Additional Examples

(E1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ this is an alternating series with $b_n = \frac{1}{\sqrt{n}}$.

Notice that $b_n > 0$ and $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right) = 0$. All we have left to show this series converges is that $b_{n+1} \leq b_n$ and that is clear since $n \leq n+1 \Rightarrow \frac{1}{n} \geq \frac{1}{n+1} \Rightarrow \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}$.

(Alternatively argue $f(x) = \frac{1}{\sqrt{x}}$ has $f'(x) = \frac{-1}{x^{3/2}} < 0$ for $x \geq 1$ thus $b_{n+1} = f(n+1) < f(n) = b_n \therefore b_{n+1} < b_n$ so they are decreasing.)

So we have shown $b_n > 0$, $b_n \to 0$ as $n \to \infty$ and $b_{n+1} \leq b_n$ for $b_n = \frac{1}{\sqrt{n}}$ hence $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by alternating series test.

(E2) $\displaystyle\sum_{m=1}^{\infty} \frac{\sin(m)}{m^3}$ consider $\displaystyle\sum_{m=1}^{\infty} \left|\frac{\sin(m)}{m^3}\right|$ with the hope of using the absolute convergence $\Rightarrow$ convergence idea. Notice that

$$\left|\frac{\sin(m)}{m^3}\right| = \frac{|\sin(m)|}{m^3} \leq \frac{1}{m^3}$$

Thus as $\sum \frac{1}{m^3}$ converges by $P=3$ series test $\Rightarrow \sum \left|\frac{\sin(m)}{m^3}\right|$ converges by the direct comparison test. Therefore $\displaystyle\sum_{m=1}^{\infty} \frac{\sin(m)}{m^3}$ is absolutely convergent hence it is convergent.

(E3) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ notice $\cos(n\pi) = \cos(\pi), \cos(2\pi), \cos(3\pi), \ldots = -1, 1, -1, 1, \ldots = (-1)^n$ this is actually the alternating harmonic series, well more precisely

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = -\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\right)$$

$\Rightarrow$ the series is the just -1 times the alternating harm. series which converges as I mentioned in lecture.

<u>Remark</u>: I have not shown why the series $\sum \frac{(-1)^{n+1}}{n}$ converges, but it does. We can use a power series expansion of $\ln$ to actually show $\sum \frac{(-1)^{n+1}}{n} = \ln(2)$.

(E4) $S = \sum_{k=1}^{\infty} \frac{1}{\ln(10^k)} = \sum_{k=1}^{\infty} \frac{1}{k\ln(10)}$ this diverges since $\frac{1}{k\ln(10)} < \frac{1}{k}$

and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges thus by direct comparison test S diverges.

(E5) $S = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$ well as $n \gg 1$ the terms in this series

look like $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ which means this is similar to a $p = 3/2$ series. Compare

$$\frac{\sqrt{n}}{n^2+n+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

Thus as $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $\underset{\text{D.C.T.}}{\Longrightarrow}$ $\sum \frac{\sqrt{n}}{n^2+n+1}$ converges.

(E6) $\sum_{n=0}^{\infty} \frac{n^2}{n!}$ whenever I see factorial I try ratio test to begin, this is clearly a series with positive terms.

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)!} \frac{n!}{n^2} \right|$$

in other words $(n+1)! = (n+1)n!$ so the $n!$'s cancel out

$$= \lim_{n\to\infty} \left| \left(\frac{n^2+2n+1}{n^2}\right)\left(\frac{\cancel{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}}{(n+1)\cancel{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}}\right) \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^3+n^2} \right|$$

$= 0 < 1$ $\therefore$ By ratio test $\sum_{n=0}^{\infty} \frac{n^2}{n!}$ converges

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(E7) $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ again use ratio test.

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{2^{k+1}}{(k+1)!} \frac{k!}{2^k} \right|$$

$$= \lim_{k\to\infty} \left(\frac{2\cdot \cancel{2^k}}{\cancel{2^k}} \frac{\cancel{k!}}{(k+1)\cancel{k!}} \right)$$

$= \lim_{k\to\infty} \left(\frac{2}{k+1}\right) = 0$ $\therefore$ $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges By ratio test $L = 0 < 1$.

(E8) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^2+k}$ note $\displaystyle\frac{2}{k^2+k} = \frac{2}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$ (partial fractions)

$$2 = A(k+1) + Bk$$

$\underline{k=-1}$ $\quad 2 = -B \Rightarrow B = -2$

$\underline{k=0}$ $\quad 2 = A \Rightarrow A = 2$

We compute the n^{th} partial sum:

$$S_n = \sum_{i=1}^{n} \left(\frac{2}{i} - \frac{2}{i+1}\right)$$

$$= \left(2 - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \cdots + \left(\frac{2}{n-1} - \frac{2}{n}\right) + \left(\frac{2}{n} - \frac{2}{n+1}\right)$$

$$= 2 - \frac{2}{n+1}$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+n} = \lim_{n\to\infty}\left(2 - \frac{2}{n+1}\right) = 2$ thus the series converges.

Remark: when we can calculate the series explicitly we avoid the divergence tests, basically this happens only for the telescoping or geometric series examples (for our course anyway.)

(E9) $\displaystyle\sum_{k=3}^{\infty} \left(\frac{e}{\pi}\right)^k = \left(\frac{e}{\pi}\right)^3 + \left(\frac{e}{\pi}\right)^4 + \left(\frac{e}{\pi}\right)^5 + \cdots$ notice $\displaystyle\frac{a_{k+1}}{a_k} = \frac{(e/\pi)^{k+1}}{(e/\pi)^k} = \frac{e}{\pi}$

this is a geometric series with $a = \left(\frac{e}{\pi}\right)^3$ and $r = \frac{e}{\pi}$. Thus as $\frac{e}{\pi} \cong \frac{2.718}{3.141} < 1$ we find it converges and it's value is $\boxed{\dfrac{(e/\pi)^3}{1 - e/\pi}}$

(E10) $\displaystyle\sum_{j=0}^{\infty} \frac{3^j + 4^j}{5^j} = \sum_{j=0}^{\infty}\left(\frac{3^j}{5^j} + \frac{4^j}{5^j}\right)$

$$= \sum_{j=0}^{\infty}\left(\frac{3}{5}\right)^j + \sum_{j=0}^{\infty}\left(\frac{4}{5}\right)^j$$

$$= 1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots + 1 + \frac{4}{5} + \left(\frac{4}{5}\right)^2 + \cdots$$

a pair of geometric series.

$$= \frac{1}{1 - 3/5} + \frac{1}{1 - 4/5}$$

$$= \frac{5}{5-3} + \frac{5}{5-4}$$

$$= \frac{5}{2} + 5$$

$$= \boxed{15/2}$$

So the series converges.

Find the I.O.C and R

E11 $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n^2 5^n}$ as usual begin with ratio test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2 5^{n+1}}\cdot\frac{n^2 5^n}{x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{n^2 x}{(n+1)^2\cdot 5}\right|$$
$$= \frac{|x|}{5}\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right|$$
$$= \frac{|x|}{5} \Longrightarrow L<1 \Leftrightarrow \frac{|x|}{5}<1 \Leftrightarrow |x|<5 \quad \therefore\ R=5$$

The series converges for $|x|<5$ for sure but what about the endpoints? We must check those to figure out the Interval of Convergence (I.O.C)

$x=-5$: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(-5)^n}{n^2 5^n} = \sum_{n=1}^{\infty}(-1)^n(-1)^n\frac{5^n}{n^2 5^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ p-series $p=2$ converges.

$x=5$: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{5^n}{n^2 5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ which converges since $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Thus the I.O.C is $[-5,5]$.

E12 $\displaystyle\sum_{n=0}^{\infty}(x^2+3)^n$, notice $\dfrac{a_{n+1}}{a_n} = \dfrac{(x^2+3)^{n+1}}{(x^2+3)^n} = x^2+3 = r$

this a geometric series, it converges when $|r|<1$ that is $|x^2+3|<1$ and there is no such $x\in\mathbb{R}$ $\therefore$ the series diverges allways.

E13 $\displaystyle\sum_{n=0}^{\infty}(2x+3)^n$ again geometric with $r = \dfrac{a_{n+1}}{a_n} = \dfrac{(2x+3)^{n+1}}{(2x+3)^n} = 2x+3$

as we proved this series converges for $|r|<1 \Rightarrow |2x+3|<1 \Rightarrow |x+\frac{3}{2}|<\frac{1}{2}$

So the I.O.C is $\left(-\frac{3}{2}-\frac{1}{2}, -\frac{3}{2}+\frac{1}{2}\right) = \boxed{(-2,-1) = \text{I.O.C}}$ with $\boxed{R=\frac{1}{2}}$

Remark: it's nice if we can write $|x-a|<R$ because then it's simple to see that the radius of convergence is R and the I.O.C. is $(a-R, a+R)$, I just used this idea above.