CALCULUS 131:	TEST 3: APPLICATIONS OF DIFFERENTIATION

Do not omit scratch work. I need to see *all* steps. Skipping details will result in a loss of credit. Thanks. You are allowed the use of a scientific (non-graphing) calculator. No electronic communication devices of any kind permitted, no IPODs, Zunes, Walkmans etc... This is a timed test and time is likely to be an issue for you, budget your time wisely. There are at least 150pts to earn on this exam.

Problem 1 [9pts] State the following theorems, make sure to include the preconditions which are necessary for the theorem to hold true.

1. Rolle's Theorem: Suppose f is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$. Then $\exists c \in (a,b)$ such that $f'(c) = 0$.

2. The Mean Value Theorem: $\ni$ f is ~~continuous on $[a,b]$,~~ diff. on (a,b) then $\exists\, c \in (a,b)$ such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

3. The Extreme Value Theorem: If f is a continuous fnct and $[a,b] \subseteq dom(f)$ then f attains its absolute max/min on $[a,b]$.

Problem 2 [11pts] Show that $\underbrace{2x - 1 - \sin(x)}$ $= 0$ has exactly one real root.

$$f(x) = 2x - 1 - \sin(x)$$
$$\frac{df}{dx} = 2 - \cos(x) \geq 1 > 0.$$

Furthermore, observe that $f(0) = -1$ whereas $f(\pi) = \pi - 1 > 0$ $\therefore$ $\exists c \in (0,\pi)$ such that $f(c) = 0$ by IVT. (clearly f is continuous)

Now suppose $\exists$ a 2^{nd} zero at c_2 then $f(c_2) = 0$. Note, f is cont., diff, & $f(c) = f(c_2) = 0$ hence Rolle's Th$^{\underline{m}}$ $\Rightarrow$ $f'(x) = 0$ for some $x \in \mathbb{R}$. However, this contradicts $f'(x) > 0$ for all $x \in \mathbb{R}$. Therefore, no second real zero can exist.

Problem 3 [15pts] Let $f(x) = \ln(x^2 + 4x + 5)$. Locate all critical numbers for f and use the second derivative test to find the local extrema for the function.

$$\frac{df}{dx} = \frac{2x+4}{\underbrace{x^2+4x+5}_{\text{nonzero always}}} \Rightarrow f'(x) = 0 \text{ iff } 2x+4 = 0$$

$$\Rightarrow \underline{x = -2}$$

since $x^2+4x+5 = (x+2)^2 + 1$.

$$f''(x) = \frac{2(x^2+4x+5) - (2x+4)(2x+4)}{(x^2+4x+5)^2}$$

$$f''(-2) = \frac{2(4-8+5) - (0)(0)}{(4-8+5)^2} = 2 > 0 \quad \therefore f(-2) \text{ is local min. by } 2^{nd} \text{ D. Test.}$$

Problem 4 [15pts] Find the intervals of increase and decrease for the function $f(x) = x + \frac{9}{x}$. Also, classify any local extreme values by applying the first derivative test.

$$\frac{df}{dx} = 1 - \frac{9}{x^2} = \frac{x^2-9}{x^2} = \frac{(x+3)(x-3)}{x^2}$$

Critical #'s clearly $x = 0, 3, -3$.

note $x = 0$ is VA $\Rightarrow$ not in dom f or dom f'

$\frac{df}{dx}$: $+++$ | -3 | $----$ | 0 | $-----$ | 3 | $+++$

f inc on $(-\infty, -3), (3, \infty)$

f dec on $(-3, 0), (0, 3)$.

$f(-3)$ is local max, $f(3)$ is local min. by 1^{st} der. test.

Problem 5 [15pts] Determine where $f(x) = x^4 - 2x^3 - 36x^2$ is concave up (CU) or concave down (CD). Also, find any inflection points.

$$f'(x) = 4x^3 - 6x^2 - 72x$$

$$f''(x) = 12x^2 - 12x - 72 = 12(x^2 - 1 - 6)$$

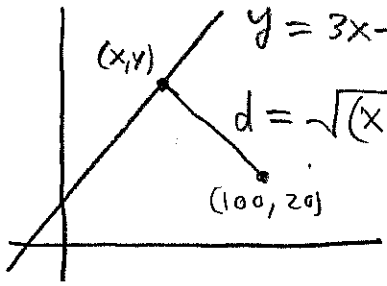
$f''(x) = 12(x-3)(x+2)$: $+++$ | -2 | $---$ | 3 | $+++$

f CU on $(-\infty, -2), (3, \infty)$

f CD on $(-2, 3)$

$(-2, f(-2))$ and $(3, f(3))$ are inflection pts.

Problem 6 [15pts] Use calculus to find the point on the line $y = 3x - 7$ which is closest to $(100, 20)$. You answer should include some indication as to how calculus shows your result is the minumum distance possible to the given line from the given point.



$y = 3x - 7$

$d = \sqrt{(x-100)^2 + (y-20)^2}$ Let $f = d^2$,

$f(x) = (x-100)^2 + (3x-27)^2$

$\frac{df}{dx} = 2(x-100) + 6(3x-27)$

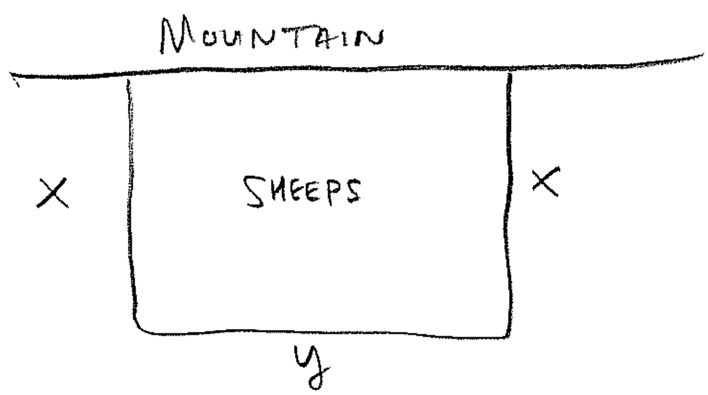
Thus $\frac{df}{dx} = 0 \Rightarrow 2x - 200 + 18x - 162 = 0$

$20x = 362 \longrightarrow x = \frac{362}{20} = \underline{18.1}$.

$f''(x) = 20 > 0$ $\therefore$ $f(18.1)$ is minimum for f and hence the minimum for d. $\Rightarrow (18.1, 3(18.1) - 7)$

$\Rightarrow \boxed{(18.1, 47.3) \text{ closest pt.}}$

Problem 7 [15pts] Suppose 300 ft of fence is used to construct a rectangular pen for some sheep. One side of the pen doesn't need a fence because there is a steep mountain which the sheep can climb. **Use calculus to find the dimensions for the fence which maximize the area enclosed.**



$2x + y = 300$

$A = xy = x(300 - 2x)$

$A = 300x - 2x^2$

$\frac{dA}{dx} = 300 - 4x = 0$

for critical #

$4x = 300$

$\underline{x = 75}$.

Note $\frac{d^2A}{dx^2} = -4 < 0$ $\therefore$ $x = 75 \Rightarrow$ max A.

$y = 300 - 2(75) = 150$ $\therefore$ $\boxed{75\text{ft} \times 150\text{ft}}$

(**please write limits where they belong, please don't make me take points off for this again... I want you to keep your points, I already have enough**)

Problem 8 [20pts] Calculate the following limits using algebra or intuition.

(*a.*) $\lim_{x\to\infty} \cos(x)$ $\boxed{\text{d.n.e.}}$ due to oscillation as $x \to \infty$.

(*b.*) $\lim_{x\to\infty} (\, 2+\tan^{-1}(x)\,)$ $= \lim_{x\to\infty} (2) + \lim_{x\to\infty} \left(\tan^{-1}(x) \right)$

$= \boxed{2 + \frac{\pi}{2}}$.

(*c.*) $\lim_{x\to -\infty} \frac{\sqrt{3x^2+2}}{x+1}$ $= \lim_{x\to-\infty} \left(\frac{\frac{-1}{x}\sqrt{3x^2+2}}{\frac{-1}{x}(x+1)} \right)$

only bring nonnegative factors into $\sqrt{}$.

$= \lim_{x\to-\infty} \left(\frac{-\sqrt{\left(\frac{-1}{x}\right)^2 (3x^2+2)}}{-1 - 1/x} \right)$

$= \lim_{x\to-\infty} \left(\frac{-\sqrt{3 + \underbrace{2/x^2}_{0}}}{-1 - \underbrace{1/x}_{0}} \right) = \boxed{-\sqrt{3}}$

(*d.*) $\lim_{x\to\infty} (\, \sqrt{9x^2+x} - 3x\,)$ $= \lim_{x\to\infty} \left(\frac{\left(\sqrt{9x^2+x} - 3x\right)\left(\sqrt{9x^2+x} + 3x\right)}{\sqrt{9x^2+x} + 3x} \right)$

$= \lim_{x\to\infty} \left(\frac{9x^2+x-9x^2}{\sqrt{9x^2+x} + 3x} \right)$

dividing numerator and denom. by x.

$= \lim_{x\to\infty} \left(\frac{1}{\sqrt{9 + \underbrace{1/x}_{0}} + 3} \right)$

$= \boxed{\frac{1}{6}}$

Problem 9 [15pts] Calculate the following limits using L'Hospital's rule where appropriate. Indicate usage of the rule either with the notation we used in lecture or in words to the side.

(a.) $\lim_{x\to-\infty}\left(xe^{x}\right) = \lim_{x\to-\infty}\left(\frac{x}{e^{-x}}\right) \underset{\left(\frac{\infty}{\infty}\right)}{\overset{\mathcal{L}}{=}} \lim_{x\to-\infty}\left(\frac{1}{-e^{-x}}\right)$

$= \lim_{x\to-\infty}\left(-e^{x}\right) = \boxed{0.}$

(b.) $\lim_{x\to0}\left[x^2e^{1/x^2}\right] = \lim_{x\to0}\left(\frac{e^{1/x^2}}{1/x^2}\right) \underset{\left(\frac{\infty}{\infty}\right)}{\overset{\mathcal{L}}{=}} \lim_{x\to0}\left(\frac{e^{1/x^2}\cancel{(-2/x^3)}}{\cancel{(-2/x^3)}}\right)$

$= \lim_{x\to0}\left(e^{1/x^2}\right)$

$= \boxed{\infty.}$

(c.) $\lim_{x\to0}\left[\frac{\sin(ax)}{\sin(bx)}\right] \underset{\left(\frac{0}{0}\right)}{\overset{\mathcal{L}}{=}} \lim_{x\to0}\left[\frac{a\cos(ax)}{b\cos(bx)}\right]$ (Let a, b be nonzero constants.)

$= \frac{a\cos(0)}{b\cos(0)}$

$= \boxed{\frac{a}{b}}$

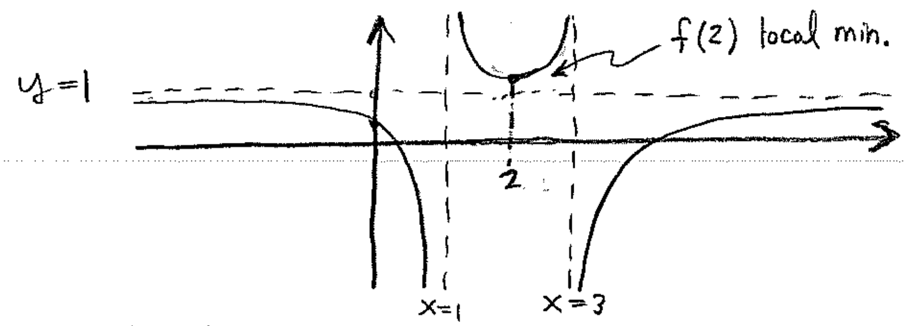
Problem 10 [10pts] Find local extrema for $f(x) = \frac{1}{x-1} - \frac{1}{x-3} + 1$ and also find the horizontal asymptotes for this function. Sketch the graph of $y = f(x)$.

$$\frac{df}{dx} = \frac{-1}{(x-1)^2} + \frac{1}{(x-3)^2} = \frac{(x-1)^2 - (x-3)^2}{(x-1)^2(x-3)^2}$$

$$= \frac{(x^2-2x+1) - (x^2-6x+9)}{(x-1)^2(x-3)^2}$$

$$= \frac{4x-8}{(x-1)^2(x-3)^2} \Rightarrow \underbrace{\begin{array}{ll} x = 2 & \text{horiz. tang} \\ x = 1,3 & \text{V.A.'s} \end{array}}_{\text{critical \#'s}}$$

$$\lim_{x \to \pm\infty} \left(\cancel{\frac{1}{x-1}}^{\to 0} - \cancel{\frac{1}{x-3}}^{\to 0} + 1 \right) = 1 \Rightarrow \boxed{\text{HA's of } y = 1 \text{ for } x \to \pm\infty}$$



f(2) local min. by 1st derivative test ∠ VA at x = 1 and x = 3 help.

$$\begin{array}{ccccccc} - & | & - \; - & | & + \; + \; + & + & \to \frac{df}{dx} \\ & 1 & & 5/2 & 3 & & \end{array}$$

Problem 11 [10pts] Let $f(x) = \sinh(x)$. Find intervals or increase, decrease and intervals of concave up/down. Use all of this information to graph $y = f(x)$.

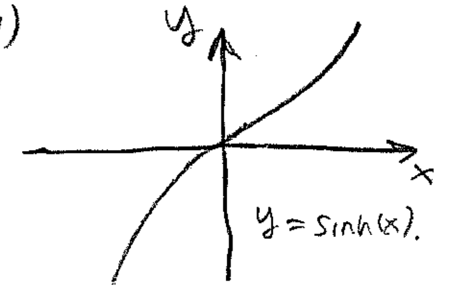
$$\frac{df}{dx} = \cosh(x) = \frac{1}{2}(e^x + e^{-x}) > 0 \quad \therefore f \text{ increasing on } \mathbb{R}.$$

$$\frac{d^2f}{dx^2} = \underbrace{\sinh(x) = \frac{1}{2}(e^x - e^{-x})}_{} \longrightarrow \begin{array}{c} - - - \; | \; + + + \to f''(x) \\ 0 \end{array}$$

note $\sinh(0) = 0$ and $\sinh(x)$ is everywhere increasing $\Rightarrow$ $\sinh(x) < 0$ for $x < 0$
$\sinh(x) > 0$ for $x > 0$

The $\sinh(x)$ fnct cannot have another real zero besides $x = 0$ since otherwise we'd get a contradiction of Rolle's Thm as in hwk.

$\therefore$ f CD on $(-\infty, 0)$
f CU on $(0, \infty)$



Problem 12 [5pts] Suppose a smooth function f has critical points at $x = -1$ and $x = 2$. Furthermore, suppose that $f''(x) < 0$ for $x > 1$ and $f''(x) > 0$ for $x < 1$.

1. what is $f''(1)$ and how do you know that to be true from the given information?
2. find the intervals of increase and decrease for f

1.) I know $f''(1) = 0$ since f'' is continuous by smoothness of f. (apply the IVT to a nbhd of $x=1$ and note $f''(x)$ changes signs on such a nbhd.)

2.)

$- \; - \; - \; - \; - \; - \; - \mid + \; + \; + \; + \quad f''(x)$

2nd D. Test ↓ max — 2nd der. test says this is min.

$+ \; + \; + \; + \mid - \; - \; - \; - \mid + \; + \; + \quad f'(x)$

$-1 \qquad\qquad 2$

$\therefore$ $\boxed{f \text{ inc. on } (-\infty, -1) \text{ and } (2, \infty),\ f \text{ dec. on } (-1, 2)}$

Problem 13 [5pts] The total energy of a free particle is a function of velocity v is given by the formula below:

$$E(v) = \frac{m_o c^{2}}{\sqrt{c^2 - v^2}} = m_o c^3 (c^2 - v^2)^{-1/2}$$

3 better

where $m_o, c > 0$ are constant with respect to v. **Find the minimum and maximum energy if possible. (if only one exists then use calculus to find it and to prove that is is an extreme value for the energy function).**

$$\frac{dE}{dV} = -\frac{M_o C^3}{2}(C^2 - V^2)^{-3/2}(-2V) = \frac{M_o C^3 V}{\left(\sqrt{C^2 - V^2}\right)^3}$$

Hence critical #'s $V = \pm C$ and $V = 0$

dom$(E) = (-C, C)$ so $V = \pm C$ not of interest.

Note. $E'(V) < 0$ for $V < 0$ and $E'(V) > 0$ for $V > 0$

$\therefore$ $E(0)$ is minimum energy : $\boxed{E(0) = M_o C^2}$

(rest energy

Problem 14 [5pts] Let p be a polynomial function of degree n for some $n > 1$. Define $f(x) = p(x)e^{cx}$ where c is a constant. Discuss how many inflection points are possible for f on $\mathbb{R}$.

$$\frac{df}{dx} = \frac{dP}{dx}e^{CX} + Pce^{CX} = (P'(x) + cP(x))e^{cx}$$

$$\frac{d^2f}{dx^2} = \Big[\underbrace{P'' + CP + (P' + CP)C}_{n^{th} \text{ order poly}}\Big]e^{CX}$$

$\Longrightarrow$ at most n real roots

For a smooth fnct an inflection point has to be a zero of f''!

$\Longrightarrow$ $\boxed{\text{at most } n\text{-inflection points}}$