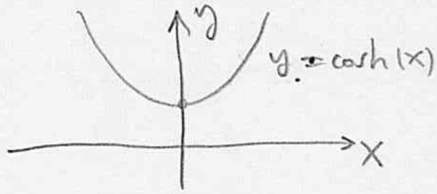
Show your work. No graphing calculators. This is a timed test, use your time wisely, some problems are worth more than others. All arguments should be given in notation used in lecture. Thanks!

Problem 1 [5pts] Define $\cosh(x)$ and $\sinh(x)$ and sketch their graphs. State the domain and range for each in interval notation.

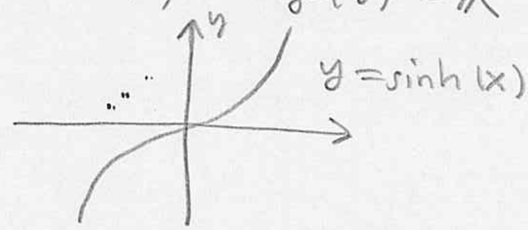
$f(x) = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$

dom$(f) = \mathbb{R}$, range$(f) = [1, \infty)$



$g(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$

dom$(g) = \mathbb{R}$, range$(g) = \mathbb{R}$



Problem 2 [10pts] State (can derive on scratch paper if you wish) the following basic derivatives:

$(a.)\ \frac{d}{dx}[\sqrt{x}\,] = \underline{\frac{1}{2\sqrt{x}}}$, $(b.)\ \frac{d}{dx}[1/x] = \underline{-1/x^2}$,

$(c.)\ \frac{d}{dx}[x^n] = \underline{nx^{n-1}}$, $(d.)\ \frac{d}{dx}[\cos(x)] = \underline{-\sin x}$,

$(e.)\ \frac{d}{dx}[\sec(x)] = \underline{\sec x \tan x}$, $(f.)\ \frac{d}{dx}[\cosh(x)] = \underline{\sinh x}$,

$(g.)\ \frac{d}{dx}[\cot(x)] = \underline{-\csc^2 x}$, $(h.)\ \frac{d}{dx}[e^x] = \underline{e^x}$,

$(i.)\ \frac{d}{dx}[\ln(x)] = \underline{1/x}$, $(j.)\ \frac{d}{dx}[10^x] = \underline{(\ln 10)10^x}$.

Problem 3 [5pts] Given that $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0}\frac{1-\cos(h)}{h} = 0$, $\Rightarrow \lim\limits_{h\to 0}\left[\frac{\cos(h)-1}{h}\right] = 0.$ prove that $\frac{d}{dx}\sin(x) = +\cos(x)$.

$$\frac{d}{dx}[\sin x] = \lim_{h\to 0}\left[\frac{\sin(x+h) - \sin(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin x}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin x\left[\frac{\cos(h)-1}{h}\right] + \cos(x)\left[\frac{\sin(h)}{h}\right]\right]$$

$$= \sin x \underbrace{\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)}_{0} + \cos x \underbrace{\lim_{h\to 0}\left[\frac{\sin(h)}{h}\right]}_{1} = \underline{\cos(x)}.$$

Problem 4 [5pts] Show that $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$.

Let $y = \sin^{-1}(x) \Rightarrow x = \sin(y) \Rightarrow 1 = \cos(y)\frac{dy}{dx}$

But, $\sin^2 y + \cos^2 y = 1 \Rightarrow \cos(y) = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$

Thus, $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1-x^2}}$. However, $y = \sin x$ is increasing on $[-\pi/2, \pi/2]$ for which $\sin^{-1}(x)$ is inverse $\boxed{\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}}$

Problem 5 [60pts] Complete the indicated derivatives. Do not simplify answers, thanks! Assume a, b, c are constants.

(a.) $\frac{d}{dx}\left[(2x+3)^2\sin(e^x)\right] = 2(2x+3)\frac{d}{dx}(2x+3)\sin(e^x) + (2x+3)^2\cos(e^x)\frac{d}{dx}[e^x]$

$= \boxed{4(2x+3)\sin(e^x) + e^x(2x+3)^2\cos(e^x)}$

$= (2x+3)\left[4\sin(e^x) + e^x(2x+3)\cos(e^x)\right].$

(b.) $\frac{d}{dx}\left[\cos\sqrt{x}+\sqrt{e^x+x}\right] = -\sin\sqrt{x}\,\frac{d}{dx}(\sqrt{x}) + \frac{1}{2\sqrt{e^x+x}}\frac{d}{dx}\left[e^x+x\right]$

$= \boxed{\frac{-\sin\sqrt{x}}{2\sqrt{x}} + \frac{e^x+1}{2\sqrt{e^x+x}}}$

(c.) $\frac{d}{dx}\left[\frac{x}{x^2+7}+\underbrace{e^3}_{0}\right] = \underbrace{\frac{1(x^2+7)-x(2x)}{(x^2+7)^2}}_{\text{also fine.}} = \boxed{\frac{7-x^2}{(x^2+7)^2}}$

(d.) $\frac{d}{dx}\left[(1+ax)^3(1+bx)\right] = 3(1+ax)^2\frac{d}{dx}(1+ax) + (1+ax)^3(b)$

$= \boxed{3a(1+ax)^2 + b(1+ax)^3}$

$= (1+ax)^2\left[3a+b(1+ax)\right]$

$= (1+ax)^2\left[3a+b+abx\right].$

Problem 6 [10pts] Calculate dy/dx via the technique of logarithmic differentiation (do not simplify answer and you may use y in your answer)

$$y = \frac{(x^2-3)^8\sqrt{42e^x}}{(3x-8)^6}$$

$$\ln y = 8\ln(x^2-3) + \tfrac{1}{2}\ln 42 + \tfrac{1}{2}x - 6\ln(3x-8)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{16x}{x^2-3} + \frac{1}{2} - \frac{18}{3x-8}$$

$$\boxed{\frac{dy}{dx} = \underbrace{\frac{(x^2-3)^8\sqrt{42e^x}}{(3x-8)^6}}_{\text{a.k.a. } y}\left[\frac{16x}{x^2-3} + \frac{1}{2} - \frac{18}{3x-8}\right]}$$

Problem 7 [10pts] Suppose $x^3 + y^3 = 5 + y^2$.

(a.) Calculate dy/dx via the technique of implicit differentiation.

(b.) find the tangent line at $(1, 2)$.

(a.) $$3x^2 + 3y^2\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-3x^2}{2y-3y^2}} = \frac{3x^2}{y(2-3y)}$$

(b.) $$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{3(1)^2}{2(2)-3(4)} = \frac{3}{4-12} = \frac{-3}{8} \Leftarrow \text{slope of tangent.}$$

$$\boxed{y = 2 - \frac{3}{8}(x-1)}$$

Problem 8 [5pts] Suppose f, g are positive-valued functions (this means $f(x), g(x) > 0$ for all x). Let $y = fg$ and derive the product rule via logarithmic-differentiation.

$\ln(y) = \ln f + \ln g$ (reasonable as $f(x), g(x) > 0$)

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{f}\frac{df}{dx} + \frac{1}{g}\frac{dg}{dx}$$

$$\Rightarrow \frac{d}{dx}(fg) = \frac{y}{f}\frac{df}{dx} + \frac{y}{g}\frac{dg}{dx} = \underline{g\frac{df}{dx} + f\frac{dg}{dx}}.$$

<u>Remark</u>: the chainrule is the most useful rule.

Problem 9 [10pts] Suppose a car travels along a path with equation $xy = 10$. If you measure $\frac{dx}{dt} = 3$ at $(-2, -5)$ then what is $\frac{dy}{dt}$ at $(-2, -5)$?

$$\frac{d}{dt}(xy) = \frac{d}{dt}(10) \Rightarrow \frac{dx}{dt}y + x\frac{dy}{dt} = 0$$

$$\Rightarrow 3(-5) + (-2)\frac{dy}{dt} = 0$$

$$\therefore \frac{dy}{dt} = \frac{-15}{-2} = \boxed{\frac{-15}{2}}$$

Choose one

Problem 12 [10pts] Suppose two trains leave a station. Train Jacob travels due West at 100mph and Train Edward travels due East at 60mph. If a Train Bella travels due North at 40mph then how fast is the sum of the distance from Bella to Jacob and Bella to Edward increasing at the time Bella has travelled 40miles.

Problem 12 [10pts] Suppose water pours into a trough which has triangular ends and a length of $10ft$. Furthermore, suppose the angles made by the upper angles in the trough are 40 degrees and 50 degrees. If the water level rises at a rate of $0.1ft/s$ when the water is $2ft$ deep then at what rate are you adding water to the trough? Please leave your answer in ft^3/s.

Choose one

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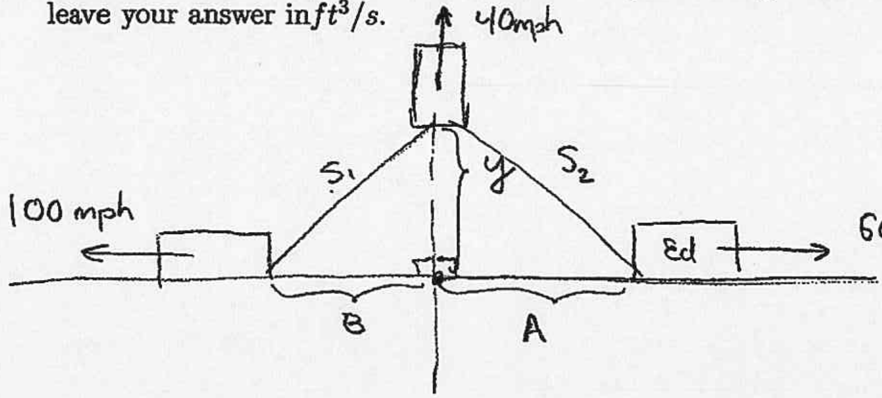

find $\frac{d}{dt}(s_1+s_2)$

when $t = 1$ hr.

$A = 60$

$B = 100$

$y = 40.$

$$y^2 + B^2 = s_1^2 \quad \text{and} \quad y^2 + A^2 = s_2^2$$

$$2s_1\frac{ds_1}{dt} = 2y\frac{dy}{dt} + 2B\frac{dB}{dt}$$

$$2s_2\frac{ds_2}{dt} = 2y\frac{dy}{dt} + 2A\frac{dA}{dt}$$

$$\frac{d}{dt}(s_1+s_2) = \frac{ds_1}{dt} + \frac{ds_2}{dt}$$

$$= \frac{y}{s_1}\frac{dy}{dt} + \frac{B}{s_1}\frac{dB}{dt} + \frac{y}{s_2}\frac{dy}{dt} + \frac{A}{s_2}\frac{dA}{dt}$$

$$= \left(\frac{1}{s_1} + \frac{1}{s_2}\right)\frac{dy}{dt} + \frac{B}{s_1}\frac{dB}{dt} + \frac{A}{s_2}\frac{dA}{dt}$$

$$\left.\frac{d}{dt}(s_1+s_2)\right|_{t=1} = \left(\frac{1}{\sqrt{40^2+100^2}} + \frac{1}{\sqrt{40^2+60^2}}\right)40 + \frac{(100)(100)}{\sqrt{40^2+100^2}} + \frac{(60)(60)}{\sqrt{40^2+60^2}}$$

$$= \left(\frac{1}{107.7} + \frac{1}{72.11}\right)(40)^2 + \frac{10000}{107.7} + \frac{3600}{72.11}$$

$$= 37.04 + 92.851 + 49.92$$

$$= \boxed{179.8\text{ mph}}$$

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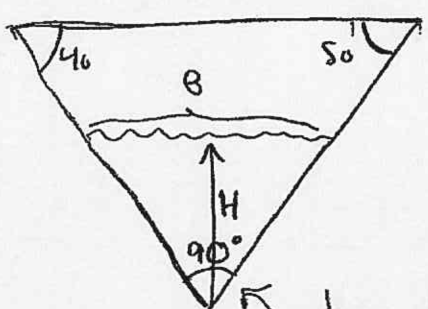
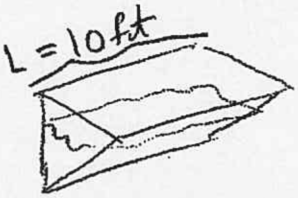
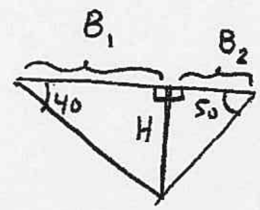


has to be 90°.



$$\frac{dH}{dt} = 0.1 \frac{ft}{s}$$

when $H = 2ft$.



$$Vol = V = \frac{1}{2} BHL$$

$$B_1 + B_2 = B$$

$$\tan 40 = \frac{H}{B_1} \quad \Big| \quad \tan 50 = \frac{H}{B_2}$$

$$B_1 = \frac{H}{\tan 40} \quad \Big| \quad B_2 = \frac{H}{\tan 50}$$

$$\Rightarrow B = H\left(\frac{1}{\tan 40} + \frac{1}{\tan 50}\right)$$

$$B \cong 2.0309\, H$$

$$V = (0.5)(2.0309)(10)H^2$$

$$\frac{dV}{dt} = 20.309\, H \frac{dH}{dt}$$

$$\left.\frac{dV}{dt}\right|_{H=2ft} = (20.309)(2)(0.1)$$

$$\Rightarrow \boxed{4.062\ ft^3/s}$$