

- Bertlmann
- Nakahara
- David Bleeker also relevant

PRINCIPAL BUNDLES, CONNECTIONS, CHERN CLASSES,
 ATIYAH-SINGER THEOREM, QUANTUM FIELD THEORETIC
 ANOMALIES

THE SCHEDULE BELOW IS ONLY POSSIBLE IF MANY OF THE RESULTS ARE STATED WITHOUT PROOF. WE WILL GIVE PROOFS WHEN THEY ARE CRUCIAL TO UNDERSTANDING THE RESULT AND SKETCHES OF PROOFS IN OTHER CASES WHICH ARE DEEMED IMPORTANT

1. We will develop notation and review certain properties of differential forms not used in previous courses. Pages 42,43,49-51,56,57 of Bertlmann.
2. There will be a quick review of principal fiber bundles and associated bundles along with examples of both.
3. Basic properties of connections on both principal bundles and vector bundles will be discussed following chapter 10 of Nakahara's book "Geometry, Topology, and Physics".
4. We will construct the Chern classes and discuss their applications to physics. This will include almost all of chapter 11 of Nakahara's book.
5. Our discussion of spin bundles will be selected from Nakahara pages 326,420-423.
6. There will be a short discussion of classical field theory with examples from gauge field theory, Dirac's theory of spin fields and their interactions. This will use much of the the material outlined above. I will then describe quantum versions of these fields by introducing n-point functions and moment generating functions of n-point functions.
7. Discussion of the moment generating function of the Lagrangian for noninteracting Dirac fermions. Additionally we will consider the moment generating function of the Lagrangian for Dirac fermions interacting with arbitrary gauge fields, pages 154-156 Bertlmann.

Differential forms $\alpha \in \Sigma^p(M)$ an p -form $\beta \in \Sigma^l(M)$ an l -form

$$\alpha = \frac{1}{p!} \alpha_{\mu_1 \mu_2 \dots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}$$

Where the sum is over all μ_i 's, technically not a basis and the $p!$ is to fix the over counting.

$$\beta = \frac{1}{l!} \beta_{\nu_1 \nu_2 \dots \nu_l} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_l}$$

The Wedge Product then: (Bertlmann will drop the "1"s)

$$\alpha \wedge \beta = \frac{1}{p!} \frac{1}{l!} \alpha_{\mu_1 \mu_2 \dots \mu_p} \beta_{\nu_1 \nu_2 \dots \nu_l} (dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_l})$$

We can express the wedge in terms of:

$$(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) \wedge (dx^{\nu_1} \wedge \dots \wedge dx^{\nu_l}) = \sum_{\lambda_1, \lambda_2, \dots, \lambda_{l+p}} \tilde{\epsilon}^{\mu_1 \mu_2 \dots \mu_p \nu_1 \nu_2 \dots \nu_l} (dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_{l+p}})$$

$$\sum_{\lambda_1, \lambda_2, \dots, \lambda_{l+p}} \tilde{\epsilon}^{\mu_1 \mu_2 \dots \mu_p \nu_1 \nu_2 \dots \nu_l} = \begin{cases} 0 & \text{if } \mu_i = \mu_j, \nu_i = \nu_j \text{ or } \mu_i = \nu_j \\ 1 & \text{if } \mu_1 \mu_2 \dots \mu_p \nu_1 \dots \nu_l \text{ is an even permutation of } \lambda_1, \lambda_2, \dots, \lambda_{l+p} \\ -1 & \text{if } \mu_1 \dots \mu_p \nu_1 \dots \nu_l \text{ is an odd permutation of } \lambda_1, \dots, \lambda_{l+p} \end{cases}$$

$$\sum_{\lambda_1, \lambda_2, \dots, \lambda_{l+p}} \tilde{\epsilon}^{\mu_1 \mu_2 \dots \mu_p \nu_1 \nu_2 \dots \nu_l} = \sum_{123 \dots (p+l)} \tilde{\epsilon}^{\mu_1 \mu_2 \dots \mu_p \nu_1 \dots \nu_l}$$

Properties of Differential Forms the Exterior Algebra

8/16/05 (3)

$$(\alpha + \beta) \wedge \gamma = (\alpha \wedge \gamma) + (\beta \wedge \gamma)$$

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$$

$$\alpha \wedge \beta = (-1)^{pq} (\beta \wedge \alpha) \quad \text{where } \alpha \in \Omega^p, \beta \in \Omega^q$$

Exterior Derivative

$$\alpha = \frac{1}{p!} \alpha_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}$$

$$d\alpha = \frac{1}{p!} \frac{\partial}{\partial x^\nu} (\alpha_{\mu_1 \mu_2 \dots \mu_p}) dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

Or skew-symmetrizing to fit the prescription of forms with skew-sym. coefficients

$$d\alpha = \frac{1}{(p+1)!} \partial_{[\nu} \alpha_{\mu_1 \mu_2 \dots \mu_p]} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

Properties of Exterior Derivative

$$d(\alpha + c\beta) = d\alpha + c d\beta$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$$

$$d^2 \alpha = 0$$

(Maslov Sign Convention
sign from deg
of push part)

Invariant formulation of form acting on vector fields,

8/18/05 (4)

α a 1-form and X, Y vector fields

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

Generalizing a bit α a p -form and X_i a vector field

$$d\alpha(X_1, X_2, \dots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i(\alpha(X_1, \dots, \hat{X}_i, \dots, X_{p+1})) + \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1})$$

$X(M)$ are vector fields on M .

$\alpha \in \text{alt}(X(M), C^\infty(M)) \leftarrow$ Module

$(\text{alt}(X(M), C^\infty(M)), d) \leftarrow$ Lie Algebra Cohomology.

Interior Products

$$i_{\mathcal{X}} = \text{"}\mathcal{X}\text{"} \leftarrow \text{Norris}$$

8/18/05 (5)

If \mathcal{X} is a vector field then $i_{\mathcal{X}}$ maps P -forms to $(P-1)$ -forms and is defined by

$$(i_{\mathcal{X}} \alpha)(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{P-1}) \equiv \alpha(\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{P-1})$$

Where $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{P-1}$ are vector fields and $P > 1$

$$i_{\mathcal{X}} \alpha \equiv \alpha(\mathcal{X}) \quad (P=1)$$

$$i_{\mathcal{X}} f \equiv 0 \quad (P=0)$$

Examples

$$(1) \quad \alpha = \alpha_{\mu} dx^{\mu} \quad \& \quad \mathcal{X} = \mathcal{X}^{\nu} \frac{\partial}{\partial x^{\nu}}$$

$$i_{\mathcal{X}} \alpha = \mathcal{X}^{\mu} \alpha_{\mu} \in C^{\infty}(M)$$

$$(2) \quad \alpha = \frac{1}{2} \alpha_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$$

$$\mathcal{X} = \mathcal{X}^{\lambda} \frac{\partial}{\partial x^{\lambda}}$$

$$\begin{aligned} i_{\mathcal{X}} \alpha &= \frac{1}{2} \alpha_{\mu\nu} i_{\mathcal{X}} (dx^{\mu} \wedge dx^{\nu}) \\ &= \frac{1}{2} \alpha_{\mu\nu} (dx^{\mu} \lrcorner dx^{\nu}) (\mathcal{X}, \cdot) \\ &= \frac{1}{2} \alpha_{\mu\nu} (\mathcal{X}^{\mu} dx^{\nu} - \mathcal{X}^{\nu} dx^{\mu}) \\ &= \frac{1}{2} \alpha_{\mu\nu} \mathcal{X}^{\mu} dx^{\nu} - \frac{1}{2} \alpha_{\mu\nu} \mathcal{X}^{\nu} dx^{\mu} \\ &= \alpha_{\mu\nu} \mathcal{X}^{\mu} dx^{\nu} \end{aligned}$$

\therefore Flip indices & use skewness.

Properties of interior product i_X

8/18/05 (6)

$$i_X(\alpha + \beta) = i_X\alpha + i_X\beta$$

hmm... what's the name of i_X ?

$$i_X(f\alpha) = f i_X\alpha$$

$$i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge i_X\beta$$

$$i_X^2 = 0$$

Metric

Let g be a metric on M , then g_x is a metric on $T_x M$ for all $x \in M$. Let $E = T_x M$ (elector)

$$g_x : T_x M \times T_x M \rightarrow \mathbb{R}$$

- symmetric
- nondegenerate
- bilinear

When g is defined on M we assume \mathcal{F} covering of M with charts (U, χ) such that the matrix $(g_{\mu\nu})$

$$g_{\alpha\beta} \equiv g(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

or we assume that g_x is positive definite, where

$$e_\alpha = f_\alpha^M \frac{\partial}{\partial x^M}$$

$$g(e_\alpha, e_\beta) = f_\alpha^M f_\beta^N g_{MN} = \eta_{\alpha\beta}$$

conditions on our manifold

Towards the Hodge Dual:

8/18/05 (7)

A metric g on M can be extended to p -forms.

(1) \tilde{g} is defined on $\Omega^p(M)$ (not quite a metric)

$$\tilde{g}(\alpha, \beta) = \frac{1}{p!} \sum g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \dots g^{\mu_p \nu_p} \alpha_{\mu_1 \mu_2 \dots \mu_p} \beta_{\nu_1 \nu_2 \dots \nu_p}$$

$$(g^{\mu\nu}) \equiv (g_{\mu\nu})^{-1}$$

$$\tilde{g}: \Omega^p(M) \times \Omega^p(M) \longrightarrow C^\infty(M).$$

- Remember: we assume our manifold supports an Euclidean or Lorentzian Metric. Meaning in nbhd of each point \exists vectorfields $\{e_a\}$ so that $g(e_a, e_b) = \eta_{ab}$

Defⁿ/ An orientation on a manifold M (with $\dim(M) = m$) is an everywhere non vanishing m -form. Call it $\mu \in \Omega^m(M)$. If μ is such a form then in a chart (X^i)

$$\mu = \mu_{12\dots m} (dx^1 \wedge dx^2 \wedge \dots \wedge dx^m)$$

We can see that $\mu_{12\dots m} > 0$ or $\mu_{12\dots m} < 0$. Moreover its clear we can make it positive, if its not to begin simply change $x_i \rightarrow \tilde{x}_i = -x_i$, and that will make $\tilde{\mu}_{12\dots m} > 0$.

$$\mu = (-\mu_{1\dots m}) (d(-x^1) \wedge dx^2 \wedge \dots \wedge dx^m)$$

If we cover M with charts (U_α, X_α) such that

$$\mu = \mu_{12\dots m}^\alpha dx_\alpha^1 \wedge \dots \wedge dx_\alpha^m$$

with $\mu_{12\dots m}^\alpha > 0$ then $\det[J(X_\beta \circ X_\alpha^{-1})] > 0$ for all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$.

Given a metric g and an orientation μ we say they are compatible iff $\forall (U_\alpha, X_\alpha)$ covering M as above

$$\mu = \sqrt{|\det g_{\mu\nu}^\alpha|} (dx_\alpha^1 \wedge \dots \wedge dx_\alpha^m)$$

$$g_{\mu\nu}^\alpha = g\left(\frac{\partial}{\partial x_\alpha^\mu}, \frac{\partial}{\partial x_\alpha^\nu}\right)$$

Th^m/Let g be a metric on a manifold M

and $\mu_g \in \Omega^m(M)$ a compatible orientation. Then

at each $x \in M \exists$ a unique linear isomorphism

$$*: \Omega^k(M) \longrightarrow \Omega^{m-k}(M)$$

such that for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{m-k}(M)$

$$\alpha \wedge (*\beta) = \tilde{g}(\alpha, \beta) \mu_g$$

Blecker's
defⁿ

Where \tilde{g} is given by

$$\begin{aligned} \tilde{g}(\alpha, \beta) &= \alpha_{\mu_1 \mu_2 \dots \mu_k} \beta^{\mu_1 \mu_2 \dots \mu_k} \\ &= \alpha_{\mu_1 \mu_2 \dots \mu_k} \beta^{\nu_1 \nu_2 \dots \nu_k} g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \dots g^{\mu_k \nu_k} \end{aligned}$$

Its the usual induced metric on k -forms.

This isn't quite Bertmann or Nakahara's take on this

Write (Bertmann's definition)

$$*(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) = \frac{1}{(m-p)!} \sum_{\mu_{p+1} \dots \mu_m} \epsilon^{\mu_1 \mu_2 \dots \mu_p \mu_{p+1} \dots \mu_m} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_m}$$

$$\sum_{\mu_{p+1} \dots \mu_m} \epsilon^{\mu_1 \mu_2 \dots \mu_p \mu_{p+1} \dots \mu_m} \equiv g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p} \epsilon_{\nu_1 \dots \nu_p \mu_{p+1} \dots \mu_m}$$

$$\underline{\epsilon}_{123 \dots m} = \tilde{\epsilon}_{12 \dots m} \sqrt{|\det(g_{\mu\nu})|}$$

scales with
the metric.

Levi-Civita Symbol

Hodge Dual of a P-form

8/23/05

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$$\omega = \frac{1}{p!} \omega_{\mu_1 \mu_2 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

$$*\omega = \frac{1}{p!} \frac{1}{(m-p)!} \omega^{\mu_1 \mu_2 \dots \mu_p} \epsilon_{\mu_1 \mu_2 \dots \mu_p \mu_{p+1} \dots \mu_m} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_m}$$

Th^y/ If g is a metric on M and ω is a p -form then

$$**\omega = (-1)^{p(m-p)} \omega \quad \text{when } g \text{ is Euclidean}$$

$$**\omega = (-1)^{p(m-p)+1} \omega \quad \text{when } g \text{ is Minkowskian}$$

Example

$$*(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) = \frac{1}{(m-p)!} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p} \epsilon_{\nu_1 \dots \nu_p \mu_{p+1} \dots \mu_m} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_m}$$

Minkowski

$$(x^0, x^1, x^2, x^3), \quad m=4$$

$$*1 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$*dx^0 = \frac{1}{(4-1)!} g^{0\nu} \epsilon_{\nu\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \quad g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$= \frac{1}{3!} \epsilon_{0\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

$$= \frac{1}{3!} 3! dx^1 \wedge dx^2 \wedge dx^3$$

$$= dx^1 \wedge dx^2 \wedge dx^3$$

$$*dx^2 = \frac{1}{3!} g^{2\gamma} \epsilon_{\gamma\alpha\beta\tau} dx^\alpha \wedge dx^\beta \wedge dx^\tau \quad (\text{wedges implicit})$$

$$= \frac{-1}{3!} \epsilon_{2\alpha\beta\tau} dx^\alpha \wedge dx^\beta \wedge dx^\tau$$

$$= \frac{-1}{3!} \cdot \epsilon_{2013} dx^0 \wedge dx^1 \wedge dx^3 \cdot 3!$$

$$= -dx^0 \wedge dx^1 \wedge dx^3$$

$$\begin{aligned}
 *(dx^2 \wedge dx^3) &= \frac{1}{2!} g^{2\mu} g^{3\nu} \epsilon_{23\alpha\beta} dx^\alpha \wedge dx^\beta \\
 &= \frac{1}{2!} (-1)(-1) \epsilon_{23\alpha\beta} dx^\alpha \wedge dx^\beta \\
 &= \frac{1}{2!} (-1)(-1) \epsilon_{2301} \cdot 2! dx^0 \wedge dx^1 \\
 &= \underline{\underline{dx^0 \wedge dx^1}}
 \end{aligned}$$

$$*(dx^0 \wedge dx^1 \wedge dx^3) = -dx^2$$

and so on...

ω — connection on principal bundle

Ω — curvature of connection

M^4 A — connection

F — field strength

$*F$ also 2-form

$\int F \wedge *F \implies$ Yang Mills Lagrangian

$$F = \frac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad *(F_{\mu\nu}) = \begin{pmatrix} E \leftrightarrow B \end{pmatrix}$$

$$dF = 0 \quad \text{and} \quad d*F = j = * \underset{\substack{\uparrow \\ \text{3-form}}}{J} \underset{\substack{\uparrow \\ \text{1-form}}}{J}$$

Define a metric on the space $\Sigma^p(M)$ of p -forms by

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∞ dim'l vector space
finitely generated
module.

$$(\alpha, \beta) \equiv \int_M [\alpha \wedge (*\beta)]$$

for $\alpha, \beta \in \Sigma^p(M)$. (Assuming α and β are suitably chosen so that the integral is sensible). Note that $*\beta$ is an $m-p$ form so $\alpha \wedge (*\beta)$ is an m -form

$$\alpha \wedge (*\beta) = f \nu_g$$

Its just some function times the volume-form. Lets expand out $\alpha \wedge (*\beta)$ to see what it really is,

$$\begin{aligned} \alpha \wedge (*\beta) &= \frac{1}{p!} \frac{1}{p!} \alpha_{\mu_1 \dots \mu_p} \beta_{\nu_1 \dots \nu_p} [dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} * (dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p})] \\ &= \frac{1}{p!} \frac{1}{p!} \alpha_{\vec{\mu}} \beta_{\vec{\nu}} [dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge \frac{1}{(m-p)!} \epsilon^{\nu_1 \dots \nu_p}_{\nu_{p+1} \dots \nu_m} dx^{\nu_{p+1}} \wedge \dots \wedge dx^{\nu_m}] \\ &= \frac{1}{(m-p)!} \frac{1}{p!} \frac{1}{p!} \alpha_{\vec{\mu}} \beta_{\vec{\nu}} \epsilon^{\nu_1 \dots \nu_p}_{\nu_{p+1} \dots \nu_m} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge dx^{\nu_{p+1}} \wedge \dots \wedge dx^{\nu_m} \\ &= \frac{1}{(m-p)!} \frac{1}{p!} \frac{1}{p!} \alpha_{\vec{\mu}} \beta_{\vec{\nu}} \epsilon^{\nu_1 \dots \nu_p}_{\nu_{p+1} \dots \nu_m} \epsilon^{\mu_1 \dots \mu_p \nu_{p+1} \dots \nu_m} dx^1 \wedge dx^2 \wedge \dots \wedge dx^m \\ &= \frac{1}{p!} \alpha_{\mu_1 \mu_2 \dots \mu_p} \beta^{\mu_1 \mu_2 \dots \mu_p} \sqrt{|\det(g_{\mu\nu})|} dx^1 \wedge dx^2 \wedge \dots \wedge dx^m \end{aligned}$$

$$\alpha \wedge (*\beta) = \frac{1}{p!} \alpha_{\mu_1 \dots \mu_p} \beta^{\mu_1 \dots \mu_p} \nu_g = \tilde{g}(\alpha, \beta) \nu_g$$

Now its fairly clear that $(\alpha, \beta) = (\beta, \alpha)$, it follows from symmetry of $\tilde{g}(\alpha, \beta)$. Likewise (α, β) inherits the properties of \tilde{g} which itself inherits the props of g (Riemannian or Lorentzian)

Using $\beta_{\nu_1 \dots \nu_p} \epsilon^{\nu_1 \nu_2 \dots \nu_p}_{\nu_{p+1} \dots \nu_m} \epsilon^{\mu_1 \mu_2 \dots \mu_p \nu_{p+1} \dots \nu_m} = \sqrt{|g|} p! (m-p)! \beta^{\mu_1 \dots \mu_p}$

Codifferential

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$$\delta = (-1)^{mp+m+1} *d*$$

$$\Omega^p \rightarrow \Omega^{m-p} \rightarrow \Omega^{m-p+1} \rightarrow \Omega^{m-(m-p+1)} \rightarrow \Omega^{p-1}$$

Its a codifferential, its not hard to see $\delta^2 = 0$, And the Laplace Beltrami Operator

$$\boxed{\Delta = \delta d + d\delta}$$

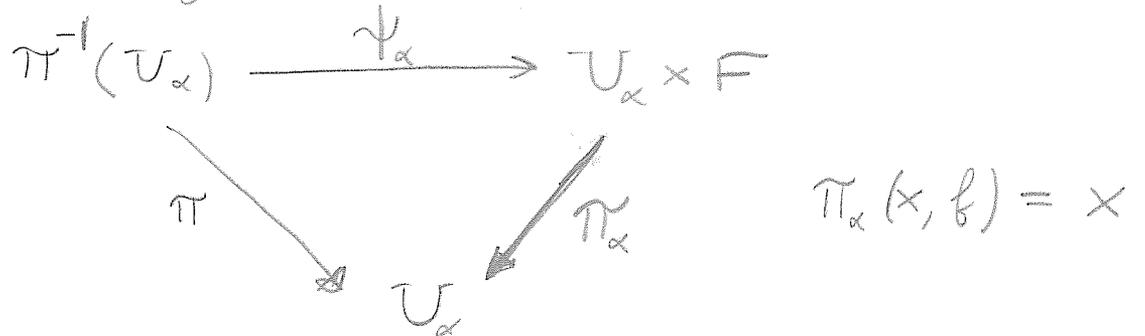
This is every where.

To say that (E, M, F, π) is a fiber bundle means

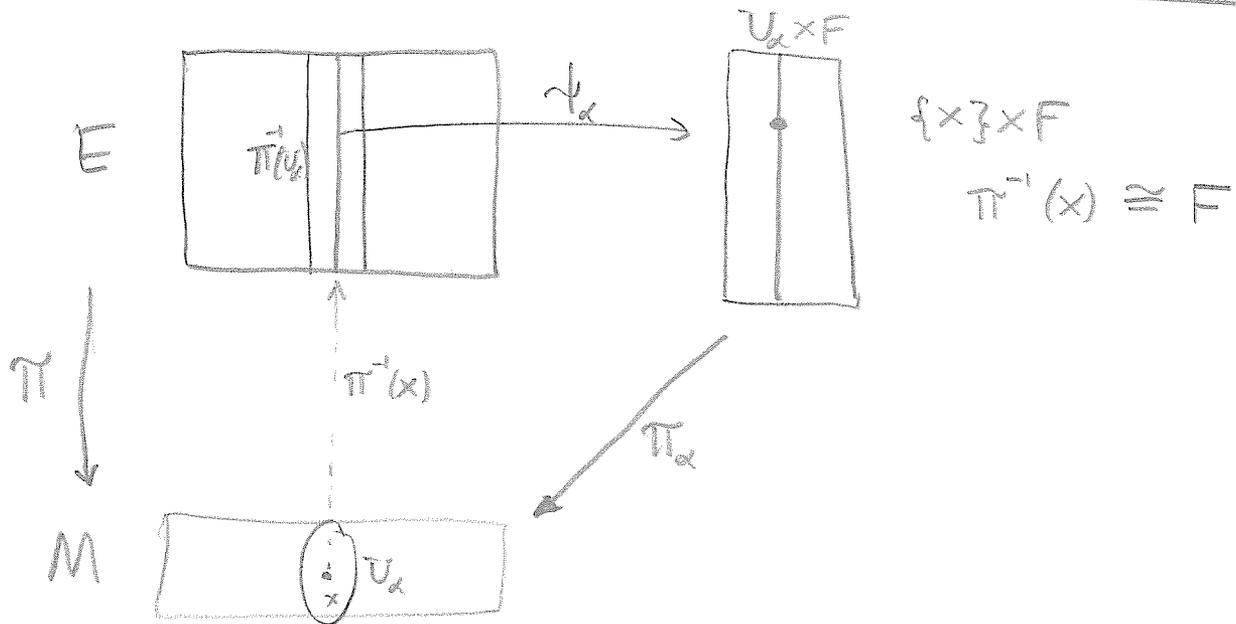
- (1.) E, M, F are manifolds. E is the bundle space
 M is the base space
 F is the fiber (standard)

(2.) $\pi: E \rightarrow M$ is a smooth surjection

(3.) \exists open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of M and maps $\psi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ called ^{local} trivializing maps such that $\forall \alpha$ the map ψ_α is a diffeomorphism and the diagram commutes



Pictorial Rep. of Fiber Bundle, locally a cartesian product



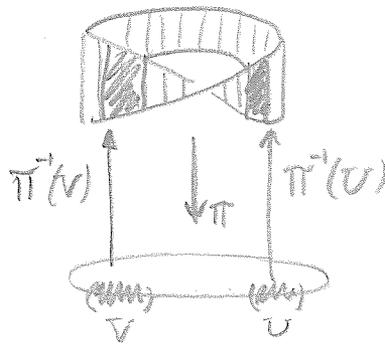
A trivial bundle is not twisted

$$M \times F$$

$$\pi \downarrow$$

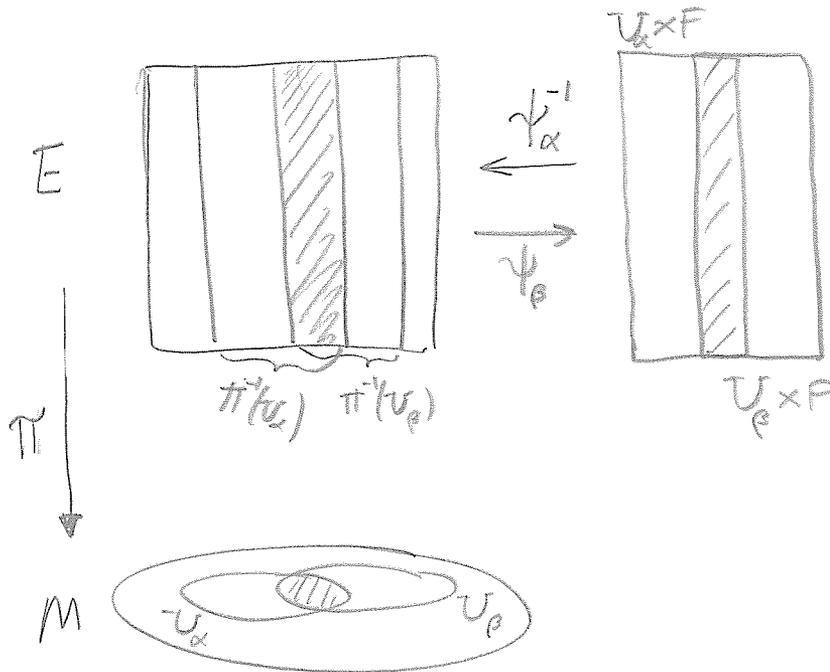
$$M$$

Möbius Band



"Transition Functions" (we'll should have principal bundle for that terminology)

$$\psi_\beta \circ \psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times F \rightarrow (U_\alpha \cap U_\beta) \times F$$



$$(\psi_\beta \circ \psi_\alpha^{-1})(x, f) = (x, \pi_F(\psi_\beta \circ \psi_\alpha^{-1})(x, f))$$

If we fix the x and let f vary we get a mapping from $F \rightarrow F$, these form a group $\text{Diff}(F)$ a huge group.

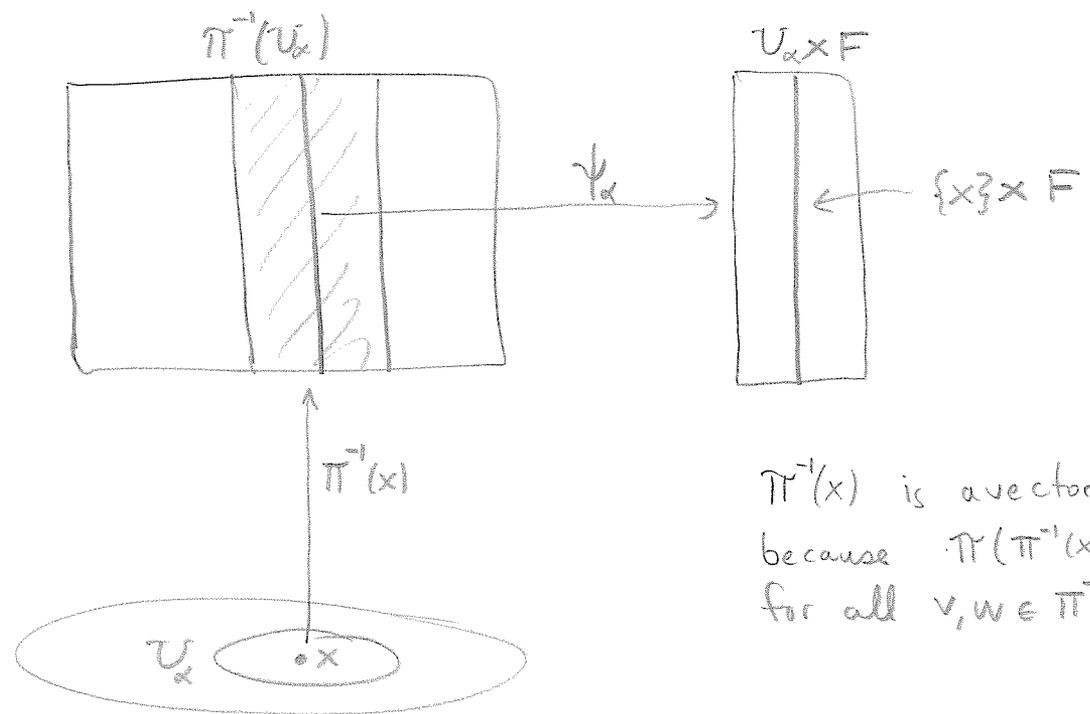
VECTOR BUNDLES

8/26/05

(16)

To say that (E, M, F, π) is a vector bundle over a field $K (= \mathbb{R} \text{ or } \mathbb{C})$ is to say it is a fiber bundle such that

- 1.) F is a vector space over K
- 2.) \exists smooth maps $+$: $\{(v, w) \in E \times E \mid \pi(v) = \pi(w)\}$ and \cdot : $K \times E \rightarrow E$ such that $(+, \cdot)$ satisfy the vector space axioms.
- 3.) \exists local trivializing maps $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ such that for each α and each $x \in U_\alpha$ the mapping from $\pi^{-1}(x)$ to F defined by
$$u \longmapsto (\pi_F \circ \psi_\alpha)(u) \quad \text{for } u \in \pi^{-1}(x)$$
 is an isomorphism (vector space)



$\pi^{-1}(x)$ is a vector space because $\pi(\pi^{-1}(x)) = x$ for all $v, w \in \pi^{-1}(x)$.

Principle Bundles

8/26/05

(17)

a principle fiber bundle (PFB) is a fiber bundle (P, M, π, G) such that

(1) G is a Lie group which acts freely on the right of P meaning \exists a smooth map $\sigma: P \times G \rightarrow P$ such that if

$$u \cdot g = \sigma(u, g)$$

has the usual action properties for $u \in P, g, g_1, g_2 \in G$

a.) $u \cdot (g_1 g_2) = (u \cdot g_1) \cdot g_2$

b.) $u \cdot e = u$ provided $e = e^2 \in G$

c.) $u \cdot g = u \Rightarrow g = e$

(the action is "free" its a strong condition on group action.)

(2) The Orbit

$$u \cdot G = \{u \cdot g \mid g \in G\}$$

is precisely for all $u \in P$

$$\pi^{-1}(\pi(u)) = u \cdot G$$

(3) \exists local trivializing maps $\psi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ such that for all $u \in \pi^{-1}(U_\alpha)$ we have

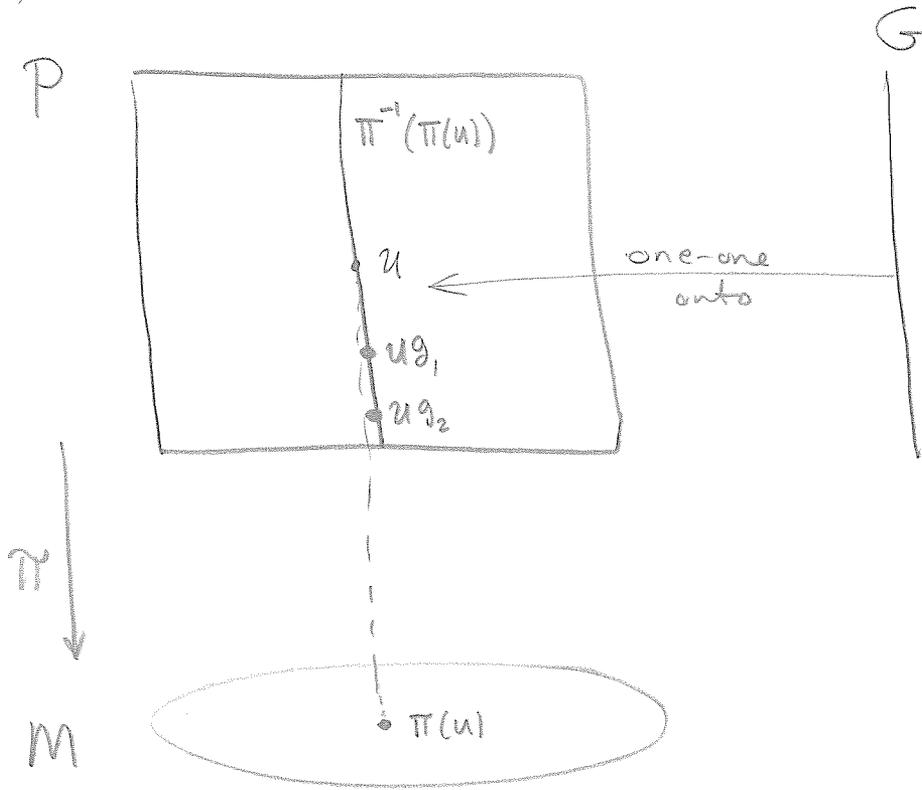
$$\psi_\alpha(u \cdot g) = \psi_\alpha(u) \cdot g \quad (g \in G)$$

where we have defined $(x, h) \cdot g \equiv (x, h \cdot g)$ for the RHS to make sense.

Principle Bundle Picture

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(1) the group action moves up/down the fiber

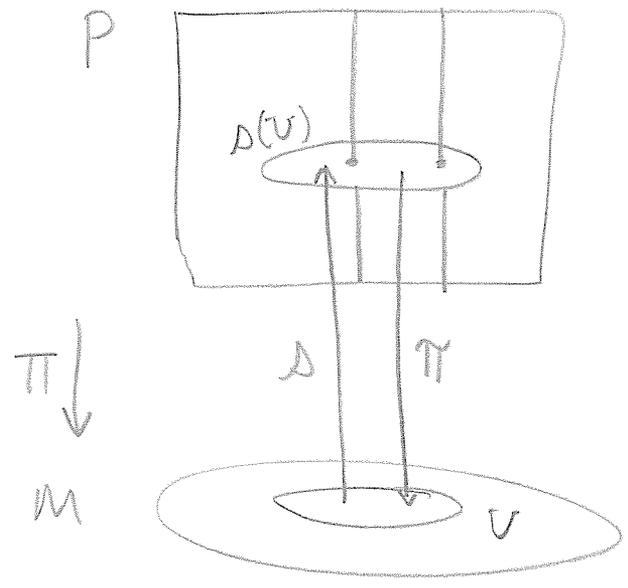
$$u g_1 = u g_2 \Rightarrow u g_1 g_2^{-1} = u \xrightarrow{(c)} g_1 g_2^{-1} = e \Rightarrow g_1 = g_2$$

thus you have a one-one onto mapping of group onto the fiber.

If (P, M, π, G) is a PFB and

$U \subseteq M$ then $D: U \rightarrow P$ is a local section

of the PFB (of π) iff $\pi \circ D = id_U$ and D is smooth



its easy to prove $D(U)$ is a submanifold. It cuts thru each fiber only once.

local sections
↓
local trivializing maps.

Now if you have a local section then you can define a local trivializing map ψ_U from $\pi^{-1}(U) \rightarrow U \times G$ defined by

$$\psi_U(D(x)g) = (x, g)$$

We can check this works, is it fair to define it on $D(x)g$? Well you can show such a ψ_U exists as follows.

1.) Define map $U \times G \rightarrow \pi^{-1}(U)$ by

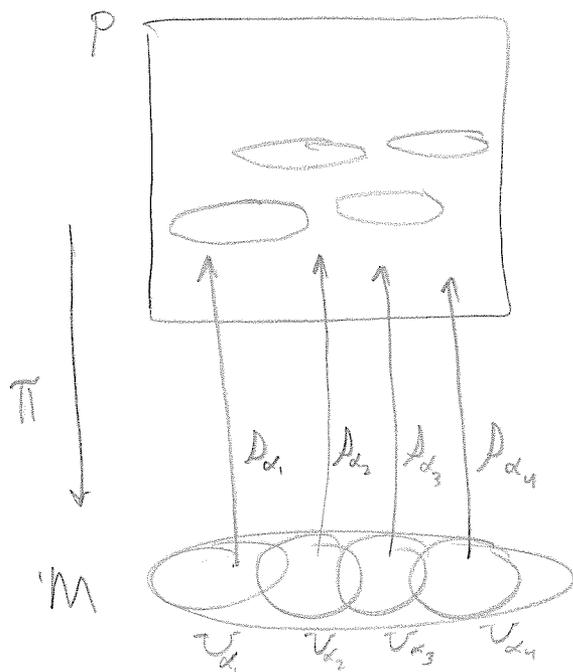
$$(x, g) \longmapsto (D(x), g) \quad (\text{no problems})$$

Can write $(D(x), g) = \sigma(D(x), g) = [\sigma \circ (D \text{ id})](x, g)$

so its smooth cause its composition of smooth maps.

then can show its 1-1 and onto. Then prove the inverse is locally smooth by inverse fct. Th^o.

Then define ψ_U as inverse of this map.



choice of gauge.

Th^m/ If $U \subseteq M$ then
 \exists a local trivializing map $\psi_U: \pi^{-1}(U) \rightarrow U \times G$
 iff \exists a local section $\Delta: U \rightarrow P$

Sketched \leftarrow last page

Conversely \Rightarrow

If $\psi_U: \pi^{-1}(U) \rightarrow U \times G$ is a local trivializing map then $\Delta_U: U \rightarrow P$ defined by $\Delta_U(x) \equiv \psi_U^{-1}(x, e)$ is a local section of $P \xrightarrow{\pi} M$

$$\pi \circ \Delta_U = \pi$$

$$\Delta_U = \pi^{-1} \circ \pi$$

$$x = \pi(\Delta_U(x, e)) = \pi(\psi_U^{-1}(x, e)) = \pi(\Delta(x))$$

By comm. of diagram.

$$\pi^{-1}(U) \xrightarrow{\psi_U} U \times G$$

$$\begin{array}{ccc} \pi \searrow & & \swarrow \pi_U \\ & U & \end{array}$$

SU(2) bundle over S^4 obtained from \mathbb{R}^4 by one-point compactification. This principal bundle represents an SU(2) instanton (§1.4). Introduce an open covering $\{U_N, U_S\}$ of S^4 ,

$$U_N = \{(x, y, z, t) | x^2 + y^2 + z^2 + t^2 \leq R^2 + \varepsilon\}$$

$$U_S = \{(x, y, z, t) | R^2 - \varepsilon \leq x^2 + y^2 + z^2 + t^2\}$$

where R is a positive constant and ε is an infinitesimal positive number. The thin intersection $U_N \cap U_S$ is essentially S^3 . Let $t_{NS}(p)$ be the transition function defined at $p \in U_N \cap U_S$. Since t_{NS} maps S^3 to SU(2), it is classified by $\pi_3(\text{SU}(2)) = \mathbb{Z}$. The integer characterising the bundle is called the **instanton number**. If $t_{NS}(p)$ is taken to be unity, we have a trivial bundle $P_0 = S^3 \times \text{SU}(2)$, which corresponds to the homotopy class 0. Non-trivial bundles are obtained as follows. We first note that $\text{SU}(2) \cong S^3$ (example 4.50). An element $A \in \text{SU}(2)$ is written as

$$A = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

where $|u|^2 + |v|^2 = 1$. Separating u and v into real and imaginary parts as $u = t + iz$ and $v = y + ix$, we find $t^2 + x^2 + y^2 + z^2 = 1$. Thus SU(2) is regarded as the unit sphere S^3 and $\pi_3(\text{SU}(2)) \cong \pi_3(S^3) \cong \mathbb{Z}$ classifies maps from S^3 to $\text{SU}(2) \cong S^3$. The identity map $f: S^3 \rightarrow S^3 \cong \text{SU}(2)$ is

$$f: (x, y, z, t) \mapsto \begin{pmatrix} t + iz & y + ix \\ -y + ix & t - iz \end{pmatrix} = t\mathbb{1} + i(x\sigma_x + y\sigma_y + z\sigma_z) \quad (9.47)$$

where

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the σ_μ are the Pauli matrices. Let us take a point $p = (x, y, z, t) \in U_N \cap U_S$. If $R = (x^2 + y^2 + z^2 + t^2)^{1/2}$ denotes the length of p , the vector $(x/R, y/R, z/R, t/R)$ has unit length. We assign an element of SU(2) to the point p as

$$t_{NS}(p) = \frac{1}{R} \left(t\mathbb{1} + i \sum x^i \sigma_i \right) \quad (9.48)$$

Let ϕ_N and ϕ_S be the local trivialisations,

$$\phi_N^{-1}(u) = (p, g_N) \quad \phi_S^{-1}(u) = (p, g_S) \quad (9.49)$$

where $p = \pi(u)$ and $g_N, g_S \in \text{SU}(2)$. On $U_N \cap U_S$, we have

$$g_N = \frac{1}{R} \left(t\mathbb{1} + i \sum x^i \sigma_i \right) g_S \quad (9.50)$$

$\phi_i^{-1}(u) = (p, g_u)$. In this local trivialisation, the section $s_i(p)$ is expressed as

$$s_i(p) = \phi_i(p, e) \quad (9.42)$$

This local trivialisation is called the **canonical local trivialisation**. By definition $\phi_i(p, g) = \phi_i(p, e)g = s_i(p)g$. If $p \in U_i \cap U_j$, two sections $s_i(p)$ and $s_j(p)$ are related by the transition function $t_{ij}(p)$ as follows

$$s_i(p) = \phi_i(p, e) = \phi_i(p, t_{ij}(p)e) = \phi_i(p, t_{ij}(p)) \quad (9.43)$$

$$= \phi_j(p, e)t_{ij}(p) = s_j(p)t_{ij}(p).$$

Example 9.11. Let P be a principal bundle with fibre $U(1) = S^1$ and the base space S^2 . This principal bundle represents the topological setting of the magnetic monopole (§1.3). Let $\{U_N, U_S\}$ be an open covering of S^2 , U_N (U_S) being the northern (southern) hemisphere. If we parameterise S^2 by the usual polar angles, we have

$$U_N = \{(\theta, \phi) | 0 \leq \theta \leq \pi/2 + \varepsilon, 0 \leq \phi < 2\pi\}$$

$$U_S = \{(\theta, \phi) | \pi/2 - \varepsilon \leq \theta \leq \pi, 0 \leq \phi < 2\pi\}.$$

The intersection $U_N \cap U_S$ is a strip which is essentially the equator. Let ϕ_N and ϕ_S be the local trivialisations such that

$$\phi_N^{-1}(u) = (p, \exp(i\alpha_N)), \quad \phi_S^{-1}(u) = (p, \exp(i\alpha_S)) \quad (9.44)$$

where $p = \pi(u)$. Take a transition function t_{NS} of the form $e^{in\phi}$, where n must be an integer so that $t_{NS}(p)$ may be uniquely defined on the equator. Since t_{NS} maps the equator S^1 to $U(1)$, this integer characterises the homotopy group $\pi_1(U(1)) = \mathbb{Z}$. The fibre coordinates α_N and α_S are related on the equator as

$$e^{i\alpha_N} = e^{in\phi} e^{i\alpha_S} \quad (9.45)$$

If $n = 0$, the transition function is the unit element of $U(1)$ and we have a trivial bundle $P = S^2 \times S^1$. If $n \neq 0$, the $U(1)$ -bundle P_n is twisted. It is remarkable that the topological structure of a fibre bundle is characterised by an integer.

Since $U(1)$ is Abelian, the right action and the left action are equivalent. Under the right action $g = e^{i\lambda}$, we have

$$\phi_N^{-1}(ug) = (p, e^{i(\alpha_N + \lambda)}) \quad (9.46a)$$

$$\phi_S^{-1}(ug) = (p, e^{i(\alpha_S + \lambda)}) \quad (9.46b)$$

The right action corresponds to the $U(1)$ -gauge transformation.

Example 9.12. If we identify all the infinite points of the Euclidean space \mathbb{R}^m , we have the one-point compactification $S^m = \mathbb{R}^m \cup \{\infty\}$. If a trivial G bundle is defined over \mathbb{R}^m , we shall have a new G bundle over S^m

$$S^2_C = \{(z^0, z^1) \in \mathbb{C}^2 \mid |z^0|^2 + |z^1|^2 = 1\}.$$

Define a map $\pi : S^2_C \rightarrow \mathbb{C}P^1$ by

$$(z^0, z^1) \mapsto [(z^0, z^1)] = \{\lambda(z^0, z^1) \mid \lambda \in \mathbb{C}\}. \tag{9.58}$$

Under this map, points of S^3 of the form $\lambda(z^0, z^1)$, $|\lambda| = 1$ are mapped to a single point of $\mathbb{C}P^1 = S^2$. This is the Hopf map $\pi : S^3 \rightarrow S^2$ obtained above. This is easily generalised to the case of the quaternion \mathbb{H} . The quaternion algebra is defined by the product table,

$$\begin{aligned} i^2 = j^2 = k^2 = -1 & & ij = -ji = k \\ jk = -kj = i & & ki = -ik = j. \end{aligned}$$

An arbitrary element of \mathbb{H} is written as

$$q = t + ix + jy + kz. \tag{9.59}$$

Clearly the unit quaternion $|q| = (t^2 + x^2 + y^2 + z^2)^{1/2} = 1$ represents $S^3 \cong \text{SU}(2)$. The quaternion one-sphere is given by

$$S^3_{\mathbb{H}} = \{q^0, q^1\} \in \mathbb{H}^2 \mid |q^0|^2 + |q^1|^2 = 1 \tag{9.60}$$

which represents S^7 . The Hopf map in this case takes the form

$$\pi : S^3_{\mathbb{H}} \rightarrow \mathbb{H}P^1 \tag{9.61}$$

where $\mathbb{H}P^1$ is the quaternion projective space whose element is

$$[(q^0, q^1)] = \{\pi(q^0, q^1) \in \mathbb{H}^2 \mid \eta \in \mathbb{H}\}. \tag{9.62}$$

Under this map, points of S^7 with $|\eta| = 1$ are mapped to a single point of $\mathbb{H}P^1 = S^4$ and we have the Hopf map

$$\pi : S^7 \rightarrow S^4. \tag{9.63}$$

The fibre is the unit quaternion $S^3 = \text{SU}(2)$. The transition function defined by the Hopf map belongs to the class 1 of $\pi_3(\text{SU}(2)) = \mathbb{Z}$. An instanton of unit strength is described in terms of this Hopf map.

Octonions define a Hopf map $\pi : S^{15} \rightarrow S^8$. This is different from other Hopf maps in that the fibre S^7 is not really a group. So far we have not found an application of this map in physics.

Example 9.14 Let H be a closed Lie-subgroup of a Lie group G . We show that G is a principal bundle with fibre H and base space $M \cong G/H$. Define the right action of H on G by $g \mapsto ga$, $g \in G$, $a \in H$. The right action is differentiable since G is a Lie group. Define the projection $\pi : G \rightarrow M = G/H$ by the map $\pi : g \mapsto [g] = \{gh \mid h \in H\}$. Clearly, $g, ga \in G$ are mapped to the same point $[g]$ hence $\pi(g) = \pi(ga) (= [g])$. To define local trivialisations, we need to define a map $f_i : G \rightarrow H$ on each chart U_i . Let s be a local section over U_i and

let $g \in \pi^{-1}([g])$. Define f_i by $f_i(g) = s([g])^{-1}g$. Since $s([g])$ is a section at $[g]$, it is expressed as ga for some $a \in H$ and accordingly $s([g])^{-1}g = a^{-1}g^{-1}g = a^{-1} \in H$. Then we define the local trivialisation $\phi_i : U_i \times H \rightarrow G$ by

$$\phi_i^{-1}(g) = ([g], f_i(g)). \tag{9.64}$$

It is easy to see that $f_i(ga) = f_i(g)a$ ($a \in H$) hence $\phi_i^{-1}(ua) = (p, f_i(u)a)$ is satisfied. Useful examples are (see example 5.65)

$$O(n)/O(n-1) = \text{SO}(n)/\text{SO}(n-1) = S^{n-1} \tag{9.65}$$

$$U(n)/U(n-1) = \text{SU}(n)/\text{SU}(n-1) = S^{2n-1}. \tag{9.66}$$

9.4.2 Associated bundles

Given a principal fibre bundle $P(M, G)$, we may construct an associated fibre bundle as follows. Let G act on a manifold F on the left. Define an action of $g \in G$ on $P \times F$ by

$$(u, f) \rightarrow (ug, g^{-1}f) \tag{9.67}$$

where $u \in P$ and $f \in F$. Then the associated fibre bundle (E, π, M, G, F, P) is an equivalence class $P \times F/G$ in which two points (u, f) and $(ug, g^{-1}f)$ are identified.

Let us consider the case in which F is a k -dimensional vector space V . Let ρ be the k -dimensional representation of G . The associated vector bundle $P \times_{\rho} V$ is defined by identifying the points (u, v) and $(ug, \rho(g)^{-1}v)$ of $P \times V$, where $u \in P$, $g \in G$ and $v \in V$. For example, associated with $P(M, \text{GL}(k, \mathbb{R}))$ is a vector bundle over M with fibre \mathbb{R}^k . The fibre bundle structure of an associated vector bundle $E = P \times_{\rho} V$ is given as follows. The projection $\pi_E : E \rightarrow M$ is defined by $\pi_E(u, v) = \pi(u)$. This projection is well defined since $\pi(ug) = \pi(u)$ implies $\pi_E(ug, \rho(g)^{-1}v) = \pi(ug) = \pi_E(u, v)$. The local trivialisation is given by $\psi_i : U_i \times V \rightarrow \pi_E^{-1}(U_i)$. The transition function of E is given by $\rho(t_{ij}(p))$ where $t_{ij}(p)$ is that of P .

Conversely a vector bundle naturally induces a principal bundle associated with it. Let $E \xrightarrow{\pi} M$ be a vector bundle with $\dim E = k$ (the fibre is \mathbb{R}^k or \mathbb{C}^k). Then E induces a principal bundle $P(E) \cong P(M, G)$ over M by employing the same transition functions. The structure group G is either $\text{GL}(k, \mathbb{R})$ or $\text{GL}(k, \mathbb{C})$. Explicit construction of $P(E)$ is carried out following the reconstruction process described in §9.1.

Example 9.15 Associated with a tangent bundle TM over an m -dimensional manifold M is a principal bundle called the frame bundle $LM \cong \cup_{p \in M} L_p M$, where $L_p M$ is the set of frames at p . We introduce

No. 16. 1970

need $t_{NS} : U_N \cap U_S \rightarrow SU(2)$

$$U_N \cap U_S = ((x, y, z, t), g_N) \sim ((x, y, z, t), g_S)$$

$$g_N = t_N(x, y, z, t) g_S$$

$$\tilde{t}_{NS} : S^3 \rightarrow SU(2)$$

$$(x, y, z, t) \mapsto t \cdot \text{id} + i\vec{r} \cdot \vec{\sigma}$$

$$= t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \left(x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} t + iz & y + ix \\ -y + ix & t - iz \end{pmatrix}$$

$$= \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \quad \text{with } |u| + |v| = 1 \quad \text{is in } SU(2).$$

Need to show \tilde{t}_{NS} is 1-1 and onto

1-1 follows from fact $\{\mathbb{1}, \{\sigma_x\}\}$ are linearly independent

onto

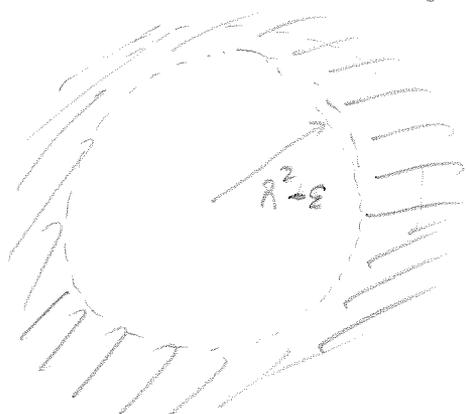
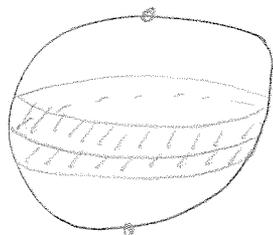
Comments on 9.12

$S^4 =$ base manifold S^4

$$U_N = \{(x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 < R^2 + \epsilon\}$$

$$U_S = \{(x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 < R^2 - \epsilon\}$$

↖ 1 pt
Compactification
regions in \mathbb{R}^4



gives an annulus which is deformation retractible to S^3

$$U_N \cap U_S \xrightarrow{\text{homotopic}} S^3$$

Principal $U_N \times SU(2)$ and $U_S \times SU(2)$ gluing
these together at $U_N \cap U_S \xrightarrow{\text{homotopy}} S^3$

Gluing $U_N \times SU(2) + U_S \times SU(2)$ need

$$t_{NS} : U_N \cap U_S \rightarrow SU(2)$$

So we look at homotopy classes from t_{NS} look
instead at $\tilde{t}_{NS} : S^3 \rightarrow SU(2)$

One fact $SU(2) \cong S^3$ so the fact $\pi_3(S^3) = \mathbb{Z}$

$$(x, y, z, t) \xrightarrow{f} t \cdot id + (x, y, z) \cdot \vec{\sigma}$$

(1) We saw f is 1-1 and onto. We think that f should generate facts that are 1-1, 2-1, 3-1, ... correspondants to $\pi_3(S^3) \cong \mathbb{Z}$. Let's see how take $P = (x, y, z, t)$

$$f(P) \cdot f(P) = f \left(\begin{array}{c} u \\ v \\ -\bar{v} \\ \bar{u} \end{array} \right)$$

$$= \left[t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + ix\sigma_x + iy\sigma_y + iz\sigma_z \right] \begin{bmatrix} u \\ v \\ -\bar{v} \\ \bar{u} \end{bmatrix}$$

$$= \begin{bmatrix} (t+iz)u - (y+ix)v & (t+iz)v + (y+ix)u \\ (-t+iz)v + (-y+ix)u & (t-iz)u + (ix-y)v \end{bmatrix}$$

$$= \begin{bmatrix} u^2 - v\bar{v} & uv + v\bar{u} \\ -\bar{u}\bar{v} - \bar{v}u & \bar{u}^2 - \bar{v}v \end{bmatrix} = \begin{bmatrix} uv \\ -\bar{v}\bar{u} \end{bmatrix}^2 = f(-P)f(-P)$$

In general can use roots of unity to see $(f(P))^n$ is an n to one mapping. Hmm roots of unity from $SU(2)$.

$$(f(P))^n = (f(\xi P))^n \quad \text{where } \xi \text{ is the } n^{\text{th}} \text{ root of unity.}$$

$$f(P) = \frac{1}{\|P\|} t \cdot id + \vec{x} \cdot \vec{\sigma}$$

$$(2) \quad g_N = f(P)^n g_S$$

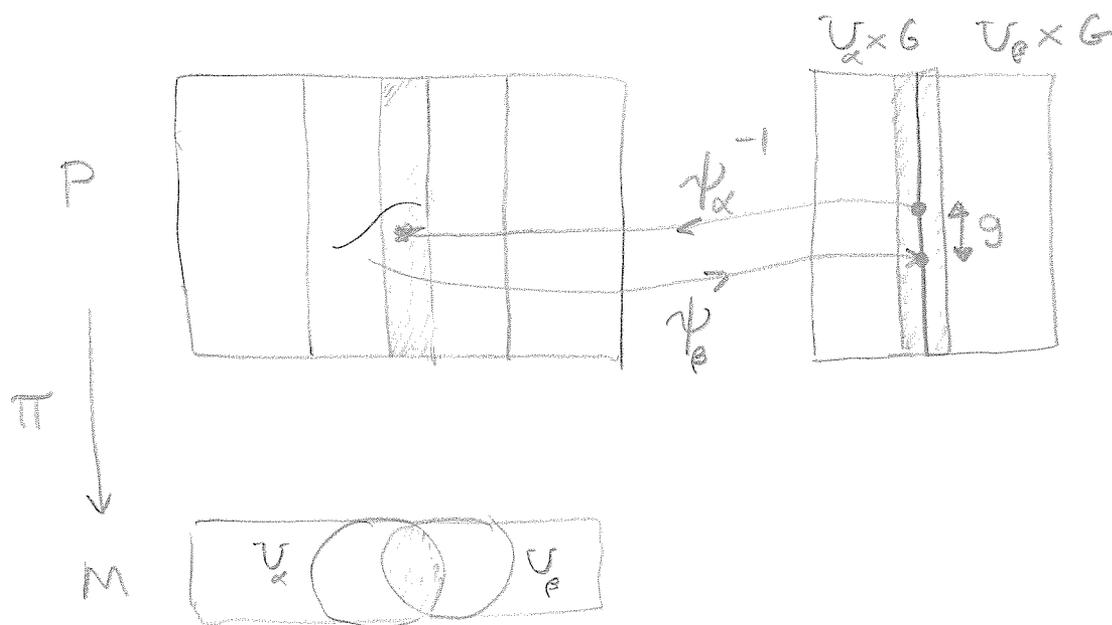
each n gives a different bundle.

Given a family $\{(U_\alpha, \psi_\alpha)\}$ of local trivializing maps of a PFB (P, M, π, G) such that M is covered (open) by $\{U_\alpha\}$, one can define then a family of maps:

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow G$$

$$(x, g_{\beta\alpha}(x)) \equiv (\psi_\beta \circ \psi_\alpha^{-1})(x, e)$$

project onto 2nd factor to get $g_{\beta\alpha}$.



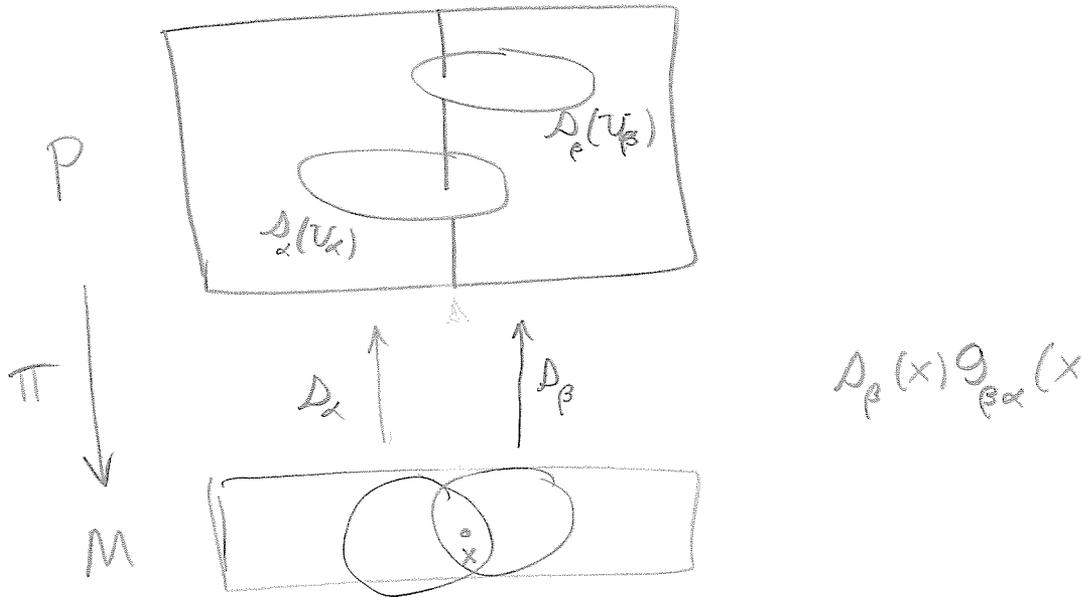
Now consider also

$$\psi_\beta^{-1}(x, g_{\beta\alpha}(x)) = \psi_\alpha^{-1}(x, e)$$

$$\underbrace{\psi_\beta^{-1}(x, e)} g_{\beta\alpha}(x) = \psi_\alpha^{-1}(x, e)$$

$$D_\beta(x) g_{\beta\alpha}(x) = D_\alpha(x)$$

transition facts can be discussed via local sections (D_α) or local trivializing facts (ψ_α)



- gauge transformations space of maps which is ∞ dim'd
- not hard to prove that

$$g_{\gamma\alpha}(x) = g_{\gamma\beta}(x) g_{\beta\alpha}(x)$$

$$g_{\alpha\alpha}(x) = e$$

fulfills

$\forall x \in U_\alpha \cap U_\beta \cup U_\gamma$
Check Cocycle Condition

(Nolan & Wallach clean approach to geometric quantization)
also Woodhouse & Simms

$$\left. \begin{array}{l} \approx \left\{ \begin{array}{l} h: U \rightarrow G \quad C^{(1)} \\ g: U \cap V \rightarrow G \quad C^{(2)} \end{array} \right\} \approx \end{array} \right\} \approx$$

$$(Sg)(x_1, x_2, x_3) = \sum g(x_i, \hat{x}_{ij}, x_k)$$

Important in geometric quantization to prove
Czech Cohomology \cong De Rham Cohomology.

Th^e/ Given $\{U_\alpha\}_{\alpha \in \Lambda}$ an open cover of M a manifold
and a family of maps $\{g_{\alpha\beta}\}_{\beta, \alpha \in \Lambda}$ where $U_\alpha \cap U_\beta \neq \emptyset$

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$$

such that

$$g_{\alpha\alpha} = \text{id}_{U_\alpha}$$

$$g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

then \exists a PFB (P, M, π, G) and a family $\{P_\alpha\}$
of local sections $P_\alpha : U_\alpha \rightarrow P$ whose transition
functions are the $\{g_{\alpha\beta}\}$

Remarks: Physicists work on M and tend to work with
local gauges defined in some nbhd of spacetime which
satisfy the above criteria \Rightarrow they can build a P.B. (or not)

Sketch

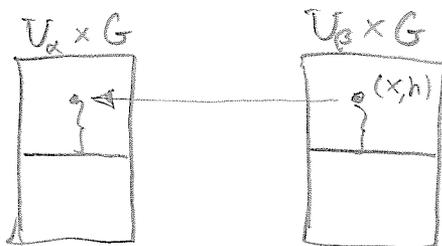
$$\text{Let } \tilde{P} = \bigsqcup_{\alpha \in \Lambda} (U_\alpha \times G)$$

(disjoint union means
 $\bigcup_{\alpha \in \Lambda} (U_\alpha \times G \times \alpha)$)

Define relation \sim on \tilde{P} by

$$(x, g) \in U_\alpha \times G \quad (y, h) \in U_\beta \times G$$

$$(x, g) \sim (y, h) \iff x = y \text{ and } g_{\alpha\beta}(x)h = g$$



• Note that $g_{\alpha\alpha}(x) = e \Rightarrow (x, g) \sim (x, g)$ (Reflexive)

• Note that $e = g_{\alpha\alpha}(x) = g_{\alpha\beta}(x) g_{\beta\alpha}(x)$

$$\Rightarrow g_{\beta\alpha}(x) = (g_{\alpha\beta}(x))^{-1}$$

Sketch Continued of Fiber Bundle Building Th^m

$$(x, g) \sim (y, h) \Rightarrow x=y \ \& \ g_{\alpha\beta}(x)h = g$$

$$\Rightarrow x=y \quad h = g_{\beta\alpha}(x)g \Rightarrow (y, h) \sim (x, g)$$

its not hard to prove \sim is transitive, thats the cocycle condition. Define then

$$P = \tilde{P} / \sim$$

P is made of \sim classes $[x, g]$. Manifold Structure can be shown by arguments involving

$$\varphi_\beta : U_\beta \times G \rightarrow \tilde{P}$$

\Rightarrow diffeomorphism... see Kobayashi & Nomizu.

$$\varphi_\beta(x, g) = [x, g]$$

Additionally define (can show π smooth)

$$\pi([x, g]) = x$$

G acts on P according to

$$[x, g] \cdot h \equiv [x, gh] \quad x \in U_\alpha, g, h \in G$$

Then define $\Delta_\alpha(x) = [x, e]_x$

$$\Delta_\alpha(x) g_{\alpha\beta}(x) = [x, e] g_{\alpha\beta}(x)$$

$$= [x, e g_{\alpha\beta}(x)] \sim (x, e) \quad \text{by } \sim \text{ immediately}$$

$$= [x, e]_e$$

$$= \Delta_\beta(x)$$

Remark: We can build a P.B. from local data.
by this Th^m

Question: What about Vector Bundles?
How to Build these

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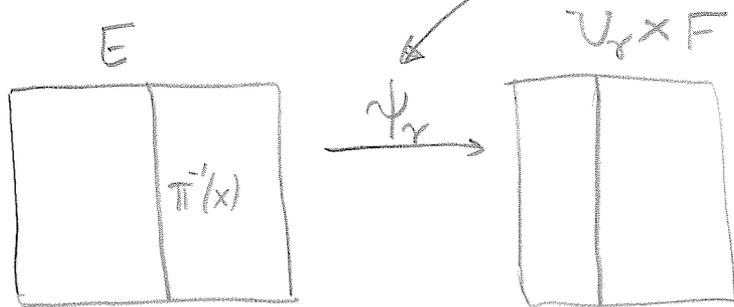
Note that if (E, M, π, F) is a vector bundle then one can define transition maps for these as well based on a local trivialization. $\{(U_\alpha, \psi_\alpha)\}$ as follows:

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow \text{aut}(F) \leftarrow (\text{is a group})$$

$$(\psi_\beta \circ \psi_\alpha^{-1})(x, f) = (x, g_{\beta\alpha}(x)(f))$$

could just be matrix multiplication

these are vector space isomorphism



$$\varphi_\gamma(x) = \left(\pi_{U_\gamma} \circ \psi_\gamma \right) \Big|_{\pi^{-1}(x)} : \pi^{-1}(x) \xrightarrow{\psi_\gamma} \{x, F\} \xrightarrow{\pi_F} F$$

Remark: $g_{\beta\alpha}(x) = \varphi_\beta(x) \circ \varphi_\alpha(x)^{-1}$

Examples:

(1) If M is a manifold then the frame bundle is a PFB. The bundle space is

$$\mathcal{F}M = \{ (p, e_i) \mid p \in M, \{e_i\} \text{ is a basis of } T_p M \}$$

• $\pi: \mathcal{F}M \rightarrow M$ where $\pi(p, e_i) = p$

• $GL(m)$ is a group and it acts on $\mathcal{F}M$ by

$$(p, e_i) \cdot g = (p, e_j g_i^j) \quad g = (g_i^j) \in GL(m)$$

($GL(m)$ is $\text{Aut}(T_p M)$ if you think about it)

• If U is the domain of chart (x^μ) of M then a local trivialization is

$$\psi_U: \pi^{-1}(U) \rightarrow U \times GL(m)$$

$$\psi_U(p, e_i) = (p, e^i(\frac{\partial}{\partial x^\mu}))$$

Where $\{e^i\}$ is a basis of $T_p^* M$ dual to $\{e_i\}$ of $T_p M$

Remark: in a vector bundle we cannot connect local sections as in the PFB. This is what the frame bundle helps with. Given E we could construct $\mathcal{F}E$ or $\mathcal{F}TM$ and so on. The frame bundle is a PFB so you can compare sections and go back down....

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(27)

(2) If L is a Lie Group and H is a closed subgroup of then $(L, L/H, \eta, H)$ is a PFB where $\eta: L \rightarrow L/H$ is the map $\eta(x) = xH \in L/H$.

Remark: By Warner $\exists!$ differentiable structure so that there is a smooth section, in $(L, L/H, \eta, H)$. Bleeker uses such an argument to show L has 4 connected components.

Add more comments here:

(3) The tangent bundle is a vector bundle

$$\pi : TM \rightarrow M$$

$$\psi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$$

$$\psi_U \left(p, v^\mu \frac{\partial}{\partial x^\mu} \right) = (p, v^\mu \Omega_\mu) \quad \text{where } \Omega_\mu \text{ is standard basis in } \mathbb{R}^n$$

$$\left(\psi_U \circ \psi_V^{-1} \right) \left(p, v^\mu \Omega_\mu \right) = \psi_U \left(p, v^\mu \frac{\partial}{\partial x^\mu} \right)$$

$$= \psi_U \left(p, v^\mu \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right)$$

$$= \left(p, v^\mu \frac{\partial y^\nu}{\partial x^\mu} \Omega_\nu \right)$$

$$p \rightarrow \frac{\partial y^\nu}{\partial x^\mu} (p) \quad \left(\begin{array}{l} \text{transition} \\ \text{fncts.} \end{array} \right)$$

we're continuing the discussion back on (13).

Recall that we had defined an inner product on p -forms

$$(\alpha, \beta) = \int_M [\alpha \wedge (*\beta)] = \frac{1}{p!} \int_M [\alpha^{\mu_1 \dots \mu_p} \beta_{\mu_1 \dots \mu_p}] \mu_g$$

Also define a codifferential δ by for $m = \dim(M)$

$$\delta \equiv (-1)^{mp + m + 1} (*d*)$$

$$\Omega^p \longrightarrow \Omega^{m-p} \longrightarrow \Omega^{m-p+1} \longrightarrow \Omega^{m-(m-p+1)} = \Omega^{p-1}$$

That is δ takes p to $p-1$ forms. And it has properties,

$$\begin{cases} \delta = (-1)^p (*d*) & m = \text{odd} \\ \delta = -(*d*) & m = \text{even} \\ \delta^2 = 0 \end{cases}$$

Where we have noticed that because $** = \pm 1$ and $d^2 = 0$ it follows

$$\begin{aligned} \delta^2 &= \pm *d**d* \\ &= \pm *dd* \\ &= \pm *d^2* \\ &= 0. \end{aligned}$$

Th^m/ If α is a $(p-1)$ -form and β is a p -form then

$$(d\alpha, \beta) = (\alpha, \delta\beta)$$

assuming the inner products make sense (M compact or α, β die-off quickly)

Proof: One can show that $\delta = (-1)^{p-1} (*^{-1}d*)$

$$\begin{aligned} (d\alpha, \beta) &= \int_M [d\alpha \wedge *\beta] \quad \text{: noting } d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{p-1} \alpha \wedge d*\beta \\ &= \int_M d(\alpha \wedge *\beta) - (-1)^{p-1} [\alpha \wedge d(*\beta)] \quad \text{: } \int_{M^k} d\gamma = \int_{\partial M^k} \gamma \quad \text{Stoke's Th}^m \\ &= (-1)^p \int_M [\alpha \wedge d*\beta] \quad \text{: our manifold is compact w/o boundary so } \partial M^k = \emptyset \\ &= (-1)^p \int_M [\alpha \wedge d(*^{-1}d*\beta)] \\ &= \int_M [\alpha \wedge *(-1)^p *^{-1}d*\beta] \\ &= \int_M [\alpha \wedge *(\delta\beta)] \\ &= (\alpha, \delta\beta) \end{aligned}$$

: Assuming $\delta = (-1)^{p-1} *^{-1}d*$ which we'll prove next page.

$$** \omega_p = (-1)^{p(m-p)} \omega_p$$

$$*^{-1} \omega_p = (-1)^{p(m-p)} * \omega_p$$

$$*^{-1} = (-1)^{p(m-p)} * \quad \leftarrow \text{for } *^{-1} \text{ applied to a } p\text{-form}$$

Now β is a p -form so consider

$$\delta \beta = (-1)^p (*^{-1} d *) \beta$$

$$= (-1)^p (-1)^{\bar{p}(m-\bar{p})} * d * \beta \quad ; \text{ where } \bar{p} = m-p+1$$

$$= (-1)^p (-1)^{(m-p+1)(m-(m-p+1))} * d * \beta$$

$$= (-1)^p (-1)^{(m-p+1)(+p-1)} * d * \beta$$

$$= (-1)^p (-1)^{(mp-p^2+p-m+p-1)} * d * \beta$$

$$= (-1)^p (-1)^{(mp-m+p-1)} * d * \beta$$

$$= (-1)^{m+mp+1} * d * \beta$$

$$\Rightarrow \delta = (-1) (*^{-1} d *)$$

$$\text{Comparing } \delta = (-1)^{mp+m+1} * d * = (-1) (*^{-1} d *)$$

The Laplace Beltrami operator Δ is the mapping from $\Omega^p(m) \rightarrow \Omega^p(m)$ defined by $\Delta = \delta d + d\delta$

Example: take f a zero-form in Euclidean Space.

$$\begin{aligned}\Delta f &= \delta(df) + d(\delta f) \stackrel{0}{=} \\ &= \delta(df) \\ &= \delta\left(\frac{\partial f}{\partial x^\mu} dx^\mu\right)\end{aligned}$$

Now consider, to calculate $\delta(df) = (-1)^m * d * df$

$$\begin{aligned}*df &= \frac{\partial f}{\partial x^\mu} * (dx^\mu) \\ &= \frac{\partial f}{\partial x^\mu} \frac{1}{(m-1)!} \varepsilon^{\mu \nu_1 \nu_2 \dots \nu_m} dx^{\nu_1} \dots dx^{\nu_m} \\ &= \frac{1}{(m-1)!} \frac{\partial f}{\partial x^\mu} g^{\mu \nu_1} \varepsilon_{\nu_1 \nu_2 \dots \nu_m} dx^{\nu_2} \dots dx^{\nu_m} \\ &= \frac{1}{(m-1)!} \left[\frac{\partial f}{\partial x^1} \varepsilon_{1 \nu_2 \dots \nu_m} + \frac{\partial f}{\partial x^2} \varepsilon_{2 \nu_2 \dots \nu_m} + \dots + \frac{\partial f}{\partial x^m} \varepsilon_{m \nu_2 \dots \nu_m} \right] dx^{\nu_2} \dots dx^{\nu_m} \\ &= \frac{1}{(m-1)!} \left[\frac{\partial f}{\partial x^1} (m-1)! \varepsilon_{12 \dots m} dx^2 \dots dx^m + \frac{\partial f}{\partial x^2} (m-1)! \varepsilon_{213 \dots m} dx^1 dx^3 \dots dx^m \right. \\ &\quad \left. \dots + \frac{\partial f}{\partial x^m} (m-1)! \varepsilon_{m12 \dots m-1} dx^1 dx^2 \dots dx^{m-1} \right]\end{aligned}$$

$$\begin{aligned}d(*df) &= \left[\frac{\partial^2 f}{\partial x^\mu \partial x^1} \varepsilon_{12 \dots m} dx^\mu dx^2 \dots dx^m \right. \\ &\quad \left. + \frac{\partial^2 f}{\partial x^\mu \partial x^2} \varepsilon_{213 \dots m} dx^\mu dx^1 dx^3 \dots dx^m \right. \\ &\quad \left. + \dots + \frac{\partial^2 f}{\partial x^\mu \partial x^m} \varepsilon_{m12 \dots m-1} dx^\mu dx^1 dx^2 \dots dx^{m-1} \right]\end{aligned}$$

$$= \left[\frac{\partial^2 f}{(\partial x^1)^2} + \dots + \frac{\partial^2 f}{(\partial x^m)^2} \right] dx^1 dx^2 \dots dx^m$$

$$\begin{aligned} &= (\Delta f) \mu_g \quad \Rightarrow \quad \delta(df) = \pm * d * df \\ &= (-1)^{m^p + m + 1} * (\Delta f) \mu_g \\ &= (-1)^{m^p + m + 1} \Delta f \\ &= -\Delta f \quad \leftarrow \text{why?} \end{aligned}$$

Theorem: for the Laplace Beltrami: $\Delta = \delta d + d\delta$

$$(1) * \Delta = \Delta *$$

$$(2) d\Delta = \Delta d$$

$$(3) \delta\Delta = \Delta\delta$$

(4) Δ is a positive operator.

Proof of (1)

We first show the action on p -forms of $*\delta d$ is

$$*\delta d = (-1)^{p+1} d*d$$

By defⁿ $\delta = (-1)^p *^{-1} d *$ on Ω^p thus if d acts on Ω^p then $d\Omega^p$ is $(p+1)$ -form and δ acts on $(p+1)$ -forms

$$*\delta d = *((-1)^{p+1} *^{-1} d *) d = \underline{(-1)^{p+1} d*d} = *\delta d \text{ lemma 1.}$$

Now since $\delta = (-1)^p *^{-1} d *$ on Ω^p we have

$$\begin{aligned} *\delta *^{-1} &= *((-1)^{m-p} *^{-1} d *) *^{-1} \\ &= \underline{(-1)^{m-p} d} = *\delta *^{-1} \text{ lemma 2.} \end{aligned}$$

Thus we find

$$\begin{aligned} *\delta d &= (-1)^{p+1} (d*d) \\ &= (-1)^{p+1} (-1)^{m-p} d * * \delta *^{-1} && \text{where } ** = (-1)^{[m-(m-p-1)](m-p-1)} \\ &= (-1)^{p+1} (-1)^{m-p} (-1)^{mp+m-p-1} d\delta *^{-1} && = (-1)^{(p+1)(m-p-1)} \\ &= (-1)^{mp-p} d\delta *^{-1} && = (-1)^{mp+m-p-1} \\ &= (-1)^{(m-p)p} d\delta *^{-1} \\ &= d\delta * \quad \longrightarrow \quad \delta d * = *\delta d \end{aligned}$$

$$\begin{aligned} \text{Thus } *\Delta &= *(\delta d + d\delta) \\ &= d\delta * + d\delta * \\ &= (\Delta) * \end{aligned}$$

I believe this is half the proof.

Proof of (4) Δ is a positive operator

9/2/05 (33)

$$\begin{aligned} (w, \Delta w) &= (w, d\delta w + \delta dw) \\ &= (w, \delta w) + (w, \delta dw) \\ &= (\delta w, \delta w) + (dw, dw) \geq 0 \end{aligned}$$

Notice this also says $\Delta w = 0 \Rightarrow \delta w = 0$ AND $dw = 0$
that is w is both closed & coclosed. The converse true? \Leftarrow
seems true...

Defⁿ φ is harmonic if $\Delta \varphi = 0$. Th^m in Warner
says every De Rham cohomology class $H_{\text{DeRham}}^k(M) = \frac{\text{closed}}{\text{exact}}$
built from $d(d\alpha) = 0$ every exact thing is closed
recall exact φ has $\varphi = d\psi$. The Hodge Th^m says
every de Rham cohomology class contains one and only one
harmonic form. The harmonic forms then are essentially
the De Rham cohomology. M compact & Riemannian

Associated Bundles

If (P, M, π, G) is a PFB and if G acts linearly on the left of a finite dim'l vector space V over \mathbb{K} , then we show \exists a vector bundle E associated to the PFB and this action

• Remark: this yields itself to many cases more than the vector bundle approach.

Proof: First define action of G on $P \times V$ by

$$(u, x) \cdot g = (ug, g^{-1} \cdot x)$$

the inverse on g comes from the assumption that we have left action
Let's see this is a right action

$$\begin{aligned} (u, x) \cdot (g_1 g_2) &= (u g_1 g_2, (g_1 g_2)^{-1} \cdot x) \\ &= ((u g_1) g_2, g_2^{-1} g_1^{-1} \cdot x) \\ &= (u g_1, g_1^{-1} x) \cdot g_2 \\ &= (u, x) g_1 \cdot g_2 \end{aligned}$$

Likewise $(u, x) \cdot e = (u, x)$. Now look at the space of orbits of the action, $xG = \{xg/g \in G\}$, $xG \cap yG \neq \emptyset \Rightarrow xG = yG$.

$$\boxed{\begin{aligned} \frac{P \times V}{G} &= \text{the associated bundle} \\ &\equiv P \times_G V \end{aligned}}$$

the orbits partition the X



$$xG = G/G_x$$

$G_x =$ little group
action free $\Rightarrow G_x = e$
So on PFB little group same every where.

Question: Why is $P \times_G V$ a manifold?

Let $\{U_\alpha\}$ be a family of local sections of π

$\Delta_\alpha: U_\alpha \rightarrow P$ such that $\{U_\alpha\}$ cover M

We'll define maps $\varphi_{U_\alpha}: U_\alpha \times V \rightarrow \tilde{\pi}^{-1}(U_\alpha)$ by

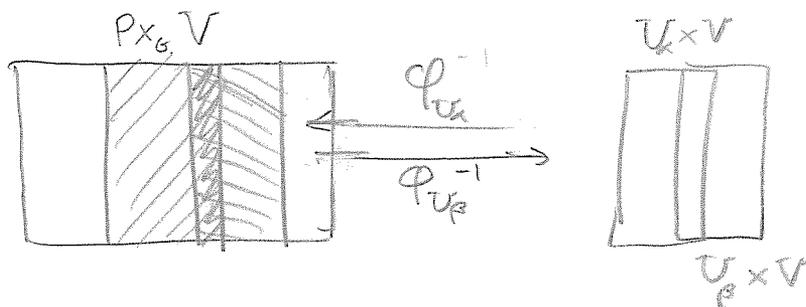
$$\varphi_{U_\alpha}(p, x) = (\Delta_\alpha(p), x)G$$

Where $\tilde{\pi}: P \times_G V \rightarrow M$ and $\tilde{\pi}((u, x)G) = \pi(u)$

$$\tilde{\pi}^{-1}(U_\alpha) \subseteq P \times_G V$$

We would force φ_{U_α} to be a diffeomorphism once we show φ_{U_α} is 1-1 and onto, by appropriately defining topology & chart structure on $P \times_G V$.

But we have 2 possible differentiable structures on $\tilde{\pi}^{-1}(U_\alpha \cap U_\beta)$ when $U_\alpha \cap U_\beta \neq \emptyset$.



You'd need to show that $\varphi_{U_\alpha} \circ \varphi_{U_\beta}^{-1}: \tilde{\pi}^{-1}(U_\alpha \cap U_\beta) \times V \rightarrow \tilde{\pi}^{-1}(U_\alpha \cap U_\beta) \times V$ is a diffeomorphism. To do this show $\varphi_{U_\beta}^{-1} \circ \varphi_{U_\alpha}$ is a diff.

$$(\Delta_\alpha(p), x) \longrightarrow (\Delta_\beta(p), x) = (\Delta_\alpha(p) g_{\alpha\beta}(x), x)$$

this is smooth.

The Associated Bundle is a Vector Bundle, why?

Define operations $+$, \cdot on $E = P \times V / G \cong P \times_G V$ by

$$\begin{aligned} (u, x)G + (u, y)G &= (u, x+y)G \\ c(u, x)G &= (u, cx)G \end{aligned}$$

Suppose we have (u, x) and (v, y) in $P \times V$ such that $\pi(u) = \pi(v)$ aka $\tilde{\pi}((u, x)G) = \tilde{\pi}((v, y)G)$

then $\exists g \in G$ such that $v = u \cdot g$ because P is a PFB.

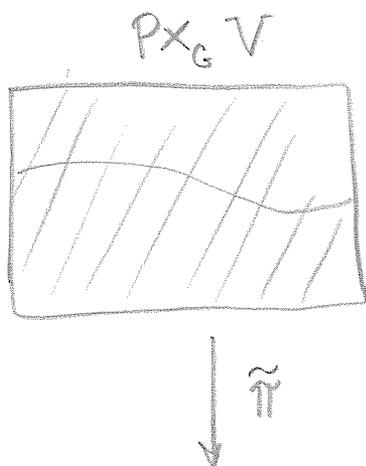
$$\begin{aligned} (u, x)G + (v, y)G &= (u, x)G + (ug, y)G \\ &= (u, x)G + (u, gy)G \end{aligned}$$

Now if $(ug, g^{-1}x)$, $(ug, g^{-1}y)$ are different representatives of the orbits $(u, x)G$ and $(u, y)G$ then

$$\begin{aligned} (ug, g^{-1}x)G + (ug, g^{-1}y)G &= (ug, g^{-1}x + g^{-1}y)G \\ &= (ug, g^{-1}(x+y))G \\ &= (u, x+y)G \\ &= (u, x)G + (u, y)G \end{aligned}$$

assuming action on V was linear.

Likewise $c(u, x)G = (u, cx)G$ is well-defined.



if a fact is constant
on leaves of foliation
then it induces a
smooth map on the
quotient space

(See Warner)

$$\begin{array}{ccc} \mathcal{F}M \times_{GL(m)} \mathbb{R}^m & \xrightarrow{\varphi} & TM \\ \tilde{\pi} \searrow & & \swarrow \pi_m \\ & M & \end{array}$$

Examples of Associated Bundles

(1) Let PFB be $(\mathcal{F}M, M, \pi, GL(m))$
the frame bundle of M . Let $GL(m) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$
 $(A, x) \rightarrow A \cdot x$

Claim $\mathcal{F}M \times_{GL(m)} \mathbb{R}^m \cong TM$.

Define then $\varphi: \mathcal{F}M \times \mathbb{R}^m \rightarrow TM$ to see this,

$\varphi((p, e_i), v) = (p, v^i e_i)$ $p \in M$, $\{e_i\}$ a basis of $T_p M$
Where $v = v^i \delta_i \in \mathbb{R}^m$ and δ_i is the standard basis.

$$\begin{aligned} \varphi((p, e_i)g, g^{-1}v) &= \varphi((p, e_j g_j^i), g_n^{-1} v^n \delta_n) \\ &= (p, (g_n^{-1} v^n) (e_j g_j^i)) \\ &= (p, v^n e_j \delta_n^j) \\ &= (p, v^n e_n) \\ &= (p, v) \\ &= \varphi(p, \end{aligned}$$

thus this map descends to a smooth mapping to TM

$$\tilde{\varphi}: \mathcal{F}M \times_{GL(m)} \mathbb{R}^m \rightarrow TM$$

Also clearly $\tilde{\varphi}$ carries
the vector space structure

(2) Let PFB be as above namely $\mathbb{F}M$. Define $GL(m)$ on $(\mathbb{R}^m)^*$ by $\alpha \in (\mathbb{R}^m)^* \Rightarrow \alpha: \mathbb{R}^m \rightarrow \mathbb{R}$

$$(g \cdot \alpha)(x) = \alpha(g^{-1}x)$$

right regular rep. in fact. analysis, QM in Hilbert Space this becomes position op. if choose things correctly. Prugvechi QM in Hilbert Space [self-adj] but if exponentiate to get Weyl-relation which are bounded & work.

$$\psi: \mathbb{F}M \times (\mathbb{R}^m)^* \longrightarrow T^*M$$

$$\psi(p, \alpha) = (p, \alpha_m e^m)$$

Where $\alpha = \alpha_m \rho^m$ and ρ^m is standard basis of $(\mathbb{R}^m)^*$ dual to ρ_m . We can go thru and verify ψ is constant on $GL(m)$ orbits so ψ descends to $\tilde{\psi}$ just like before.

(3) Take your favorite tensor bundle on $T^r_s M$ its just the associated bundle from $T^r_s \mathbb{R}^m$. The action on $T^r_s \mathbb{R}^m$ of $GL(m)$ is naturally extracted from (1) & (2).