Don't even think about working these problems out on this page alone. The solutions should be written neatly on lined or unlined paper with the work clearly labeled. Do not omit scratch work. I need to see all steps. Thanks and enjoy. Notice that the A,B,C labels refer to the system described in the syllabus. I expect everyone to do the type-A problems however you may pick and choose from the type-B or C problems. This is due 4-29-2010 at start of lecture.

Problem A37 [35pts] Suppose $g = e^1 \otimes e^1 + e^2 \otimes e^2$ where $\{e^1, e^2\}$ form the standard dual basis on $(\mathbb{R}^2)^*$. Calculate $g(x, y)$ and identify this mystery tensor as a familiar friend that has been with us since day one in this course. (multilinear maps are tensors).

Problem A38 [35pts] Show that $Ae_1 \wedge Ae_2 \wedge Ae_3 = det(A)e_1 \wedge e_2 \wedge e_3$ reproduces the usual formula for the determinant of a $3x3$ matrix. For the sake of specificity and simplicity lets do this exercise for my favorite matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(this problem is just the $n = 3$ extension of Example 11.4.2)

Problem A39 [35pts] Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, z) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} x_i y_j z_k$$

Furthermore, define $g : \mathbb{R}^{3\times 3} \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ by

$$g(A) = (col_1(A), col_2(A), col_3(A))$$

Show that f is multilinear. My advice, use Σ notation, do not expand in terms of components one-by-one ! Finally, identify the mapping $f \circ g : \mathbb{R}^{3\times 3} \to \mathbb{R}$ in terms of a familiar mapping from matrices to numbers. **What is $f(g(A))$?**

Problem A40 [35pts] Let $\alpha = x^2 dx + 2zdy$ and $\beta = xdx \wedge dt + ydy \wedge dt + zdz \wedge dt$. Calculate $\alpha \wedge \beta$ and $\alpha \wedge \alpha$ and $\beta \wedge \beta$. Assume that x, y, z, t are independent Cartesian coordinates on $\mathbb{R}^4$.

Problem A41 [35pts] Suppose $f(x, y, z) = \phi$ where $z = (\sqrt{x^2 + y^2 + z^2}) \cos(\phi)$ implicitly defines the azmuthial angle ϕ. Calculate df.

Problem A42 [35pts] Let $Vol = dx \wedge dy \wedge dz$. Suppose $y = s \cos \beta$ and $z = s \sin \beta$. Calculate the form of the volume element Vol in the (x, s, β) coordinate system. *coordinate change is just a matter of substitution in our notation!*

Problem A43 [35pts] Show that $\Phi_{\vec{v}+f\vec{w}} = \Phi_{\vec{v}} + f\Phi_{\vec{w}}$ for vector fields $\vec{v}$ and $\vec{w}$ and a function f. Here Φ denotes the "flux-form" correspondance as I detail in the notes.

Problem A44 [35pts] Use Proposition 13.3.5 to prove ${}^*\omega_{\vec{v}} = \Phi_{\vec{v}}$. You may also assume linearity of the Hodge dual operation. Also, show that ${}^*\Phi_{\vec{v}} = \omega_{\vec{v}}$.

Problem A45 [35pts] Calculate the integral of $\alpha = dx \wedge dy$ over the upper half of the unit sphere oriented outwards.

Problem A46 [35pts] Calculate the integral of $\alpha = ydx \wedge dz + xdy \wedge dz$ over the upper half of the unit sphere oriented outwards.

Problem A47 [35pts] Explain what vector-calculus identities follow from $d(d\alpha) = 0$ and the correspondances $df = \omega_f$, $d\omega_{\vec{F}} = \Phi_{\nabla \times \vec{F}}$ and $d\Phi_{\vec{G}} = (\nabla \cdot \vec{G}) dx \wedge dy \wedge dz$. I did one of these in lecture.

Problem A48 [35pts] The Hodge dual operation on $\mathbb{R}^3$ allows us to intoduce another way to differentiate a p-form α:

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha = (-1)^{3p} * d * \alpha$$

where I have set $n = 3$ since I intend to use this **coderivative** δ on $\mathbb{R}^3$. Let $\alpha = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$ where a, b, c are smooth functions on $\mathbb{R}^3$. Calculate the formula for $\delta\alpha$.

Problem A49 [35pts] De Rahm, Hodge and others developed a theory to analyze closed vs. exact differential forms. See my notes for an example of how the shape of the domain can come into play. One interesting theorem Hodge proved was that if ω was any p-form on a Riemannian manifold then there exists a $(p-1)$-form α and a $(p+1)$-form β and a *harmonic form* γ such that

$$\omega = d\alpha + \delta\beta + \gamma.$$

In the special case $M = \mathbb{R}^3$ it is the case $\gamma = 0$. **Use the theorem due to Hodge to prove that any vector field can be written in terms of the gradient of a scalar function and the curl of some vector field; that is, for any vector field $\vec{F}$ there exists another vector field $\vec{G}$ and a function g such that $\vec{F} = \nabla g + \nabla \times \vec{G}$.** I think if you examine the case $\omega = \omega_{\vec{F}}$ then it ought to be about a line or two once you unravel the notation. I let Hodge do the really hard part. (you need to use Problem 48 to understand the coderivative part)

Problem B4 [100pts] Use differential forms on $\mathbb{R}^3$ to find an interesting identity from the Generalized Stokes Theorem for the following surface integral:

$$\int_{\partial V} \vec{A} \times \vec{B} \cdot d\vec{S}$$

(ingredients include: flux-form and work-form correspondances, the remark connecting the flux, and work form with the cross product and the product rule for exterior derivatives of wedge product)

Problem B5 [100pts] Surface area of a surface S in $\mathbb{R}^3$ can be calculated by a certain double integral involving the parametrization of S. In particular,

$$Area(S) = \iint_S \vec{N} \cdot d\vec{S}$$

where $\vec{N}$ is the unit-normal of S. Reformulate the area in terms of the integral of a particular differential form. (I think this will help make B6 easier to start)

Problem B6 [100pts] Show that the (surface) volume of the hypersphere $S_3 = \{(x,y,z,w) \mid x^2+y^2+z^2+w^2 = R^2\}$ is simply $2\pi^2 R^3$. To do this I would recommend you use generalized spherical coordinates as in my notes. Also, you'll need to propose a good definition of the surface volume of a hypervolume in $\mathbb{R}^4$.

Problem B7 [200pts] Define $\mathcal{C}_p^\infty$ be the set of all smooth functions whose range includes a neighborhood of $p \in \mathbb{R}^2$ and whose domain includes zero which maps to p. We **define** two curves in $\mathcal{C}_p^\infty$ to be **equivalent** iff

$$\gamma_1 \sim_p \gamma_2 \quad \Leftrightarrow \quad \gamma_1, \gamma_2 \in \mathcal{C}_p^\infty \text{ with } \gamma_1'(0) = \gamma_2'(0)$$

Given this definition prove:

1. $\sim_p$ is an equivalence relation.
2. Let $[\gamma]_p = \{\phi \mid \phi \sim_p \gamma\} \subseteq \mathcal{C}_p^\infty$. Define addition of equivalence classes by addition of their tangent vectors:
$$[\gamma_1]_p + [\gamma_2]_p = [\gamma]_p$$
where $\gamma'(0) = \gamma_1'(0) + \gamma'(0)$. Prove this operation is well defined and offer a similar definition for scalar multiplication.
3. If $T_p\mathbb{R}^2_{curves} = \{[\gamma]_p \mid \gamma \in \mathcal{C}_p^\infty\}$ then show this set is in 1-1 correspondance with the set of tangents at p defined in 11.5.1. In particular, find a bijection between $T_p\mathbb{R}^2_{curves}$ and $T_p\mathbb{R}^2$.

I have made the dimension $n = 2$ for convenience of notation. There is no technical necessity for this choice of dimension, you can just as well do the problem for $n \in \mathbb{N}$. The point of this problem is to build your intuition about tangent vectors, if you take a course in manifold theory later on this correspondance will likely be used often and sometimes without warning.

Problem B8 [100pts] Let the vector potential $\vec{A}(t,x,y,z) =< 0, 0, xtA_o >$ and the scalar potential $V(t,x,y,z) = V_o \tan^{-1}(tx)$. Calculate the Faraday tensor given these potentials. What are $\vec{B}$ and $\vec{E}$ for the given potentials?

Problem C7 [100pts] does Mathematica do form calculations?

Wildcard Problem [100pts] students choice, do something interesting.