Do not omit scratch work. I need to see all steps. Skipping details will result in a loss of credit. Thanks. You are allowed the use of a scientific (non-graphing) calculator. No electronic communication devices of any kind permitted, no IPODs, Zunes, Walkmans etc... This is a timed test and time is likely to be an issue for you, budget your time wisely. There are at least 150pts to earn on this exam.

Problem 1 [7pts] Let $f(x) = x^2 + x$. Calculate:

(a.) $f'(x) = \underline{2x+1}$, (b.) $\frac{d}{dx}[f'(x)] = \underline{2}$.

Problem 2 [63pts] Complete the indicated derivatives. If more than one variable is present in an expression you may assume that the variables are independent.

(a.) $\frac{d}{dx}\left[\frac{x^2+3x+7}{\sqrt{x}}+\pi^3\right] = \frac{d}{dx}\left[x^{3/2} + 3x^{1/2} + 7x^{-1/2}\right] = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-1/2} - \frac{7}{2}x^{-3}$

(b.) $\frac{d}{db}\left[ax^2+bx+c\right] = x.$

(c.) $\frac{d}{dx}\left[(1-3x)^6(x^2+x)^3\right] = 6(1-3x)^5(-3)(x^2+x)^3 + (1-3x)^6 3(x^2+x)^2(2x+1)$

(d.) $\frac{d}{dx}\left[\sqrt{3-x}\right] = \frac{1}{2\sqrt{3-x}}(-1)$

(e.) $\frac{d}{dx}\left[\sqrt[3]{x+\sqrt{4x^2}}\right] = \frac{1}{3}(x+\sqrt{4x^2})^{-2/3}\left(1+\frac{4x}{\sqrt{4x^2}}\right)$

(f.) $\frac{d}{dx}\left[\ln(Ae^{Bx+C})\right] = \frac{d}{dx}\left[\ln(A) + Bx + C\right] = B.$

(g.) $\frac{d}{dx}\left[\sin[x^3+2\cos(7x))]\right] = \cos(x^3+2\cos 7x)\cdot(3x^2-14\sin 7x).$

(h.) $\frac{d}{dx}\left[sin^{-1}(3e^x)\right] = \frac{1}{\sqrt{1-(3e^x)^2}}(3e^x).$

(i.) $\frac{d}{dx}\left[\frac{\cot(x)}{x-e^{-2x}}\right] = \frac{-\csc^2(x)(x-e^{-2x}) - \cot(x)(1+2e^{-2x})}{(x-e^{-2x})^2}$

Problem 3 [10pts] Let $a > 0$ with $a \neq 1$. **Show that** $\frac{d}{dx}[a^x] = \ln(a)a^x$.

Let $y = a^x \Rightarrow \ln(y) = \ln(a^x) = x\ln(a)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln(a)$

$\Rightarrow \frac{dy}{dx} = \ln(a)\, y$

$\Rightarrow \underline{\frac{dy}{dx} = \ln(a)\, a^x}.$

Problem 4 [10pts] You are given standard trigonometric identities and $\lim_{h\to 0}\left[\frac{sin(h)}{h}\right] = 1$ and $\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right] = 0$. **Prove that** $\frac{d}{dx}[\sin(x) = \cos(x)]$.

$$\frac{d}{dx}\left(\sin(x)\right) = \lim_{h\to 0}\left(\sin(x+h) - \sin(x)\right)\frac{1}{h}$$

$$= \lim_{h\to 0}\left[\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}\right]$$

$$= \sin x \underbrace{\lim_{h\to 0}\left[\frac{\cos h - 1}{h}\right]}_{\to 0} + \cos(x)\underbrace{\lim_{h\to 0}\left(\frac{\sin h}{h}\right)}_{\to 1}$$

$$= \underline{\cos(x)}.$$

Problem 5 [5pts] Given $y = x + [\cos(x)]^{\sin(x)}$ calculate dy/dx.

$$\frac{dy}{dx} = 1 + \frac{d}{dx}\left(\underbrace{\cos(x)^{\sin(x)}}_{z}\right) = \boxed{1 + \left[\cos(x)\ln(\cos(x)) - \frac{\sin^2 x}{\cos x}\right]\cos x^{\sin x}}$$

$$z = \cos(x)^{\sin(x)}$$

$$\ln(z) = \sin(x)\ln(\cos x)$$

$$\frac{1}{z}\frac{dz}{dx} = \cos x \ln(\cos x) - \frac{\sin^2(x)}{\cos(x)}$$

Problem 6 [10pts] Calculate dy/dx via the technique of logarithmic differentiation (do not simplify answer and you may use y in your answer)

$$y = \frac{3e^{2x+1}(x+1)(x^2-3)^7}{\sqrt{1+\sin(x)}}$$

$$\ln(y) = \ln(3) + 2x + 1 + \ln(x+1) + 7\ln(x^2-3) - \frac{1}{2}\ln(1+\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[2 + \frac{1}{x+1} + \frac{14x}{x^2-3} - \frac{\cos(x)}{2(1+\sin x)}\right]$$

Problem 7 [10pts] Suppose $(x+y)^3 = y^3 + 2x + 17$.

(a.) Calculate dy/dx via the technique of implicit differentiation.

(b.) find the tangent line at $(1, 2)$.

(a.) $$3(x+y)^2 \frac{d}{dx}(x+y) = 3y^2 \frac{dy}{dx} + 2$$

$$3(x+y)^2 + 3(x+y)^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2 - 3(x+y)^2}{3(x+y)^2 - 3y^2}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{2-3(9)}{27-3(4)} = \frac{-25}{15} = \frac{-5}{3}$$

(b.) $$y = 2 + \left.\frac{dy}{dx}\right|_{(1,2)} (x-1)$$

$$y = 2 - \frac{5}{3}(x-1)$$

Problem 8 [5pts] Approximate $\sqrt[4]{17}$ by considering the linearization of $f(x) = \sqrt[4]{x}$ centered at $a = 16$.

$L_f^{16}(x) = f(16) + f'(16)(x-16)$

$f'(x) = \frac{1}{4} x^{-3/4}$

$f'(16) = \frac{1}{4(\sqrt[4]{16})^3}$

$= \frac{1}{4(8)}$

$L_f^{16}(x) = 2 + \frac{1}{32}(x-16)$

$L_f^{16}(17) = 2 + \frac{1}{32}(17-16) = \boxed{\frac{65}{32} \approx \sqrt[4]{17}}$

Problem 9 [5pts] Argue for or against the following claim: "f is a differentiable function on $\mathbb{R}$" where f is given as follows:

$$f(x) = \begin{cases} \cos(x) + 1 & if\ x < \pi/2 \\ \cos(x) - 1 & if\ x \geq \pi/2 \end{cases}$$

$2.03125\ldots$

$32 \overline{)65}$
64
100
96
40
32
80

Notice $\lim_{x \to \pi/2^-} (f(x)) = 2$

whereas $\lim_{x \to \pi/2^+} (f(x)) = 0$

Thus f not continuous at $x = 0$

$\therefore f$ not diff. at $x = 0$. $\Rightarrow f$ not diff on $\mathbb{R}$.

Problem 10 [10pts] Suppose some guy named Frank tells you that $P = mv$. If you are given that $\frac{dP}{dt} = 10$ and $\frac{dv}{dt} = 3$ when $m = 1$ and $v = 2$ then at what rate is m changing?

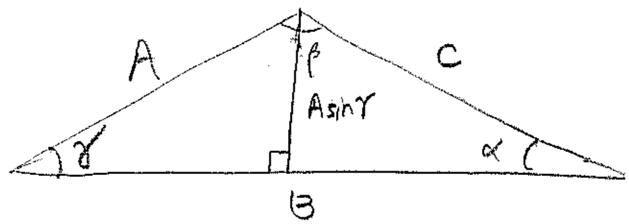
$P = mV \Rightarrow \frac{dP}{dt} = \frac{dm}{dt} V + m\frac{dV}{dt}$

$\Rightarrow 10 = \frac{dm}{dt}(2) + (1)(3)$

$\Rightarrow \boxed{\frac{dm}{dt} = \frac{7}{2}}$

Problem 11 [20pts] Suppose that three sticks are joined to make a triangle. Furthermore, suppose these three sticks of lengths A, B, C are placed on a flat surface. Label the angles of this triangle by α, β, γ which are opposite sides A, B, C respective. You observe that α is increasing at rate of 1*degree* per second and β is decreasing at a rate of 3 degrees per second. **Find the rate of change of the area in this triangle.**

$\alpha + \beta + \gamma = 180^\circ$



$$\mathcal{A} = \text{AREA} = (\text{BASE})(\text{HEIGHT})/2$$

$$= \frac{B \cdot A \sin(\gamma)}{2}$$

$$\frac{d\mathcal{A}}{dt} = \frac{AB}{2}\cos\gamma \frac{d\gamma}{dt}$$

BUT, THIS PROBLEM IS NOT CORRECT SINCE

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$

$$\Rightarrow \underbrace{0 = 0 + 2AB\sin\gamma \frac{d\gamma}{dt}}_{\text{Contradiction!}} \quad \text{if } \frac{dA}{dt} = \frac{dB}{dt} = \frac{dC}{dt} = 0.$$

(I asked in lecture to choose an option

1.) explain why impossible

2.) allow $\frac{dA}{dt} \neq 0$ and solve.

Problem 12 [5pts] If a function f is continuously differentiable at $a \in dom(f)$ such that $f'(a) \neq 0$ then what does that tell you about the function? What interesting result follows given this data?

If $f'(a) \neq 0$ then either $f'(a) > 0$ near a or $f'(a) < 0$ near a applying Bolzano's Thm to the continuous derivative function. Thus f is monotonic near a hence $f|_{B_\delta(a)}^{-1}$ exists for some $\delta > 0$.

Problem 13 [5pts] Let p be a polynomial function of degree n for some $n > 1$. Define $f(x) = p(x)e^{cx}$ where c is a constant. Discuss how many horizontal tangents are possible for f on $\mathbb{R}$.

$$\frac{df}{dx} = \frac{d}{dx}\left[Pe^{cx}\right] = \frac{dP}{dx}e^{cx} + Pce^{cx} = \underbrace{\left(\frac{dP}{dx} + cP\right)}_{\text{degree } n \text{ polynomial}}e^{cx}$$

$$\frac{df}{dx} = 0 \Rightarrow \underbrace{\frac{dP}{dx} + cP}_{n^{th} \text{ order polynomial has at most } n\text{-real zeros}} = 0 \quad \text{since } e^{cx} \neq 0 \ \forall x \in \mathbb{R}$$

$\therefore$ There are j-horizontal tangents where $0 \leq j \leq n$.

Problem [10pts] Suppose that $u \in C^2(\mathbb{R})$ in this problem. Suppose $g(u) = \sinh(u^2)$. Define $h(x) = g(u(x))$.

1. calculate $\frac{dh}{dx}$
2. calculate $h''(x)$
3. does $y = h(x)$ have any horizontal tangents?

1.) $\dfrac{dh}{dx} = \dfrac{d}{dx}\left[\sinh(u^2)\right] = \cosh(u^2)\dfrac{d}{dx}(u^2) = \underline{2u\cosh(u^2)\dfrac{du}{dx}}$.

2.) $h''(x) = 2u'\cosh(u^2)\dfrac{du}{dx} + 2u\sinh(u^2)\,2u\left(\dfrac{du}{dx}\right)^2 + 2u\cosh(u^2)\dfrac{d^2u}{dx^2}$

$= \underline{\left[2\left(\dfrac{du}{dx}\right)^2 + 2u\dfrac{d^2u}{dx^2}\right]\cosh(u^2) + 4u^2\sinh(u^2)\left(\dfrac{du}{dx}\right)^2}$.

3.) Horizontal tangents have $\frac{dh}{dx} = 0$. Now $\cosh(u^2) \neq 0$ because $\cosh(u^2) = \frac{1}{2}(e^{u^2} + e^{-u^2}) > 0$ hence we either need $u = 0$ or $\frac{du}{dx} = 0$ (or both). Thus, h may have horizontal tangents at $x = x_o$ if $u(x_o) = 0$ or $\left.\frac{du}{dx}\right|_{x_o} = 0$.