Don't even think about working these problems out on this page alone. The solutions should be written neatly on lined or unlined paper with the work clearly labeled. Do not omit scratch work. I need to see all steps. Thanks and enjoy. Notice that the A,B,C labels refer to the system described in the syllabus. I expect everyone to do the type-A problems however you may pick and choose from the type-B or C problems. This is due 2-11-2010 at start of lecture.

Problem A1 [35pts] Let $v = <1,2,3>$ and $w = <c,4,5>$ where $c \in \mathbb{R}$,

1. calculate $v + w$ $= <1,2,3> + <c,4,5>$
   $= \boxed{<c+1, 6, 8>}$

2. calculate $v \cdot w$ $= c + 8 + 15$
   $= \boxed{c+23}$

3. find $||v||$ and $||w||$
   $$||v|| = \sqrt{1+4+9} = \boxed{\sqrt{14} = ||v||}$$
   $$||w|| = \sqrt{c^2+16+25} = \boxed{\sqrt{c^2+41} = ||w||}$$

4. what choice of c makes v orthogonal to w
   $$v \cdot w = c + 23 = 0 \Rightarrow \boxed{c = -23,}$$

5. what value of c makes the angle between v and w equal to θ
   $$v \cdot w = ||v|| \, ||w|| \cos\theta$$
   $$c + 23 = \sqrt{14(c^2+41)} \cos\theta$$
   $$(c+23)^2 = 14(c^2+41)\cos^2\theta$$
   $$c^2 + 46c + (23)^2 - 14\cos^2\theta \, c^2 - 14(41)\cos^2\theta = 0$$
   $$\boxed{\underbrace{(1-14\cos^2\theta)}_{\alpha} c^2 + \underbrace{46}_{\beta} c \underbrace{- 14(41)\cos^2\theta + 23^2}_{\gamma} = 0}$$
   $$\boxed{c = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}}$$

Problem A2 [35pts] Supply the implied domains and codomains for mappings with the formulas given below. Also, for (5),(6) and (7) find their formula.

1. $G(x, y, z) = (x^2, \underbrace{\sqrt{x-y}}, \frac{1}{z})$

need $x - y \geq 0$
and $z \neq 0$

$G : dom(G) \longrightarrow \mathbb{R}^3$

$dom(G) = \{(x,y,z) \mid x \geq y \text{ and } z \neq 0\}$

2. $L(x, y) = (x + y, x - y)$

$dom(L) = \mathbb{R}^2$ and $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

3. $f(x, y, z) = (x, z)$

$dom(f) = \mathbb{R}^3$ and $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$.

4. $T(x, y, z) = (x + y, z - y, x)$

$dom\, T = \mathbb{R}^3$ and $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

5. $G + T$

$dom(G+T) = dom(G)$.

from part 1. $G(x,y,z) = (x+y+x^2, z-y+\sqrt{x-y}, x+\frac{1}{z})$.

6. $f \circ \xi + L$ where $\xi(x, y) = (x, y, 0)$

$(f \circ \xi + L) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

$$
\begin{aligned}
(f \circ \xi + L)(x,y) &= f(\xi(x,y)) + L(x,y) \\
&= f(x,y,0) + L(x,y) \\
&= (x, 0) + (x+y, x-y) = (2x+y, x-y) = (f \circ \xi + L)(x,y)
\end{aligned}
$$

7. $L \circ f$

$$
\begin{aligned}
(L \circ f)(x,y,z) &= L(f(x,y,z)) \\
&= L(x, z) \\
&= (x+z, x-z).
\end{aligned}
$$

$L \circ f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$.

Problem A3 [35pts] State which mappings in the previous problem are linear. Then prove their linearity by find a matrix which induces the mapping; that is find a matrix A such that $F(x) = Ax$ for the mapping F.

① $$L(x,y) = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\substack{\text{matrix of } L:\mathbb{R}^2 \to \mathbb{R}^2 \\ \text{is a } 2\times 2 \text{ matrix}}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = (x+y,\ x-y).$$

② $$f(x,y,z) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{matrix of } f:\mathbb{R}^3 \to \mathbb{R}^2 \\ \text{is a } 2\times 3 \text{ matrix.}}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ z \end{bmatrix} = (x,z)$$

③ $$T(x,y,z) = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{\substack{\text{matrix of } T:\mathbb{R}^3 \to \mathbb{R}^3 \\ \text{is a } 3\times 3 \text{ matrix.}}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ -y+z \\ x \end{bmatrix} = (x+y,\ z-y,\ x)$$

④ $$(f\circ\xi + L)(x,y) = (2x+y,\ x-y) = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}}_{\substack{\text{matrix of } f\circ\xi + L:\mathbb{R}^2 \to \mathbb{R}^2 \\ \text{is a } 2\times 2 \text{ matrix.}}}\begin{bmatrix} x \\ y \end{bmatrix}$$

Note $\xi(x,y) = (x,y,0)$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underline{[f][\xi] + [L] = [f\circ\xi + L]}$$ nice.

⑤ $$(L\circ f)(x,y,z) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ x-z \end{bmatrix} = (x+z,\ x-z).$$

the other mappings are nonlinear.

Problem A4 [45pts] An unusual coordinate system on $\mathbb{R}^2$ is given by the formulas below:

$$x = r\cosh(\phi) \qquad y = r\sinh(\phi)$$

Invert these equations to find $r = r(x,y)$ and $\phi = \phi(x,y)$. Describe some subset of $\mathbb{R}^2$ for which (Φ is 1-1 (oops.)) the mapping $\Phi(r,\phi) = (r\cosh(\phi), r\sinh(\phi))$. Graph the coordinate curves $\phi = c_1$ and $r = c_2$ for a few values of c_1, c_2. Is this an orthogonal coordinate system? Is it curvelinear or rectilinear?

Recall $\cosh\phi = \frac{1}{2}(e^{\phi} + e^{-\phi})$ and $\sinh\phi = \frac{1}{2}(e^{\phi} - e^{-\phi})$ from which it follows $\cosh^2\phi - \sinh^2\phi = 1$. We observe that

$$x^2 - y^2 = r^2\cosh^2\phi - r^2\sinh^2\phi = r^2$$

Thus we find $\underline{r = \sqrt{x^2 - y^2}}$ provided $x^2 - y^2 \geq 0$.

or, we could solve for $\underline{r = -\sqrt{x^2+y^2}}$ again for $x^2 - y^2 \geq 0$.

Note $\cosh\phi \geq 1$ for all $\phi \in \mathbb{R}$ which leads to two cases

1.) $x = r\cosh\phi$ with $r = \sqrt{x^2 - y^2}$ (covers $x \geq 0$)

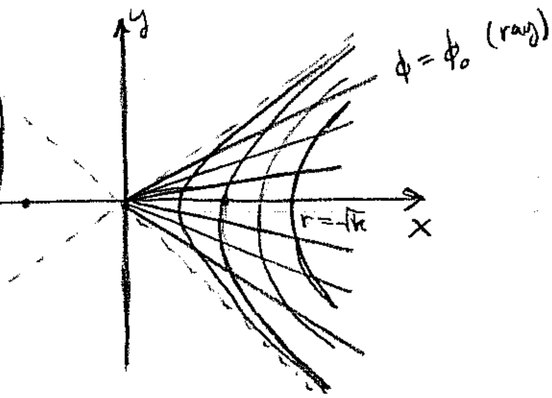
2.) $x = r\cosh\phi$ with $r = -\sqrt{x^2+y^2}$ (covers $x \leq 0$)

In both cases $y = r\sinh\phi$ covers all values for y since $\sinh(\mathbb{R}) = \mathbb{R}$. (in contrast $\cosh(\mathbb{R}) = [1,\infty)$). Solving for ϕ is not too hard here, if $x \neq 0$ then:

$$\frac{y}{x} = \frac{r\sinh\phi}{r\cosh\phi} = \tanh\phi \Rightarrow \underline{\phi = \tanh^{-1}(y/x)}$$

To be careful if $r = 0$ the formula above is also suspect, note $r = 0$ where $x^2 - y^2 = 0$ a.k.a $\underline{y = \pm x}$.

this is not an orthogonal or rectilinear coord. system. However, it is curvelinear.



$x^2 - y^2 = k > 0$

$x = \sqrt{k + y^2}$

$r = \sqrt{k}$

$\phi =$ constant

coordinate curves in xy-coord. we have

$r = \sqrt{k}$: $x = \sqrt{k+y^2}$

$\phi = \varphi_o$: $y = (\tanh\varphi_o)x$

$\Phi : (0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}^2$ is injective
moreover range $(\Phi) = \{(x,y) \mid x > 0,\ x^2 - y^2 > 0\}$
$\Phi^{-1}(x,y) = \left(\sqrt{x^2 - y^2},\ \tanh^{-1}(y/x)\right)$

(to prove not orthogonal could show $(\nabla r) \cdot (\nabla\phi) \neq 0$, true for $n = 2 \ldots$)

Problem A5 [35pts] Level curves can be understood in terms of inverse images. Describe the curves or surfaces given by the inverse images below: if it's not much trouble then illustrate your answer with a simple hand-drawn graph (or use Mathematica if you prefer). Assume that $a, b, c, d \in \mathbb{R}$.

1. let $F(x, y, z) = x^2 + y^2 + z^2$ what is $F^{-1}(\{R\})$ for $R > 0$. What is $F^{-1}(\{0\})$?

$$(x, y, z) \in F^{-1}(\{R\}) \Rightarrow \underbrace{x^2 + y^2 + z^2 = R}_{\text{sphere of radius } \sqrt{R} \text{ centered at } (0,0,0).}$$

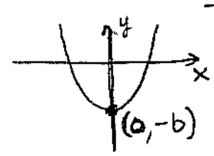
$F^{-1}(\{0\})$ is the set of sol^{n}s to $x^2+y^2+z^2=0$ a.k.a. $\{(0,0,0)\}$ the set containing the origin.

2. let $G(x, y, z) = ax + b\cancel{y} + cz + d$ what is $G^{-1}(\{0\})$?

$$(x, y, z) \in G^{-1}(\{0\}) \Rightarrow \underbrace{ax + by + cz + d = 0}_{\text{plane with normal vector } \langle a, b, c\rangle.}$$

3. let $H(x, y) = x^2 - y$ what is $H^{-1}(\{b\})$ for $b \in \mathbb{R}$?

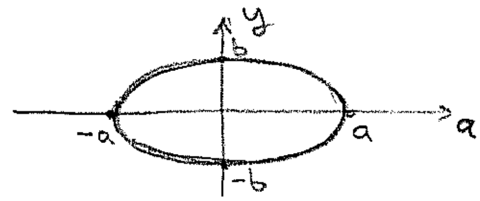
$$(x, y) \in H^{-1}(\{b\}) \Rightarrow x^2 - y = b \Rightarrow \underline{y = x^2 - b}$$



parabola with vertex $(0, -b)$ which opens upward.

4. let $f(x, y) = x^2/a^2 + y^2/b^2$ what is $f^{-1}(\{1\})$?

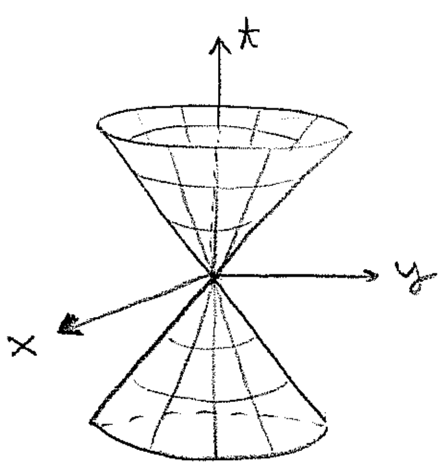
$$(x, y) \in f^{-1}(\{1\}) \Rightarrow \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}_{\text{ellipse with major/minor axes along } x \text{ or } y \text{ axis}}$$



(assuming $a, b > 0$)

5. let $\eta(t, x, y) = -t^2 + x^2 + y^2$ what is $\eta^{-1}(\{0\})$?

$$(t, x, y) \in \eta^{-1}(\{0\}) \Rightarrow -t^2 + x^2 + y^2 = 0$$
$$\Rightarrow t^2 = x^2 + y^2$$
$$\Rightarrow \underbrace{t = \pm\sqrt{x^2 + y^2}}_{\text{Cone}}$$



<u>Note:</u>

this is the light-cone from the Minkowski metric in $(2+1)$ spacetime. It's the set of <u>null</u>-vectors (non zero vectors with zero length)

$$\| \bar{v} \| = \sqrt{|\bar{v}^T [\eta] \bar{v}|} \leftarrow \text{"length" of relativistic vectors.}$$

matrix of η

Problem A6 [35pts] Give parametrizations of the curves, surfaces and volume described in the preceding problem. In each case clearly describe the domain of the mapping which defines the parametrization. Remember that $X : U \to \mathbb{R}^n$ gives a parametrization of $S \subseteq \mathbb{R}^n$ if X is a mostly one-one mapping such that $X(U) = S$.

1.) $\Sigma(\varphi, \theta) = \langle \sqrt{R}\cos\theta\sin\varphi, \sqrt{R}\sin\theta\sin\varphi, \sqrt{R}\cos\varphi \rangle$ for $\underbrace{0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi}_{\text{describes dom}(\Sigma)}$.

$(x^2+y^2+z^2 = R \Rightarrow \text{radius } \sqrt{R})$

2.) If $a \neq 0$ then can use y, z as parameters,

$$\Sigma(y, z) = \left\langle \frac{-d-by-cz}{a}, y, z \right\rangle \qquad \text{dom}(\Sigma) = \mathbb{R}^2$$

3.) Let $\vec{r}(x) = \langle x, x^2 - b \rangle$ for all $x \in \mathbb{R}$.

4.) Let $\vec{r}(\theta) = \langle a\cos\theta, b\sin\theta \rangle$ for $0 \leq \theta \leq 2\pi$.

5.) $\Sigma(\theta, \lambda) = \langle \lambda, \lambda\cos\theta, \lambda\sin\theta \rangle$ for $0 \leq \theta \leq 2\pi$, $\lambda \in \mathbb{R} - \{0\}$.

The reason 5.) is a parametrization of $-t^2+x^2+y^2 = 0$ is that $t = \lambda$, $x = \lambda\cos\theta$, $y = \lambda\sin\theta$ yields,

$$-\lambda^2 + (\lambda\cos\theta)^2 + (\lambda\sin\theta)^2 = -\lambda^2 + \lambda^2 = 0.$$

I threw out $\lambda = 0$ to avoid the singularity of the cone.

Remark: there are other correct answers. I just give an answer I can't give all answers especially here. Parametrizations will play an important role in integration of differential forms later in the course. For now the main importance is parametrized curves $t \longmapsto \phi(t)$ helps us understand geometry of many shapes.

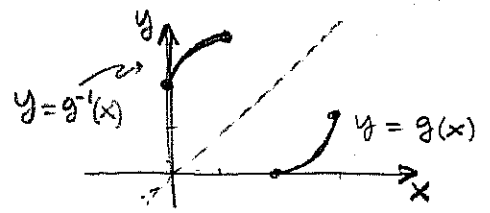
Problem A7 [70pts] Examples of finding inverse or implicit mappings. I hope these help illustrate the inverse and implicit mapping theorems which we will study later in the course. The tools you need to solve these are precalculus and an understanding of the definitions in Chapter 3.

1. suppose $f(x) = (x-2)^2$ for all $x \in \mathbb{R}$. Find a local inverse for f relative to some subset of $dom(f)$. Graph the local inverse and the function.

Let $g : [2,3] \longrightarrow \mathbb{R}$ be defined by $g(x) = (x-2)^2$.

Note $g(x) = (x-2)^2 = y \Rightarrow x = 2 + \sqrt{y}$ thus,

$$\boxed{g^{-1}(y) = 2 + \sqrt{y} \quad \text{for } y \in \text{range}(g) = [0,1].}$$



$g = f|_{[2,3]}$ ← restriction of f to $[2,3]$.

2. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $F(x,y,z) = (e^x, zy^2)$ for all $(x,y,z) \in \mathbb{R}^3$. Find a local inverse for F the given mapping. State the domain U for which $F|_U$ is invertible.

Need to restrict F to $U \subseteq \mathbb{R}^3$ such that $F|_U$ is 1-1. This means we need unique sol$^{\underline{n}}$'s to $F(x,y,z) = F(a,b,c)$ meaning

$$e^x = e^a \quad \text{and} \quad zy^2 = cb^2 \quad \Rightarrow x = a,\ y = b,\ z = c.$$

I fix $y = b = 1$. Let $U = \mathbb{R} \times \{1\} \times \mathbb{R}$.

$$F|_U(x,y,z) = F|_U(a,b,c) \Rightarrow e^x = e^a,\ zy^2 = cb^2 \text{ with } y = b = 1$$

Hence $x = a$, $y = b$, $z = c$.

Construct then,

$$\boxed{(F|_U)^{-1}(s,t) = (\ln(s),\ 1,\ t)}$$

Remark: there are many other choices for U.

3. Given that $x^2 + y^2 + z^2 = 1$ find $y = f(x,z)$ for some function f. State the $dom(f)$.

Solve for $y^2 = 1 - x^2 - z^2 \Rightarrow y = \pm\sqrt{1 - x^2 - z^2}$

We need $1 - x^2 - y^2 \geq 0$, define

$$\boxed{\begin{aligned} f(x,z) &= \sqrt{1 - x^2 - z^2} \\ dom(f) &= \overline{B_1(0,0)} = \{(x,z) \mid x^2 + y^2 \leq 1\} \end{aligned}}$$

4. Given that $x^2 + y + z = 1$ find $x = f(y,z)$ for some function f. State the $dom(f)$.

$$x^2 = 1 - y - z \Rightarrow x = \pm\sqrt{1-y-z}$$

Let $f(y,z) = \sqrt{1-y-z}$ with $dom\, f = \{(y,z) \mid \underbrace{1-y-z \geq 0}_{\text{a.k.a. } y+z \leq 1}\}$

5. Suppose $F(x,y,z) = (x^2+y^2, y+z)$. If $F(x,y,z) = (1,4)$ then find $x = g(z)$ and state $dom(g)$.

$$x^2 + y^2 = 1$$
$$y + z = 4 \Rightarrow y = 4 - z \Rightarrow x^2 + (4-z)^2 = 1$$

The algebra allows at least two choices,

$$x = \pm\sqrt{1-(4-z)^2}$$

Let $dom(g) = \{z \in \mathbb{R} \mid 1-(4-z)^2 \geq 0\}$ and define

$$\boxed{g(z) = \sqrt{1-(4-z)^2}}$$

Notice $\underline{dom(g) = [3,5]}$.

6. Suppose $F(x,y,z) = (x^2+y^2, y+z)$. If $F(x,y,z) = (1,4)$ then find $x = g(y)$ and state $dom(g)$.

$$x^2 + y^2 = 1 \Rightarrow x^2 = 1 - y^2 \Rightarrow x = \pm\sqrt{1-y^2}$$
$$y + z = 4$$

$$\Rightarrow \boxed{\begin{array}{l} x = g(y) \text{ where} \\ g(y) = \sqrt{1-y^2} \\ \text{and } dom(g) = [-1,1] \end{array}}$$

Problem A8 [35pts] It is at times useful to realize:

1. dot-product is a continuous mapping
2. cross-product is a continuous mapping.

Prove these assertions by utilizing the theorems in my notes. In other words, you can argue in the style of examples 3.2.20, 3.2.21, 3.2.22. You may find my ϵ_{ijk} notation useful for this problem. (if you want to try an epsilon-delta argument feel free, but I don't recommend it upon further thought)

Define $F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by $F(x,y) = x \cdot y = \sum_{j=1}^{n} x_j y_j$

Likewise define $G : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by the ϵ_{ijk} - symbol,

$G(x,y) = \sum_{i,j,k=1}^{3} x_i y_j \epsilon_{ijk} e_k$. The formulas for the dot and cross-products consist of finite linear combinations of the coordinate functions which we proved continuous previously. Therefore, F & G are continuous.

(in the case of $G = (G_1, G_2, G_3)$ each component function is continuous hence G is continuous.)

Problem A9 [45pts] Hint, choose $\delta = 1$ for the first part.

1. Give an epsilon-delta argument to prove contant mappings are continuous.
2. Give an epsilon-delta argument to prove $f(x) = \sqrt{x}$ is continuous for $x > 0$
3. Given $g(x)$ is continuous at x and $g(x) > 0$ prove $h(x) = \sqrt{g(x)}$ is continuous.

For part (3.) you could just prove that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$ then use the theorem from Edwards about the composite of continuous functions.

1.) Let $F: U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and suppose $F(z) = c_o \ \forall z \in U$.
Let $\epsilon > 0$ choose $\delta = 1$. Let $x \in U$ and note for $x_o \in U$
$|F(x) - c| = |c - c| = 0 < \epsilon$. Thus $0 < |x - x_o| < \delta$
$\Rightarrow |F(x) - c| < \epsilon \quad \therefore \lim_{x \to x_o} F(x) = c$. //

2.) Let $\epsilon > 0$ and $a > 0$. Let $\delta = \min(1, \epsilon(\sqrt{a} + \sqrt{a-1}))$ for $a > 1$
and if $a < 1$ then we choose $\delta = \min(\epsilon, 1)$ Suppose $x \in \mathbb{R}$ such that
$0 < |x - a| < \delta$ and consider,

$$|\sqrt{x} - \sqrt{a}| = \left| \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}} \right| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \frac{\delta}{\sqrt{x} + \sqrt{a}}$$

Next, note $\delta \leq 1$ thus $0 < |x - a| < \delta \leq 1$ yields
$-1 < x - a < 1 \Rightarrow a - 1 < x < a + 1$ consequently
either we have

1.) $a > 1$ thus $\sqrt{a-1} < \sqrt{x} \Rightarrow \sqrt{a-1} + \sqrt{a} < \sqrt{x} + \sqrt{a}$
and $|\sqrt{x} - \sqrt{a}| < \dfrac{\delta}{\sqrt{x} + \sqrt{a}} < \dfrac{\delta}{\sqrt{a-1} + \sqrt{a}} < \epsilon$.

2.) $a < 1$ then $|\sqrt{x} - \sqrt{a}| < \dfrac{\delta}{\sqrt{x} + \sqrt{a}} < \dfrac{\delta}{\sqrt{x} + 1} < \delta < \epsilon$

Thus in all possible cases we find
$0 < |x - a| < \delta \Longrightarrow |\sqrt{x} - \sqrt{a}| < \epsilon$
Therefore, $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ for all $a > 0$.

3.) h is the composite
of continuous functions and is therefore continuous.
(using 2. and a Th$^{\underline{m}}$)

Problem A10 [50pts] Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ and $f(a) \neq 0$.

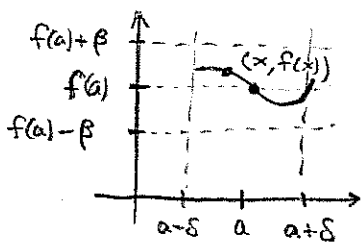
1. Give an epsilon-delta arguments that $h = 1/f$ is continuous at a.
2. Give an epsilon-delta argument that $g = f^2$ is continuous at a.
3. Use a result from the notes about the product function to prove that $g = f^2$ is continuous at a.

sol^n left to reader

1.) Consider that we may choose $x \in \mathbb{R}$ s.t. $|f(x) - f(a)| < \beta$.

$$\left| \frac{1}{f(x)} - \frac{1}{f(a)} \right| = \frac{|f(x) - f(a)|}{|f(x)||f(a)|} < \frac{\beta}{|f(x)||f(a)|}$$

We need to find M s.t. $1/|f(x)| < M$ for appropriately chosen x.



$f(x) > f(a) - \beta$

I simplified the argument by doing this part alone. The argument for $f(a) < 0$ follows same argument ≈

If $f(a) > 0$ then choose β small enough so that $\overbrace{f(a) - \beta > 0}^{f(a) > \beta}$ then $f(x) > 0$ and $\frac{1}{|f(x)|} < \frac{1}{f(a) - \beta}$, this condition gives

$$\left| \frac{1}{f(x)} - \frac{1}{f(a)} \right| < \frac{\beta}{|f(x)||f(a)|} < \frac{\beta}{(f(a) - \beta) f(a)}$$

We'd like $\epsilon = \frac{\beta}{(f(a) - \beta) f(a)} \Rightarrow \epsilon f(a)(f(a) - \beta) = \beta$

$$\Rightarrow \beta = \frac{\epsilon f(a)^2}{1 + \epsilon f(a)}$$

Suppose $f(a) > 0$. Let $\epsilon > 0$ note $\beta = \min\left(\frac{\epsilon f(a)^2}{1 + \epsilon f(a)}, \frac{f(a)}{1+\epsilon} \right)$ hence by continuity of f at a $\exists \delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \beta$. Also $\beta < \frac{f(a)}{1+\epsilon} < f(a)$ hence $f(a) - \beta > 0$. Consider then

$$\left| \frac{1}{f(x)} - \frac{1}{f(a)} \right| = \frac{|f(x) - f(a)|}{f(x) f(a)} < \frac{\beta}{(f(a) - \beta) f(a)} = \underbrace{\frac{\beta}{f(a)^2 - \beta f(a)} < \epsilon}_{*}.$$

* note $\beta < \frac{\epsilon f(a)^2}{1 + \epsilon f(a)} \Rightarrow \beta + \epsilon \beta f(a) < \epsilon f(a)^2$

$$\Rightarrow \beta < \epsilon f(a)(f(a) - \beta)$$

$$\Rightarrow \frac{\beta}{(f(a) - \beta) f(a)} < \epsilon.$$

Therefore, $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{f(a)}$. If $f(a) < 0$ the argument is similar. //

Problem B1 [100pts] orthogonal transformations. A **line $\mathcal{L}$ with direction** v is defined to be the set $\mathcal{L} = \{tv + b \mid t \in \mathbb{R}, \}$ where $v, b \in \mathbb{R}^n$ and $v \neq 0$. A **line segment $\mathcal{L}_{pq}$ from** $p \in \mathbb{R}^n$ **to** $q \in \mathbb{R}^n$ is defined to be the set $\mathcal{L}_{pq} = \{p + t(q-p) \mid t \in [0,1], \}$.

1. show that if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation such that $L(v) \neq 0$ then $L(\mathcal{L})$ is a line in $\mathbb{R}^m$.

$$L(\mathcal{L}) = \{L(v) \mid v \in \mathcal{L}\}$$
$$= \{A(tv+b) \mid t \in \mathbb{R}\} \quad \text{where } A \text{ is matrix of } L.$$
$$= \{tAv + Ab \mid t \in \mathbb{R}\}$$
$$= \{t\bar{v} + \bar{b} \mid t \in \mathbb{R}\} \quad \text{where } Av = \bar{v} \neq 0 \text{ since } L(v) = Av \neq 0$$

Finally note $Av = \bar{v} \in \mathbb{R}^m$ and $\bar{b} = Ab \in \mathbb{R}^m$ and we have shown $L(\mathcal{L})$ is precisely a line in $\mathbb{R}^m$ with direction vector $L(v) = \bar{v}$.

2. show that if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation such that $L(p-q) \neq 0$ then $L(\mathcal{L}_{pq})$ is a line segment from $L(p)$ to $L(q)$ in $\mathbb{R}^m$.

Almost the same calculation as part 1. Note

$$L(\mathcal{L}_{pq}) = \{L(v) \mid v \in \mathcal{L}_{pq}\}$$
$$= \{L(p + t(q-p)) \mid t \in [0,1]\}$$
$$= \{L(p) + L(t(q-p)) \mid t \in [0,1]\} \quad : \text{since } L \text{ linear transform}$$
$$= \{L(p) + tL(q-p) \mid t \in [0,1]\} \quad : \text{using homogeneity of } L.$$
$$= \underbrace{\{L(p) + t(L(q) - L(p)) \mid t \in [0,1]\}}_{\text{this is precisely } \mathcal{L}_{L(p)L(q)}} \text{ in } \mathbb{R}^m.$$

[Remark: the reference to the matrix A in my sol$^{\underline{n}}$ for 1 is not necessarily necessary.]

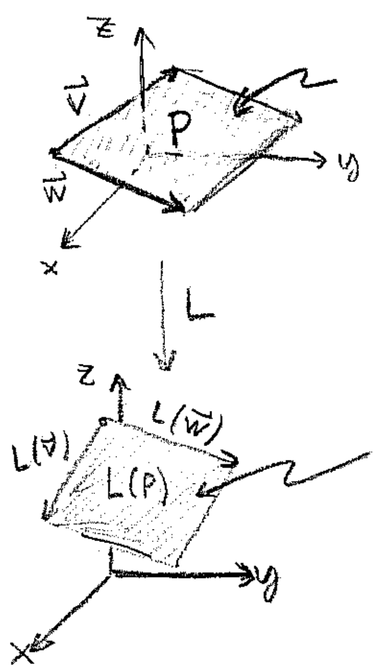
3. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is an injective linear transformation.**Show that if $P \subseteq \mathbb{R}^3$ is a paralellogram then $L(P)$ is a paralellogram and**

$$Area(L(P)) = |det(A)|Area(P).$$

Hint: the area of a paralellogram is given by $|v \times w|$ where v, w are vectors corresponding to the sides of P. ~~*Moreover, $det(A) = \sum_{ij} \epsilon_{ij} A_{1i} A_{2j}$. Two properties of the cross product are $x \times (v + w) = x \times v + x \times w$ and $x \times (cv) = c(x \times v)$. Repeated application of those properties justifies the calculation below:*~~

$$\cancel{x \times Aw = x \times (\sum_{ij} A_{ij} w_j e_i) = \sum_{ij} A_{ij} w_j (x \times e_i).}$$

Moreover, $e_i \times e_j = \sum_k \epsilon_{ijk} e_k$. ~~*All of this said, this is a 2×2 problem, if you are not comfortable with the ϵ-notation you could just work it out algebraically in terms of an arbitrary matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.*~~ (oops !)

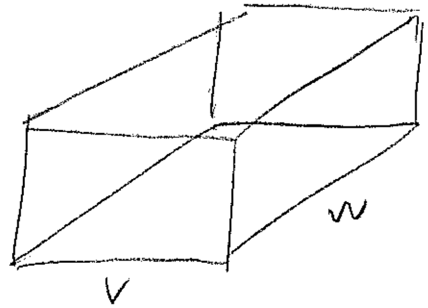


Area of $P = \| \vec{v} \times \vec{w} \|$

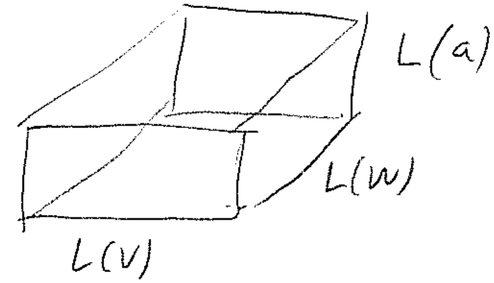
$$\text{Area of } L(P) = \| L(\vec{v}) \times L(\vec{w}) \|$$
$$= |\det(A)| \, \| \vec{v} \times \vec{w} \|$$
$$= |\det A| \, (\text{Area of } P).$$

jump !

<u>Remark</u>: there may be a simple way to tweak part 4. to get part 3. but it currently escapes me.



$a \in \operatorname{span}\{v, w\}^{\perp}$



$$\operatorname{vol}(L(R)) = \det\big(L(v) \,|\, L(w) \,|\, L(a)\big)$$

$$= \det(L)\, \det(v \,|\, w \,|\, a)$$

$\operatorname{area}(L(P)) \cdot \|L(a)\|$

$$= \det(L)\, |a|\, \operatorname{area} |P|$$

$$\operatorname{area}(L(P)) = \frac{|a|}{|L(a)|} \,|\det(L)|\, \operatorname{area}(P)$$

choose a s.t. $\underline{|L(a)| = |a|}$.

$$|L(v)| = \lambda |v|$$

$$\longmapsto w = \frac{v}{\lambda}$$

$$|L(w)| = \frac{1}{\lambda} |L(v)| = \frac{\lambda |v|}{\lambda} = |v|.$$

4. The volume of n-dimensional paralellogram can be defined in terms of the determinant. If the n-paralellogram is placed with a corner at the origin such that it has vectors $v_1, v_2, \ldots, v_n$ pointing along the sides then $vol(P) = |det[v_1|v_2|\cdots|v_n]|$. **Show that if $L : \mathbb{R}^n \to \mathbb{R}^n$ is an injective linear transformation with standard matrix A then $vol(L(P)) = |det(A)|vol(P)$.**

Notice that in the case of a box B with sides le_1, we_2, he_3 we can easily recover the handy-dandy formula $vol(B) = det(le_1|we_2|he_3) = lwhdet(e_1|e_2|e_3) = lwhdet(I) = lwh$.

$$
\begin{aligned}
\mathrm{Vol}\,(L(P)) &= \left| \det [\, AV_1 \,|\, AV_2 \,|\cdots|\, AV_n \,] \right| \\
&= \left| \det A\, [V_1|V_2|\cdots|V_n] \right| \\
&= \left| \det A \det [V_1|V_2|\cdots|V_n] \right| \\
&= |\det A|\, |\det [V_1|V_2|\cdots|V_n]| \\
&= |\det A|\, \mathrm{vol}\,(P).
\end{aligned}
$$

: since $L(P)$ has sides $L(V_1), L(V_2)$ etc... and we have $L(V_j) = AV_j$.

By $(*)$

Lemma $(*)$: Let $A \in \mathbb{R}^{n\times n}$ and $V_1, V_2, \ldots, V_n \in \mathbb{R}^n$ then

$$[AV_1|AV_2|\cdots|AV_n] = A[V_1|V_2|\cdots|V_n].$$

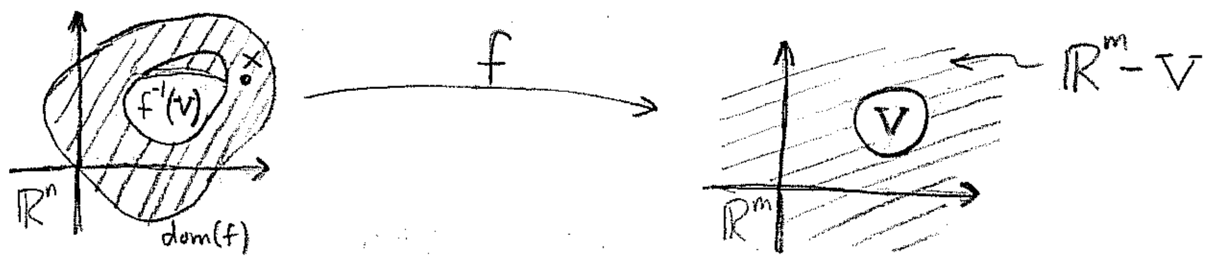
Proof: to show matrices equal we need to show ij-components match for LHS and RHS,

$$
\begin{aligned}
[AV_1|AV_2|\cdots|AV_n]_{ij} &= (AV_j)_i \quad \leftarrow \text{ i-component of the j^{th} column AV_j} \\
&= \Big(\sum_{k,l} A_{kl}(V_j)_l e_k\Big)_i \\
&= \sum_l A_{il}(V_j)_l \\
&= \sum_l A_{il}\, [V_1|V_2|\cdots|V_n]_{lj} \\
&= (A[V_1|V_2|\cdots|V_n])_{ij}
\end{aligned}
$$

the matrix identity follows.

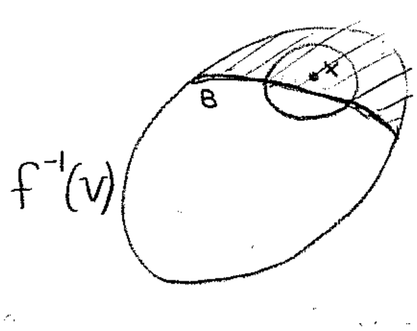
Problem B2 [100pts] Prove the closed case of Theorem 3.3.1, Prove that $\overline{B_1(0)} \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$

We wish to prove the inverse image of closed sets is closed. We defined $V \subseteq \mathbb{R}^n$ to be closed iff $\mathbb{R} - V$ was open. I proved the inverse image of open sets is open in the notes. Suppose $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and let V be some closed set in $\mathbb{R}^m$. It follows that $\mathbb{R}^m - V$ is open in $\mathbb{R}^m$. Moreover, by my notes proof of open case Thm 3.3.1, $f^{-1}(\mathbb{R}^m - V)$ is open in $\mathbb{R}^n$. We wish to show $f^{-1}(V)$ is closed, this means we need to show $\mathbb{R}^n - f^{-1}(V) \equiv W$ is open.



I've hatched $\mathbb{R}^m - V$ and $f^{-1}(\mathbb{R}^m - V)$. Notice we might have some nonempty overlap of $f^{-1}(V)$ and $f^{-1}(\mathbb{R}^m - V)$, call it S

Let $x \in \mathbb{R}^n - f^{-1}(V)$ then $x \in f^{-1}(\mathbb{R}^m - V)$ hence $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subset f^{-1}(\mathbb{R}^m - V)$. If $B_\epsilon(x) \cap S = \phi$ then we're done, otherwice choose $\delta = \frac{1}{2} \min\{\|x - y\| \mid y \in B\}$



(where B = boundary of $f^{-1}(V)$)

then $B_\delta(x) \subset f^{-1}(\mathbb{R}^m - V)$.

Thus, every point in $\mathbb{R}^n - f^{-1}(V) = W$ has an open ball contained within W. //

Remark: I think there's a small gap in my logic. Can you find it?

Remark: I have not shown $\overline{B_1(0)} \approx \mathbb{R}^n$.