + 2x^2 - 2(y-1)^2 + \cdots\right]$$

$$\Rightarrow f(0+h, 1+k) = \frac{1}{e} + \underbrace{2h^2 - 2k^2}_{Q(h,k)} + \cdots$$

$Q(h,k)$ has e-values $\lambda_1 = 2$ and $\lambda_1 = -2$

$\therefore$ neither max nor min

$z = f(0,1)$ is saddle point of $z = f(x,y)$.

Next $(0,-1)$ Want expand in x and $(y+1)$.
I'll use Taylor's Formula directly this time

$$f_{xx} = \frac{\partial}{\partial x}\left[\left(2x + 2x^3 + 2xy^2\right)e^{x^2-y^2}\right]$$

$$= \left[2(1+3x^2+2y^2) + \underbrace{(2x+2x^3+2xy^3)(2x)}_{x=0,\ \text{vanishes.}}\right]e^{x^2-y^2}$$

$$\Rightarrow f_{xx}(0,-1) = 2(1+2(-1)^2)e^{-1} = 2(3)e^{-1} = 6/e$$

$$f_{yy} = \frac{\partial}{\partial y}\left[(2y - 2yx^2 - 2y^3)e^{x^2-y^2}\right]$$

$$= (2 - 2x^2 - 6y^2)e^{x^2-y^2} + (2y - 2yx^2 - 2y^3)e^{x^2-y^2}(-2y)$$

$$\Rightarrow f_{yy}(0,-1) = \left[2 - 6 + 2\big(\cancel{-2+2}\big)^{0}\right]e^{-1} = -4e^{-1} = -4/e.$$

$$f_{xy} = \frac{\partial}{\partial y}\left[(2x + 2x^3 + 2xy^2)e^{x^2-y^2}\right]$$

$$= \cancel{\left[2 + 6x^2 + 2y^2 + (2x+2x^3+2xy^2)e^{x^2}\right.}$$

$$= \left[2x(2y)e^{x^2-y^2} + (2x+2x^3+2xy^2)e^{x^2-y^2}(-2y)\right]$$

$$\Rightarrow \underline{f_{xy}(0,-1) = 0.}$$

thus as $f(0,-1) = (0+1)e^{0-1} = \frac{1}{e}$ and $f(x,y) = f(a,b) + f_x(a,b)(x-a) + \cdots$

$$f(x,y) = \frac{1}{e} + \frac{3}{e}x^2 - \frac{2}{e}(y+1)^2 + \cdots$$

$$\Rightarrow f(h,-1+k) = \frac{1}{e} + \underbrace{\frac{3}{e}h^2 - \frac{2}{e}k^2}_{Q(h,k) = [h,k]\begin{bmatrix} 3/e & 0 \\ 0 & -2/e \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix}} + \cdots$$

$\Rightarrow z = f(0,-1)$ is saddle point of graph $z = f(x,y)$ since $\lambda_1 = 3/e$ & $\lambda_2 = -2/e$.

$\boxed{\text{PROBLEM A28-II}}$ Edwards # 7.12 pg. 141

Classify critical point $(0,0,0)$ for

$$f(x,y,z) = x^2+y^2+e^{xy} - y\tan^{-1}(x) + \sin^2(z)$$

$$= x^2+y^2+1+\cancel{xy}+\tfrac{1}{2}(xy)^2+\cdots$$

$$-y\left(\cancel{x}-\frac{x^3}{3}+\frac{x^5}{5}\right)+\cdots+z^2-\frac{z^4}{3}+\frac{2}{45}z^6+\cdots$$

using Edwards 7.5 pg. 141

$$= 1+\underbrace{x^2+y^2+z^2}_{\text{all the quadratic terms}}+\cdots$$

$$\Rightarrow \lambda_1=\lambda_2=\lambda_3=1 \quad \therefore \boxed{f(0,0,0) \text{ is local minimum for } f(x,y,z).}$$

$\boxed{\text{PROBLEM A29}}$ Edwards 8.3 p. 158

Use method of Ex. 2 to diagonalize the quadratic form...
Sketch graph of the eq$^{\underline{n}}$ $\underbrace{2x^2+5y^2+2z^2+2xz}_{Q(x,y,z)} = 1$.

$Q(x,y,z) = 1$ ← level surface.

Matrix of Q is,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{then} \quad Q(\vec{r}) = \vec{r}^{\,T} A \vec{r}.$$

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 5-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(5-\lambda)(2-\lambda)+1(-(5-\lambda)(1))$$

$$= (5-\lambda)\left[(2-\lambda)^2-1\right]$$

$$= -(\lambda-5)(\lambda^2-4\lambda+3)$$

$$= -(\lambda-5)(\lambda-3)(\lambda-1)$$

$$\underline{\lambda_1 = 5,\ \lambda_2 = 3,\ \lambda_3 = 1}.$$

PROBLEM A29 Continued

$\lambda_1 = 5$ | $A\vec{V}_1 = 5\vec{V}_1$ or $(A - 5I)\vec{V}_1 = 0$

$$\begin{bmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} -3u + w = 0 \\ u - 3w = 0 \end{matrix} \longrightarrow u = w = 0$$

but v free

$$\therefore \vec{V}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$\lambda_2 = 3$ | $(A - 3I)\vec{V}_2 = 0$,

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} u = w \\ v = 0 \end{matrix} \quad \therefore \vec{V}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$\lambda_3 = 1$ | $(A - I)\vec{V}_3 = 0$

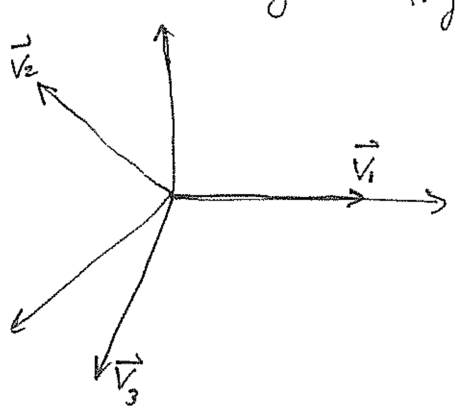
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} u = -w \\ 4v = 0 \end{matrix} \quad \therefore \vec{V}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Hence $P = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ has $P^T P = I$

since $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ forms an orthonormal set of vectors.

$$\begin{aligned} Q(\vec{r}) = \vec{r}^T A \vec{r} &= \vec{r}^T P^T P A P^T P \vec{r} \\ &= \vec{y}^T (P A P^T) \vec{y} \\ &= (\bar{x}, \bar{y}, \bar{z}) \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \\ &= 5\bar{x}^2 + 3\bar{y}^2 + \bar{z}^2 = 1 \quad \leftarrow \underline{\text{ellipsoid}}\ ! \end{aligned}$$

Principle axes of symmetry are $\vec{V}_1, \vec{V}_2, \vec{V}_3$



(I'm not going to ruin this sol$^{\underline{n}}$ with a sketch.)

PROBLEM A30

$\boxed{\text{pg. 194 \#3.1}}$ Show that $f(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ is locally invertible $\forall (x,y) \in \mathbb{R}^2$ except $(x,y) = (0,0)$. Compute $f^{-1}(x,y)$.

$$f'(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] & \frac{\partial}{\partial y}\left[\frac{x}{x^2+y^2}\right] \\ \frac{\partial}{\partial x}\left[\frac{y}{x^2+y^2}\right] & \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right] \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1(x^2+y^2)-2x^2}{(x^2+y^2)^2} & \frac{-2yx}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \end{bmatrix}$$

$$= \frac{1}{(x^2+y^2)^2}\begin{bmatrix} y^2-x^2 & -2xy \\ -2xy & x^2-y^2 \end{bmatrix}$$

$$\det(f'(x,y)) = \frac{1}{(x^2+y^2)^4}\left[(y^2-x^2)(x^2-y^2) - 4x^2y^2\right]$$

$$= \frac{1}{(x^2+y^2)^2}\left[\cancel{y^2x^2} - x^4 - y^4 - \cancel{x^2y^2} - 4x^2y^2\right]$$

$$= \frac{-1}{(x^2+y^2)^2}\left[x^4+4x^2y^2+y^4\right] \neq 0 \quad \forall (x,y)$$

However, it is divergent when $x=y=0$. Thus the inverse mapping theorem $\Rightarrow$ f^{-1} exists on some open set near each nonzero pt. in $\mathbb{R}^2$.

$$A = \frac{x}{x^2+y^2} \longrightarrow x^2+y^2 = \frac{x}{A}$$
$$B = \frac{y}{x^2+y^2} \longrightarrow x^2+y^2 = \frac{y}{B}$$
$$\longrightarrow \frac{x}{A} = \frac{y}{B} \longrightarrow y = \frac{B}{A}x \quad \text{(I)}$$

Then using (I) we find $B = \frac{y}{x^2+y^2} = \frac{\frac{B}{A}x}{x^2+\frac{B^2}{A^2}x^2} = \frac{ABx}{A^2x^2+B^2x^2}$

which simplifies to $B = \frac{AB}{(A^2+B^2)x} \Rightarrow x = \frac{A}{A^2+B^2}$ (II)

Finally, using (I) & (II) we obtain $y = \frac{B}{A^2+B^2}$.

Therefore, $\boxed{f^{-1}(A,B) = \left(\frac{A}{A^2+B^2}, \frac{B}{A^2+B^2}\right) \text{ for } (A,B) \in f(\mathbb{R}-\{(0,0)\})}$

Check: $f(f^{-1}(A,B)) = f\left(\frac{A}{A^2+B^2}, \frac{B}{A^2+B^2}\right) = \left(\frac{\frac{A}{A^2+B^2}}{\left(\frac{A}{A^2+B^2}\right)^2+\left(\frac{B}{A^2+B^2}\right)^2}, \frac{\frac{B}{A^2+B^2}}{\left(\frac{A}{A^2+B^2}\right)^2+\left(\frac{B}{A^2+B^2}\right)^2}\right)$

$$= \left(\frac{(A^2+B^2)A}{A^2+B^2}, \frac{(A^2+B^2)B}{A^2+B^2}\right) = (A,B).$$

[PROBLEM A31]

Problem 3.5 from pg. 194

Show that the equations

$$\sin(x+z) + \log(yz^2) = 0$$
$$e^{x+z} + yz = 0$$

implicitly define z near $z=-1$ as a function of (x,y) near $(1,1)$.

$$G(x,y,z) = (\underbrace{\sin(x+z)+\log(yz^2)}_{G_1}, \underbrace{e^{x+z}+yz}_{G_2})$$

$$\frac{\partial(G_1,G_2)}{\partial(x,y)} = \det\begin{bmatrix} G_{1x} & G_{1y} \\ G_{2x} & G_{2y}\end{bmatrix}$$

$$= \det\begin{bmatrix} \cos(x+z) & 1/y \\ e^{x+z} & z\end{bmatrix}$$

$$\Rightarrow \left.\frac{\partial(G_1,G_2)}{\partial(x,y)}\right|_{\substack{x=1\\y=1\\z=-1}} = \det\begin{bmatrix} \cos(0) & 1 \\ e^0 & -1\end{bmatrix} = -1-1 = -2 \neq 0$$

Therefore, by implicit mapping Th^{m} we can find $z=h(x,y)$ for some smooth function of (x,y) near $(1,1)$.

Problem 3.6 from pg. 194

Can the surface whose equation is

$$xy - y\log(z) + \sin(xz) = 0$$

be represented in the form $z=f(x,y)$ near $(0,2,1)$?

Let $G(x,y,z) = xy - y\log(z) + \sin(xz)$ and calculate

$$\frac{\partial G}{\partial z} = \frac{-y}{z} + x\cos(xz)$$

Notice $G_z(0,2,1) = \frac{-2}{1} + 0\cos(0) = -2 \neq 0$. Therefore, by the implicit fnct. theorem it follows $\exists f(x,y)$ such that $G(x,y,f(x,y)) = 0 \quad \forall (x,y)$ near $(0,2)$.

PROBLEM A32

problem 3.9 from pg. 195

Determine approximate sol^{n} of eq^{n}

$$z^3 + 3xyz^2 - 5x^2y^2z + 14 = 0$$

for z near $z=2$ as function of (x,y) near $(1,-1)$

Let $G(x,y,z) = z^3 + 3xyz^2 - 5x^2y^2z + 14$ and note
$G(1,-1,2) = 8 + 3(1)(-1)(4) - 5(1)(-1)^2(2) + 14 = 8 - 12 - 10 + 14 = 0.$
Next, calculate

$$\frac{\partial G}{\partial z} = 3z^2 + 6xyz - 5x^2y^2$$

$$\Rightarrow \frac{\partial G}{\partial z}(1,-1,2) = G_z(1,-1,2) = 3(4) + 6(-1)(2) - 5 = -5.$$

Use Theorem 1.5 for guidance here. We suppose (x,y) near $(1,-1)$

$$f_0(x,y) = 2$$

$$f_1(x,y) = f_0(x,y) - \frac{G(x,y,f_0(x,y))}{G_z(1,-1,2)} =$$

$$= 2 + \frac{1}{5}\left(G(x,y,2)\right)$$

$$= 2 + \frac{1}{5}\left(8 + 12xy - 5x^2y^2(2) + 14\right)$$

$$= 2 + \frac{22}{5} + \frac{12}{5}xy - 2x^2y^2$$

$$= \frac{32}{5} + \frac{12xy}{5} - 2x^2y^2$$

We could continue, $f_2(x,y) = f_1(x,y) + \frac{1}{5}G(x,y,f_1(x,y))$ etc...
but, I'll stop here,

$$\boxed{z \cong \frac{32}{5} + \frac{12}{5}xy - 2x^2y^2}$$

Solves $z^3 + 3xyz^2 - 5x^2y^2z + 14 = 0$ near $(1,-1,2)$.
As a check, $\frac{32}{5} + \frac{12}{5}(-1) - 2(1)(1) = \frac{32-12-10}{5} = \frac{10}{5} = 2.$

PROBLEM A33 Edwards problems 1.1 & 1.8 of p. 171

1.1 Show that $x^3 + xy + y^3 = 1$ can be solved for $y = f(x)$ in a neighborhood of $(1, 0)$.

Define $G(x,y) = x^3 + xy + y^3 - 1$ and note $G(x,y) = 0$ is equivalent to the given eq$^{\underline{n}}$. Moreover, note $G(1,0) = 1 + 0 + 0 - 1 = 0$ and

$$\frac{\partial G}{\partial y} = x + 3y^2 \implies \left.\frac{\partial G}{\partial y}\right|_{(1,0)} = 1 + 3(0)^2 = 1 \neq 0$$

Therefore, by the implicit fnct. Th$^{\underline{m}}$, we can find $f(x)$ such that $G(x, f(x)) = 0$ for x in some nbhd of $x = 1$. This means we've ~~solved our~~ shown $\exists$ a sol$^{\underline{n}}$ $y = f(x)$ to $x^3 + xy + y^3 = 1$ near $(1,0)$.

1.8 Show that $x + y - z + \cos(xyz) = 0$ can be solved for $z = f(x,y)$ near $(0,0)$.

Let $G(x,y,z) = x + y - z + \cos(xyz)$. Notice that if $x = y = 0$ then $z = 1$ solves $G(x,y,z) = 0$ since $G(0,0,1) = 0 + 0 - 1 + \cos(0) = -1 + 1 = 0$. Consider then that

$$G_z = -1 - \sin(xyz)\frac{\partial}{\partial z}(xyz)$$
$$= -1 - xy\sin(xyz) \implies G_z(0,0,1) = -1 \neq 0.$$

Thus, by the implicit fnct. Th$^{\underline{m}}$ $\exists f(x,y)$ defined near $(0,0)$ such that $G(x,y,f(x,y)) = 0$. In other words we've shown $\exists$ sol$^{\underline{n}}$ $z = f(x,y)$ to the given eq$^{\underline{n}}$ $x + y - z + \cos(xyz) = 0$ near $(0,0)$.

$\boxed{\text{PROBLEM 34}}$ Find coordinate chart for mobius strip. In other words find inverse for the patch given below:

$$\Phi(t,\lambda) = \left(\left[1+\tfrac{1}{2}\lambda\cos(t/2)\right]\cos t,\ \left[1+\tfrac{1}{2}\lambda\sin(\tfrac{t}{2})\right]\sin t,\ \tfrac{1}{2}\lambda\sin(\tfrac{t}{2}) \right)$$

for $(t,\lambda) \in [0,2\pi]\times[-1,1]$

<u>Would like to solve for λ & t.</u>

$$x = \left(1+\tfrac{\lambda}{2}\cos(\tfrac{t}{2})\right)\cos t$$

$$y = \left(1+\tfrac{\lambda}{2}\sin(\tfrac{t}{2})\right)\sin t \longrightarrow \underbrace{y = (1+z)\sin t}$$

$$\underbrace{z = \tfrac{\lambda}{2}\sin(\tfrac{t}{2})}$$

$$\sin t = \frac{y}{1+z}$$

$$\lambda = \frac{2z}{\sin(\frac{t}{2})} \qquad \therefore\ \underline{t = \sin^{-1}\left[\frac{y}{1+z}\right]}.$$

$$\Rightarrow \underline{\lambda = \frac{2z}{\sin\left(\frac{1}{2}\sin^{-1}\left(\frac{y}{1+z}\right)\right)}}.$$

$$\boxed{\Phi^{-1}(x,y,z) = \left(\sin^{-1}\left(\frac{y}{1+z}\right),\ \frac{2z}{\sin\left(\frac{1}{2}\sin^{-1}\left(\frac{y}{1+z}\right)\right)} \right)}$$

Check my algebra,

$$\Phi(\Phi^{-1}(x,y,z)) = \Phi\Big(\overbrace{\sin^{-1}\left(\tfrac{y}{1+z}\right)}^{t},\ \overbrace{2z/\left[\sin\left(\tfrac{1}{2}\sin^{-1}\left(\tfrac{y}{1+z}\right)\right)\right]}^{\lambda}\Big)$$

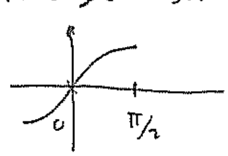
$$= \left(\left[1+\frac{1}{2}\frac{2z}{\sin\left(\frac{1}{2}\sin^{-1}\left(\frac{y}{1+z}\right)\right)}\right]\cdots,\ -,\ \ldots\right)$$

Oh, on second thought, I'll just check z-part,

$$\frac{1}{2}\lambda\sin\left(\frac{t}{2}\right) = \frac{1}{\cancel{2}}\frac{\cancel{2}z}{\cancel{\sin\left[\frac{1}{2}\sin^{-1}\left(\frac{y}{1+z}\right)\right]}}\cancel{\sin\left[\frac{1}{2}\sin^{-1}\left[\frac{y}{1+z}\right]\right]}$$

$$= z.$$

To give a complete answer I should explain what the domain Φ^{-1} includes. Since I used inverse sine, we ought to chop ~~$[0,\pi/2]$ from $[0,2$~~ $(0,2\pi)$ down to $[0,\pi/2)$ use



$$\underline{\mathrm{dom}(\Phi^{-1}) = \Phi\left([0,\pi/2]\times[-1,1]\right)}.$$

PROBLEM 35 Use Ex. 6.2.9 to construct a 3-dim'l manifold in $\mathbb{R}^3$

We noted $F(x,y,z) = (xyz, y, z)$ has

$$F'(x,y,z) = \begin{bmatrix} yz & xz & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $\det(F'(x,y,z)) = yz$ so if we choose $\text{dom}(F) = (0,\infty)^3$ then $\det(F'(x,y,z)) \neq 0$ $\forall (x,y,z) \in \text{dom}(F)$ hence F' is invertible at all points. We're told in Def$^{\underline{n}}$ 10.0.12 $\Phi_1 : V_1 \subseteq \mathbb{R}^n \to M \subseteq \mathbb{R}^m$ defines a manifold M provided Φ_1 is 1-1 and $\text{rank}(\Phi_1') = n$ at each point in $\text{dom}(\Phi_1)$. Since we only have one chart compatibility is trivially satisfied. We need only show $F = \Phi_1$ is 1-1 on $(0,\infty)^3$. Suppose $(x,y,z), (a,b,c) \in (0,\infty)^3$ and let

$$F(x,y,z) = F(a,b,c) \Rightarrow (xyz, y, z) = (abc, b, c)$$
$$\Rightarrow b = y,\ c = z \text{ and } xyz = abc$$
$$\Rightarrow (x,y,z) = (a,b,c) \quad \therefore F \text{ is 1-1 and } F((0,\infty)^3) \text{ is a manifold.}$$

PROBLEM 36 Find connected subset of $\mathbb{R}^3$ which can be viewed as a manifold with coordinate chart F from Ex. 6.2.10.

We had $F(x,y,z) = (x^2+z^2, yz)$. We hope to interpret F as the coordinate chart of a 2-dim'l manifold in $\mathbb{R}^3$. I only required $F: U \to \mathbb{R}^2$ have U connected and F be 1-1. Technically, I think I should also insist $\text{rank}(F') = 2$, but I won't complain if you didn't worry about that since my def$^{\underline{n}}$ is flawed.

$$F'(x,y,z) = \begin{bmatrix} 2x & 0 & 2z \\ 0 & z & y \end{bmatrix}$$

I explain in Ex 6.2.10 that $(0,0,0)$ is the only point where $\text{rank}(F'(x,y,z)) \neq 2$. Therefore, choose $U = \mathbb{R}^3 - \{(0,0,0)\}$ would work, except, we need F 1-1, thus use

$$U = \{(x,y,z) \mid y = 1,\ z > 0,\ x > 0\}$$

Then $F^{-1}(u,v) = (\sqrt{u-v^2}, 1, v)$ is easily seen to be inverse of F and $F: U \to \mathbb{R}^2$ defines coordinate chart on U.

$$\left(\begin{array}{c} x^2+z^2 = u \\ yz = v \\ x = \sqrt{u-v^2} \end{array} \right)$$