(ALLOWED TO SKIP ONE OF 8, 9, 10, 11 AND ONE OF 14, 15, 16, 17)

Show your work. No graphing calculators. This is a timed test, use your time wisely, some problems are worth more than others. You can use work from one problem to support the answer of another. You are not allowed to use theorems we have not yet covered concerning derivatives...unless, you prove the rule before you use it. All arguments should be given in the style in which we discussed in lecture. Thanks!

Problem 1 [3pts] Let A, B be sets. Define carefully what it means when we say "$f : A \to B$ is a function".

f is a function from A to B iff for each $a \in A$ f assigns a single-element $f(a) \in B$.

Problem 2 [3pts] Suppose $a \in \text{int}(\text{dom}(f))$ and $L \in \mathbb{R}$. Define carefully what is meant by stating "$\lim_{x\to a} f(x) = L$".

We say $\lim_{x\to a} f(x) = L$ iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in \mathbb{R}$ with $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Problem 3 [3pts] Suppose f is continuous at an interior point $a \in \text{dom}(f)$. Define continuity of f at $x = a$ carefully.

f is continuous at $a \in \text{int}(\text{dom}(f))$ iff $\lim_{x\to a} f(x) = f(a)$.

Problem 4 [3pts] Simplify the expressions below into the format cx^n where c, n are constants.

(a.) $\dfrac{1}{3\sqrt[5]{x^2}} = \underline{\tfrac{1}{3}x^{-2/5}}$, (b.) $\dfrac{2}{7x^3} = \underline{\tfrac{2}{7}x^{-3}}$, (c.) $\sqrt[3]{5x^8} = \underline{\sqrt[3]{5}\,x^{8/3}}$.

$\frac{1}{3}\frac{1}{x^{2/5}} = \frac{1}{3}x^{-2/5}$

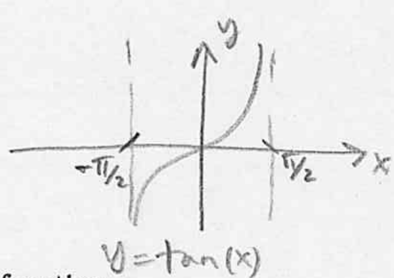
Problem 5 [4pts] State the domain and range for each of the inverse trigonometric functions:

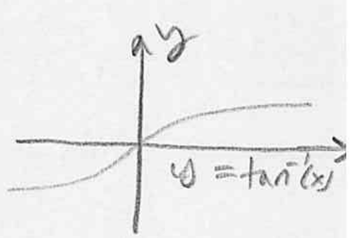
(a.) $\text{dom}(\sin^{-1}(x)) = \underline{[-1, 1]}$,

(b.) $\text{dom}(\tan^{-1}(x)) = \underline{(-\infty, \infty)}$,

(d.) $\text{range}(\sin^{-1}(x)) = \underline{[-\pi/2, \pi/2]}$,

(f.) $\text{range}(\tan^{-1}(x)) = \underline{(-\pi/2, \pi/2)}$.

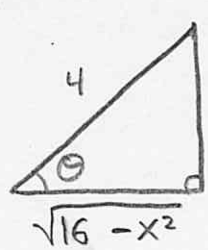




Problem 6 [4pts] Express $\cos(\underbrace{\sin^{-1}(x/4)}_{\theta})$ as an algebraic function.

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

$\sin\theta = \frac{x}{4}$



$\Rightarrow$ $\boxed{\cos\left(\sin^{-1}\left(\frac{x}{4}\right)\right) = \frac{\sqrt{16-x^2}}{4}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$

Problem 7 [8pts] Given $a, b, c, d \in \mathbb{R}$ with $a, b, c, d \neq 0$ determine if the following are true or false (circle T or F accordingly):

(i.) **T** | **F** $\quad (a+b)c = cb+ac$

(ii.) **T** | **F** $\quad \dfrac{a+b}{c} = \dfrac{a}{c}+\dfrac{b}{c}$

(iii.) **T** | **F** $\quad \dfrac{c}{a+b} = \dfrac{c}{a}+\dfrac{c}{b}$ $\quad : 1 = \frac{4}{2+2} \neq \frac{4}{2} + \frac{4}{2} = 2+2 = 4.$

(iv.) **T** | **F** $\quad \sqrt{a^2} = a$ $\quad : \sqrt{(-5)^2} = \sqrt{25} = 5 \neq -5$

(v.) **T** | **F** $\quad \sqrt[3]{a^3} = a$

(vi.) **T** | **F** $\quad \sin(a+b) = \sin(a)+\sin(b)$ $\quad 0 = \sin\left(\frac{\pi}{2}+\frac{\pi}{2}\right) \neq \sin \pi/2 + \sin \pi/2 = 2.$

(vii.) **T** | **F** $\quad$ given that $a, b > 0$, $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ $\quad \sqrt{1+1} \neq \sqrt{1}+\sqrt{1}$

(viii.) **T** | **F** $\quad$ If $\sin(\theta) = a$ then $\theta = \sin^{-1}(a) \in \mathbb{R}$. $\quad \sin^{-1}(2)$ d.n.e. (see 5a.)

Problem 8 [8pts] Find the limit, if indeterminant show all work needed to resolve limit. If the limit does not converge then explain why it fails and write ∞ or $-\infty$ where appropriate.

$$\lim_{x\to-3} \frac{x^2+4x+3}{x^2-9}. = \lim_{x\to-3}\left(\frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}(x-3)}\right)$$

$$= \lim_{x\to-3}\left(\frac{x+1}{x-3}\right)$$

$$= \frac{-3+1}{-3-3}$$

$$= \frac{-2}{-6}$$

$$= \boxed{1/3}$$

Problem 9 [8pts] Find the limit, if indeterminant show all work needed to resolve limit. If the limit does not converge then explain why it fails and write ∞ or $-\infty$ where appropriate.

$$\lim_{x\to2}\left[\frac{1}{x-2} - \frac{2}{x^2-2x}\right] = \lim_{x\to2}\left[\frac{1}{x-2} - \frac{1}{x(x-2)}\right]$$

$$= \lim_{x\to2}\left[\frac{1}{x-2}\left[1 - \frac{2}{x}\right]\right] \qquad \left(\begin{matrix}\text{type}\\ \infty\cdot 0\end{matrix}\right)$$

$$= \lim_{x\to2}\left[\frac{1}{\cancel{x-2}}\left[\frac{\cancel{x-2}}{x}\right]\right]$$

$$= \boxed{\frac{1}{2}}$$

Problem 10 [8pts] Find the limit, if indeterminant show all work needed to resolve limit. If the limit does not converge then explain why it fails and write ∞ or $-\infty$ where appropriate.

$$\lim_{x\to 0+}\left[\frac{\frac{1}{x^3}+3}{2-\frac{1}{x^2}}\right] = \lim_{x\to 0^+}\underbrace{\left[\frac{1+3x^3}{2x^3-x}\right]}_{\text{type } \frac{1}{0}}$$

multiplied by $\frac{x^3}{x^3}$

As $x\to 0^+$ we have $x\approx\delta$ for small, positive δ,

$$\frac{1+3x^3}{2x^3-x} = \frac{1+3x^3}{x(2x^2-1)} \approx \frac{1+3\delta^3}{\delta(2\delta^2-1)} \approx \frac{1}{-\delta}$$

Thus $\boxed{\lim_{x\to 0^+}\left[\frac{1/x^3+3}{2-1/x^2}\right] = -\infty}$

Problem 11 [8pts] Find the limit, if indeterminant show all work needed to resolve limit. If the limit does not converge then explain why it fails and write ∞ or $-\infty$ where appropriate.

$$\lim_{\theta\to 0+}(\cos\theta\sec\theta+\sin\theta\csc\theta) = \lim_{\theta\to 0^+}\left(\cos\theta\frac{1}{\cos\theta}+\sin\theta\frac{1}{\sin\theta}\right)$$

$$= \lim_{\theta\to 0^+}(1+1)$$

$$= \boxed{2.}$$

Problem 12 [8pts] Find the limit, if indeterminant show all work needed to resolve limit. If the limit does not converge then explain why it fails and write ∞ or $-\infty$ where appropriate.

$$\lim_{x\to 5}\frac{\sqrt{x+11}-4}{x-5} = \lim_{x\to 5}\left[\frac{\sqrt{x+11}-4}{x-5}\right]\left[\frac{\sqrt{x+11}+4}{\sqrt{x+11}+4}\right]$$

$$= \lim_{x\to 5}\left(\frac{x+11-16}{x-5(\sqrt{x+11}+4)}\right)$$

$$= \lim_{x\to 5}\left(\frac{1}{\sqrt{x+11}+4}\right)$$

$$= \frac{1}{\sqrt{16}+4} = \boxed{\frac{1}{8}.}$$

Problem 13 [8pts] Let $f(x)=\sqrt{x+11}$ and calculate $f'(5)$. $f(5)=\sqrt{16}=4.$

Observe $f'(5) = \lim_{x\to 5}\left(\frac{f(x)-f(5)}{x-5}\right) = \lim_{x\to 5}\left(\frac{\sqrt{x+11}-4-\cancel{(4-4)}}{x-5}\right) = \boxed{\frac{1}{8}}.$

Problem 14 [5pts] Use a theorem we discussed in lecture to show $\lim_{x\to 0} x^2\sin(1/x^2) = 0$.

$$-1 \le \sin\left(\tfrac{1}{x^2}\right) \le 1 \quad \text{for} \quad x \neq 0$$

$$\Rightarrow -x^2 \le x^2\sin\left(\tfrac{1}{x^2}\right) \le x^2 \quad \text{for} \quad x\neq 0$$

But, $\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$ hence by Squeeze Th$^{\underline{m}}$ we find $\lim_{x\to 0} x^2\sin\left(\tfrac{1}{x^2}\right) = 0.$

Problem 15 [5pts] Show there exists a solution to the equation below: hint: think about the interval [0, 4].

$$\underbrace{\sqrt{4x} - \sqrt[3]{x/4}}_{f(x)} = 1 \qquad (*)$$

$f(x)$ is clear f continuous on $[0,4]$.
Consider, $f(0) = 0$ whereas $f(4) = \sqrt{16} - \sqrt[3]{1} = 3$.
Observe 1 is between 0 and 3 hence by IVT $\exists c \in (0,4)$ s.t. $f(c) = 1$. This value c is a sol$^{\underline{n}}$ to $(*)$.

Problem 16 [10pts] Let $f(x) = x^2$. Show, by the definition, that $f'(x) = 2x$ for $x \neq 0$.

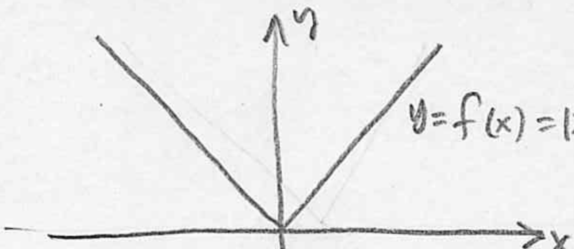
$$f'(x) = \lim_{h\to 0}\left[\frac{(x+h)^2 - x^2}{h}\right] = \lim_{h\to 0}\left[\frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\cancel{h}(2x+h)}{\cancel{h}}\right] = \lim_{h\to 0}(2x+\underset{0}{\underset{\downarrow}{h}}) = 2x. //$$

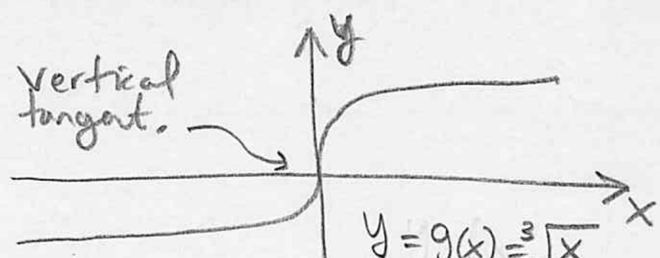
Problem 17 [4pts] Suppose the tangent line to $y = f(x)$ at $x = 2$ has equation $y = 42 + 13(x-2)$. Fill in the blanks:

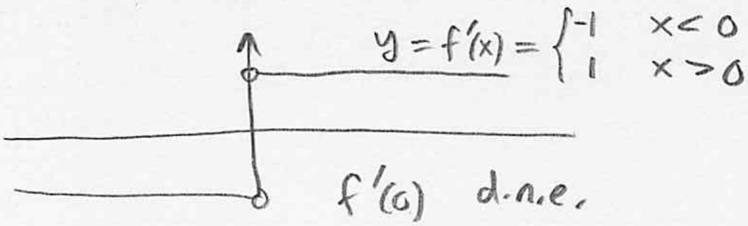
$f(2) =$ ___42___. $\qquad f'(2) =$ ___13___.

Problem 18 [4pts] Give examples of functions which are continuous at $x = 0$, but fail to be differentiable for two distinct reasons. Graph or formulas will do, but if you graph, be careful.

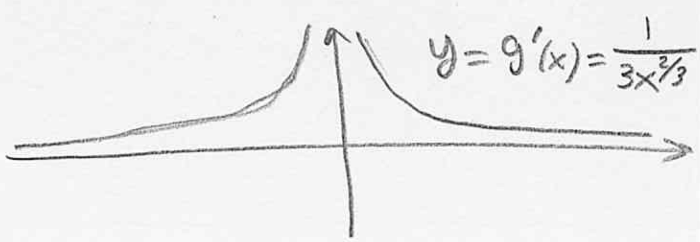


kink, or corner in graph

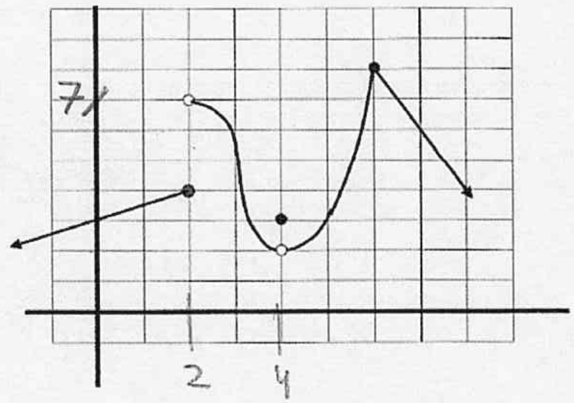




$f'(0)$ d.n.e.

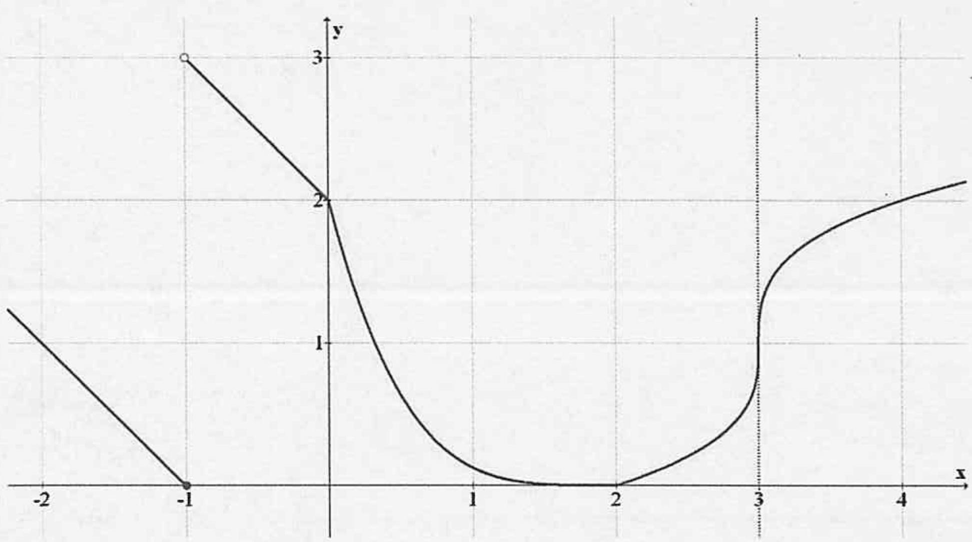


Problem 19 [8pts] Use the graph of $y = f(x)$ to determine the limits following the graph (assume a 1-unit grid):



(a.) $\lim_{x \to 2^+} f(x) = \boxed{7}$ (b.) $\lim_{x \to 2^-} f(x) = \boxed{4}$ (c.) $\lim_{x \to 4} f(x) = \boxed{2}$ (d.) $f(4) = \boxed{3}$

Problem 20 [8pts] Read the derivatives from the graph of $y = f(x)$ given below, if possible. If not possible briefly explain why.



(a.) $f'(-1)$
d.n.e
f is not even continuous at x = -1.

(b.) $f'(0)$
d.n.e.
there is kink in graph. We could say left/right derivatives do not agree.

(c.) $\underbrace{f'(1.7) = 0}$
best guess from graph looks horizontal

(d.) $\underbrace{f'(3) = \text{d.n.e.}}$
vertical tangent at x = 3.

Problem 21 [4pts] Completely factor $f(x) = x^4 + 4x^3 - 4x^2 - 36x - 45$ given that $x = -2 + i$ is a complex zero of this polynomial.

$x = -2+i$ a sol^{n} $\Rightarrow (x+2)^2+1$ is factor.

$$\begin{array}{r} x^2 - 9 \\ x^2+4x+5 \,\overline{)\, x^4+4x^3-4x^2-36x-45} \\ x^4+4x^3+5x^2 \\ \hline -9x^2-36x-45 \\ -9x^2-36x-45 \\ \hline \boxed{0} \end{array}$$

$\therefore f(x) = (x^2+4x+5)(x^2-9)$

$\boxed{f(x) = (x^2+4x+5)(x+3)(x-3)}$

Problem 22 [4pts] Let $f(x) = x^3 + 5x^2 + 6x$. Find $f^{-1}(0, \infty)$.

$x^3+5x^2+6x = x(x^2+5x+6) = x(x+2)(x+3)$

$$\begin{array}{ccccccccl} --- & | & +++ & | & --- & | & +++ & \to & f(x) \\ & -3 & & -2 & & 0 & & & \end{array}$$

$f^{-1}(0,\infty) = \{x \in \mathbb{R} \mid f(x) \in (0,\infty)\}$

$= \boxed{(-3,-2) \cup (0,\infty)}$ by sign-chart & Bolzano's Th^{m}.

Problem 23 [4pts] Suppose f is even (this means $f(-x) = f(x)$ for all $x \in \text{dom}(f)$ and continuous on $\mathbb{R}$. Show that f' is an odd function in this case.

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\left(\frac{f(-x+h)-f(-x)}{h}\right) && : \text{def}^{n} \text{ of } f' \\ &= \lim_{h\to 0}\left(\frac{f(-(x-h))-f(-x)}{h}\right) && : \text{algebra.} \\ &= \lim_{h\to 0}\left(\frac{f(x-h)-f(x)}{h}\right) && : f \text{ is even.} \\ &= \lim_{k\to 0}\left(\frac{f(x+k)-f(x)}{-k}\right) && : \text{substitute } k=-h \\ &= -f'(x) && : \text{def}^{n} \text{ of } f' \quad \therefore f'(-x) = -f'(x) \end{aligned}$$

Thus f' odd. //

Problem 24 [4pts] Find the limit, if indeterminant show all work needed to resolve limit. If the limit does not converge then explain why it fails and write ∞ or $-\infty$ where appropriate.

$$\lim_{x\to 0^-} \sin(x)\sqrt{1 - \frac{4}{\cos^2(x) - 1}} = \lim_{x\to 0^-} -|\sin x|\sqrt{1-\frac{4}{\cos^2 x - 1}}$$

[for $x \approx -\delta$ for small $\delta > 0$ we have $\sin x < 0$ $\Rightarrow |\sin x| = -\sin x$

$$= \lim_{x\to 0^-} -\sqrt{|\sin x|^2\left[1-\frac{4}{\cos^2 x-1}\right]}$$

$$= -\lim_{x\to 0^-}\sqrt{\sin^2 x\left[1+\frac{4}{\sin^2 x}\right]}$$

[noted $|\sin x|^2 = \sin^2 x$, $\cos^2 x - 1 = -\sin^2 x$

$$= -\lim_{x\to 0^-}\sqrt{\sin^2 x + 4} = -\sqrt{\sin^2(0)+4} = \boxed{-2.}$$