2. FUNCTIONS AND ALGEBRA

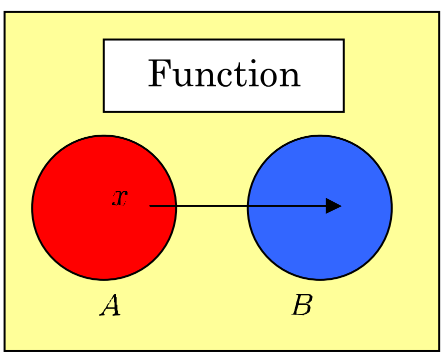
You might think of this chapter as an icebreaker. Functions are the primary participants in the game of calculus, so before we play the game we ought to get to know a few functions. I start the chapter with a few rather abstract notions. Then I transition to the concrete setting we will spend the majority of our time. I assume that you know at least a little algebra and precalculus or advanced mathematics, I'll try to alert you to my assumptions about your background.

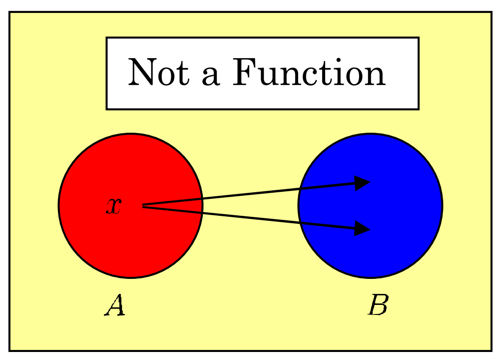
2.1. ABSTRACT IDEA OF A FUNCTION

To begin we need to settle a few common terms and definitions for future use. I warn you that I assume that you know what is meant by the terms "set", "subset" and "element".

> Definition: $f : A \to B$ is a function from A to B if for each $x \in A$ the function assigns is a unique element $f(x) \in B$. We say that f maps x to $f(x)$. This can be denoted $x \mapsto f(x)$.

In other words, a function is a rule that outputs $f(x)$ if it is given the input x. There can be no ambiguity in what the output is, if there was more than one output for a given input then we would not call that rule a function.





The rules illustrated above are schematic, sometimes I find such pictures to be conceptually useful. For the beginning of the calculus sequence we will be almost entirely interested in functions which map sets of real numbers to sets of real numbers. Sometimes we may also dabble with functions which map to and from subsets of complex numbers.

Definition: Let $f : A \rightarrow B$ is a function from A to B then
- A is the domain of the function; $dom(f) = A$.
- B is the codomain of the function.
- $f(A) = \{f(x) \mid x \in A\} = range(f)$ is the range of f

when the codomain is the same as the range the function is said to be "surjective" or "onto".

2.2. DOMAIN, RANGE AND GRAPH

In this section we consider "real-valued functions of a real variable". This sort of function takes a real number input and returns a real number output. Often such a function is defined by some formula. When a function is defined by a formula then we may not need to explicitly state the domain. Also the codomain can just be taken to be the range.

Definition: Let f be a real-valued function of a real variable x which is defined by a formula $f(x)$ and nothing more,
- $dom(f) = \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}$
- $range(f) = \{f(x) \mid x \in dom(f)\}$

in other words the domain is the largest set of real numbers for which the formula $f(x)$ yields a real number. In this case we can let the codomain just match the range; $f : dom(f) \rightarrow range(f)$.

Let me give a few typical examples just to illustrate how a formula might fail to return a real number.

Example 2.2.1: Let $f(x) = \frac{2}{x^2+1}$. Observe that $x^2 + 1$ cannot be zero since $x^2 \geq 0$ so there is no way to cancel the 1. Thus there is no real number x which will result in division by zero. Consequently, $dom(f) = \mathbb{R} = (-\infty, \infty)$.

Example 2.2.2: Let $f(x) = \frac{1}{x-1}$. Observe that $x - 1 = 0$ when $x = 1$. Thus the formula makes no sense when $x = 1$ due to division by zero. Consequently, $dom(f) = (-\infty, 1) \cup (1, \infty)$. This notation means that $x \in dom(f)$ if $x < 1$ or $x > 1$.

Example 2.2.3: Let $f(x) = \sqrt{x-2}$. Observe that $x - 2 \geq 0$ when $x \geq 2$. Thus the formula will return a real number so long as the input of the square root is nonnegative. So, $dom(f) = [2, \infty)$. Note the formula still makes sense for $x < 2$ but then $f(x)$ is not a real number so we exclude such x from the domain.

Sometimes there are considerations beyond the formula for the function. These might come from the interpretation assigned to the real variable.

Example 2.2.4: Let $f(x) = \frac{2}{x^2+1}$ model the number of toys owned by per household in a country with an average of x children per household. Since we cannot have a negative number of children it follows that $dom(f) = [0, \infty)$. In this example there was extra information beyond the formula. Probably we could let the domain be $dom(f) = [0, 20]$ without danger of ignoring many cases. My point in this silly example is that the domain can be adjusted.

Sometimes a picture can be used to describe a function.

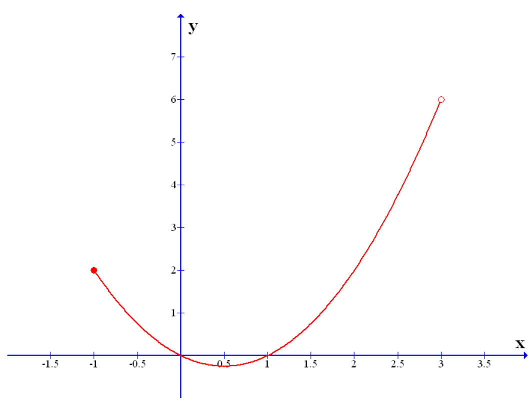
Definition: Let f be a real-valued function of a real variable x

- $graph(f) = \{(x, y) \mid x \in dom(f),\ y = f(x)\}$

In other words, the graph of a function is the set of points in the xy -plane such that $y = f(x)$

The graph of a function is equivalent to its formula and domain. Given the formula we can construct the graph. Conversely, given the graph we can find $f(x)$ for each x in the domain. However, we may be unable to actually find a nice simple formula for the function.

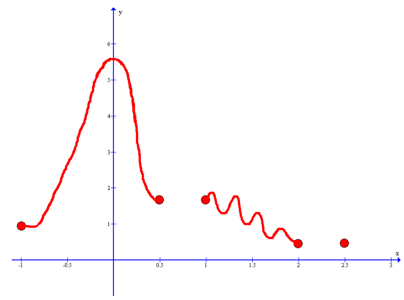
Example 2.2.5: Consider the graph of $y = f(x)$ below,



We can see that the domain begins at -1 which is included and continues right up to 3, so $dom(f) = [-1, 3)$. We can also see the values of the function from the graph. For example, $f(-1) = 2$ can be deduced from the leftmost point on the graph. This graph is pretty nice, we can see it is a parabola. Bonus point if you can tell me the formula and the range of this function soon.

The next example will not allow you to find a nice formula for the function, but the graph still contains lots of useful information about the function.

Example 2.2.6: Consider the graph of $y = f(x)$ below,



This is an example of a discontinuous function, it would be hard to find the formula for this function explicitly. Observe that $dom(f) = [-1, 0.5] \cup [1, 2] \cup \{2.5\}$. The notation $\{2.5\}$ means a set with just the number 2.5 inside it. The range of this function is approximately [0.4, 5.6], the crudeness of my graph does not allow for a better estimate.

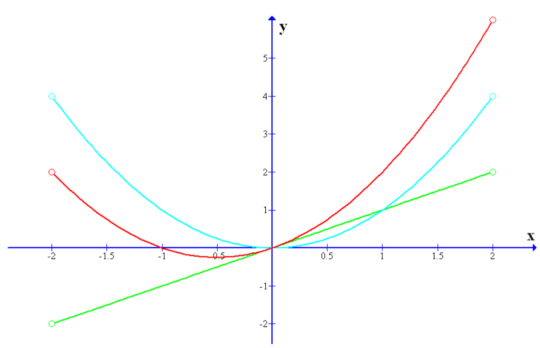
2.3. TYPES AND PROPERTIES OF FUNCTIONS

We begin by defining even and odd functions.

Definition 2.3.1: Let f be a real-valued function of a real variable,

- f is an even function if $f(-x) = f(x)$ for each $x \in dom(f)$
- f is an odd function if $f(-x) = -f(x)$ for each $x \in dom(f)$

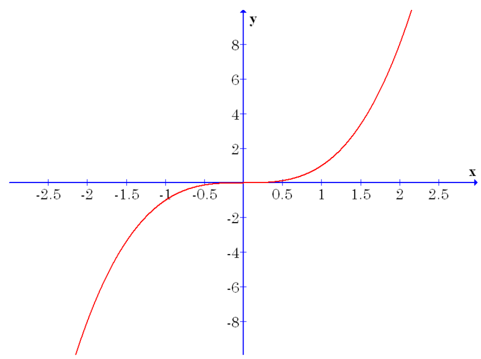
It turns out that many functions can be written as the sum of an even and an odd function. For example, let $f(x) = x^2 + x$. Notice that $f_{even}(x) = x^2$ is an even function and $f_{odd}(x) = x$ is an odd function. Thus for my example function $f = f_{even} + f_{odd}$. For a bonus point show me how to do this in general. Notice the even function (cyan graph) is symmetric about the y-axis while the odd function(green graph) is symmetric about the origin.



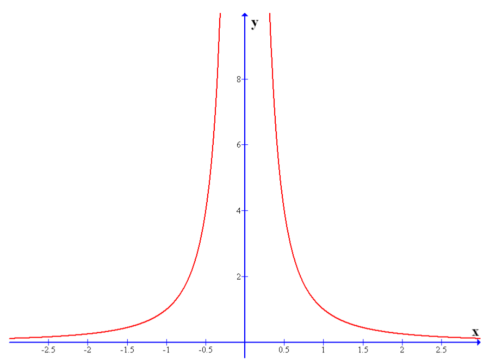
The concepts of even and odd functions concern the total domain of the function, they are *global concepts*. In contrast, the concepts of increasing and decreasing apply to some open subset of a function's domain. So a function may be increasing for some points and decreasing elsewhere.

> <u>*Definition 2.3.2:*</u> Let $I \subseteq dom(f)$,
> - f is increasing on I if $f(a) < f(b)$ for each pair $a, b \in I$ with $a < b$.
> - f is decreasing on I if $f(a) > f(b)$ for each pair $a, b \in I$ with $a < b$.

The graph of $f(x) = x^3$ gives us an example of a function which is increasing everywhere except at zero. It increases on $(-\infty, 0)$ and $(0, \infty)$. We will learn that calculus gives us a nice way to figure out where a function increases and decreases without even drawing the graph! But for now, we graph.



The graph that follows is of the function $f(x) = 1/x^2$. This function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. The point where the function changes from increasing to decreasing happens to be a vertical asymptote which means that zero is not even in the domain of the function in this case.



> <u>*Definition 2.3.3:*</u> We say that a function is *injective* or *one to one* (1-1) on $I \subseteq dom(f)$ if $f(a) = f(b)$ implies that $a = b$ for any pair $a, b \in I$. If $I = dom(f)$ then graphically this is equivalent to saying the function passes the horizontal line test. Recall that a function passes the horizontal line test if any horizontal line drawn through the graph intersects the graph only once.

We will discuss one to one functions further later this chapter when we define inverse functions. I want to have a few more interesting examples before we say more.

> *<u>Definition 2.3.4:</u>* A function f has a zero at $c \in dom(f)$ if $f(c) = 0$. Graphically this means that the graph of the function intersects the x-axis at $(c, 0)$.

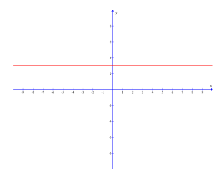
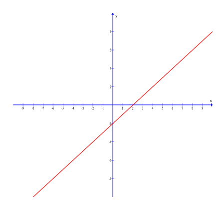
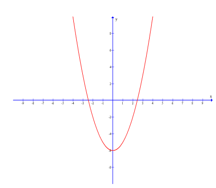
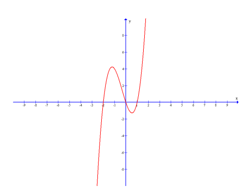
Let me give a nontrivial example of how to find the zeros of a cubic function.

> *<u>Example 2.3.1:</u>* Let $f(x) = 3x^3 - 2x^2 - 1$. Our mission is to find all the zeros of this function with a minimum of computer aid. To begin notice that $f(1) = 3 - 2 - 1 = 0$. (I might tell you this or give a graph which reveals this fact, guessing a zero is not generally an easy thing to do, but hey I just made this up so it was pretty easy for me). Think about what $f(1) = 0$ says about the function, it tells us we can factor out $(x - 1)$,
>
> $$f(x) = (x-1)(ax^2 + bx + c).$$
>
> We knew there had to be a quadratic left since we started with a cubic, but we don't know what the coefficients a, b, c are without some work. Lets multiply out what we just wrote,
>
> $$\begin{aligned} f(x) &= ax^3 + bx^2 + cx - ax^2 - bx - c \\ &= ax^3 + (b-a)x^2 + (c-b)x - c \end{aligned}$$
>
> This must match the given function so,
>
> $$ax^3 + (b-a)x^2 + (c-b)x - c = 3x^3 - 2x^2 - 1$$
>
> Now perhaps you haven't seen such an equation before, but it's not as bad as you might imagine. In fact we can just equate the coefficients of like powers of x,
>
> $$\boxed{x^3 :}\ a = 3, \qquad \boxed{x^2 :}\ b - a = -2, \qquad \boxed{x :}\ c - b = 0, \qquad \boxed{1 :}\ -c = -1$$
>
> The first and last of these equations are really easy to solve, clearly $a = 3$ and $c = 1$. The equation from the x-coefficient shows $b = c = 1$. So we find,
>
> $$f(x) = (x-1)(3x^2 + x + 1)$$
>
> The only way $f(x) = 0$ is if either $x - 1 = 0$ or $3x^2 + x + 1 = 0$. So the zeros come from $x = 1$ or the solutions to the quadratic equation. Notice,
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 12}}{6} = \frac{-1 \pm i\sqrt{11}}{6}$$
>
> Therefore there are no additional roots besides the one we guessed to begin with; $f(x) = 3x^3 - 2x^2 - 1$ has one zero at $x = 1$. You might ask why not just learn these things with a graphing calculator? That is another valid approach for this problem, but the approach we took here helps us learn some algebra and builds character. After all, there is little in the calculus sequence that cannot be done on a computer. Then why do we do such calculations? We do them to gain a deeper understanding of math.

2.4. ELEMENTARY FUNCTIONS

The functions we discuss in this section are the most common functions used in calculus. We can model a great variety of phenomena with these functions.

1.) We say p is a ***polynomial function*** of degree n if it has the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ where $a_n \neq 0$ and we call $a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \in \mathbb{R}$ the *coefficients* of the polynomial.

Formula	Name	Zeros	Graph of p
$p(x) = c$	constant function	None, unless $c = 0$ in which case there are infinitely many.	
$p(x) = mx + b$	linear function	$x = -\frac{b}{m}$ we assume $m \neq 0$.	
$p(x) = ax^2 + bx + c$	quadratic function	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ if $b^2 - 4ac \geq 0$, $a \neq 0$.	
$p(x) = ax^3 + bx^2 + cx + d$	cubic function	No simple formula. There is always one zero. In some cases there are 3 zeros.	

2.) We say f is a ***power function*** if $f(x) = x^a$ where a is a fixed constant. There are a few special cases with added labels,

i.) $a = n \in \mathbb{N}$ then $f(x) = x^n$ is a *polynomial.*
ii.) $a = \frac{1}{n}$ with $n \in \mathbb{N}$ then $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is the n^{th} *root function.*
iii.) $a = -1$ then $f(x) = x^{-1} = \frac{1}{x}$ is the *reciprocal function.*

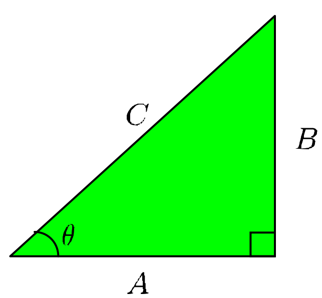
3.) We say that f is a ***rational function*** if it has the form $f(x) = p(x)/q(x)$ for a pair of polynomial functions p and q. The zeros of f occur where the zeros of p occur, except possibly some of those are cancelled by the q (could have holes in the graph or a vertical asymptote). The domain of a rational function is simply all the points where we avoid division by zero; $dom(\frac{p}{q}) = \{x \mid q(x) \neq 0\}$. The *reciprocal function* is a rational function.
A typical example of a rational function is

$$f(x) = \frac{x(x-1)(x-3)}{x(x^2-5x+6)}$$

this function has a hole in the graph at zero and three. It has a vertical asymptote at two. It has a zero at one. So $dom(f) = (-\infty, 0) \cup (0,2) \cup (2,3) \cup (3,\infty)$. *Can you tell me the formula for a function g that agrees with f on $dom(f)$ but has no holes? It's not a hard question.*

4.) We say that f is an ***algebraic function*** if it has a formula which is comprised of finitely many algebraic operations. By "algebraic" we mean you may add, subtract, multiply, divide and raise to powers or take roots. This category of functions includes all the preceding examples in 1,2 and 3. The domain for an algebraic function is simply all the inputs which result in a real number output. That means we must avoid taking the square root of a negative number and also division by zero. *A silly example of an algebraic function is $f(x) = \sqrt{\sqrt{x} - \sqrt{x}}$. What is the difference between this function and $g(x) = 0$? I'll give you a clue, it's just the domain that is different.*

5.) ***Trigonometric functions*** such as sine, cosine and tangent are based on the geometry of triangles. Recall a *right triangle* is one for which an angle measures 90 degrees (or $\pi/2$ radians, or 100 grads, etc...).



$\cos(\theta) = \frac{A}{C} = \frac{adj}{hyp}$

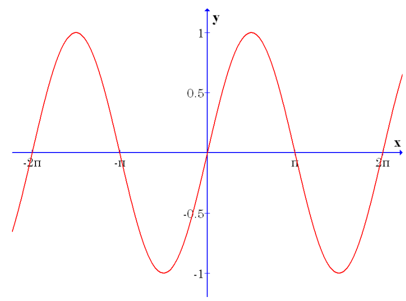
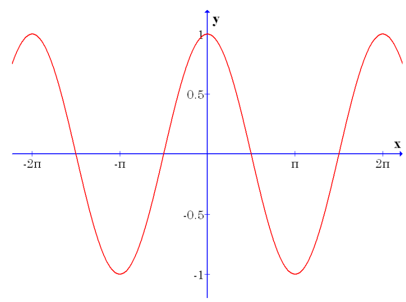
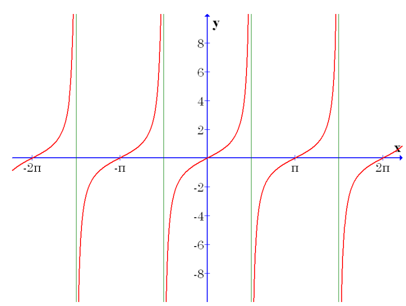
$\sin(\theta) = \frac{B}{C} = \frac{opp}{hyp}$

$\tan(\theta) = \frac{B}{A} = \frac{opp}{adj}$

In the picture above we assume that $A, B, C > 0$ and we have drawn the triangle so that $0 < \theta < \pi/2$, it is an *acute* angle. You may recall that the side A is *adjacent* to the angle θ while the side B is *opposite* the angle θ. The longest side C is called the *hypotenuse*.

Theorem 2.4.1: (Pythagorean Theorem) Let A, B, C be the sides of a right triangle with hypotenuse C then $A^2 + B^2 = C^2$.

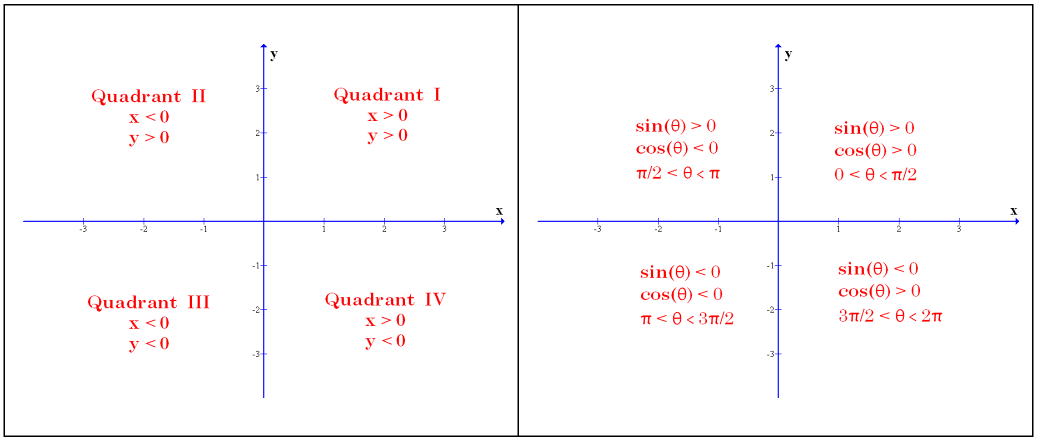
5.) ***Trigonometric functions continued:*** We could go on and list many more facts that are known about triangles and the geometric ratios of sine, cosine and tangent. Instead, I now introduce you to the functions $\cos, \sin, \tan$ which are defined for any value of θ.

notation	name	Zeros	graph
$\sin(x)$	sine	$x = 0, \pm\pi, \pm 2\pi, \ldots$ Equivalently, $x = n\pi,\ n \in \mathbb{Z}$	
$\cos(x)$	cosine	$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ Equivalently, $x = n\pi + \frac{\pi}{2},\ n \in \mathbb{Z}$	
$\tan(x)$	tangent	Same as sine. The green lines are the vertical asymptotes which happen where cosine is zero.	

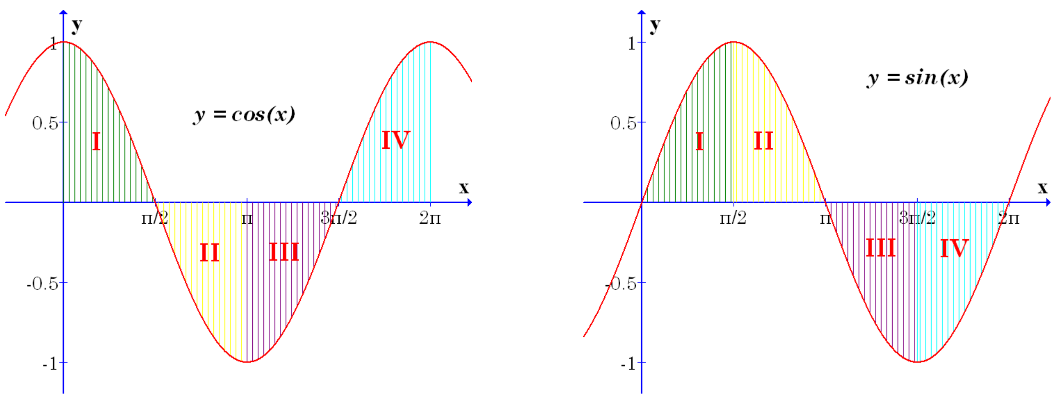
These functions extend the quadrant I geometric quantities to the other three quadrants. The definitions make *polar coordinates* work. The xy-plane is the set of points of the form $P = (x, y)$. We say x is the x-coordinate of P while y is the y-coordinate of P. The *polar coordinates* of P are r, θ where

$$\boxed{x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad r^2 = x^2 + y^2, \qquad \tan(\theta) = \frac{y}{x}}$$

and r is the *radial coordinate* and θ is the *standard angle*. There are a number of conventions as to what particular values the polar coordinates should be allowed to take. We will insist that $r \geq 0$ and $0 \leq \theta \leq 2\pi$. The xy-plane is divided into four *quadrants*. See below how the sine and cosine of the standard angle matches the signs of x and y.



Or perhaps the following diagrams make more sense to you,



Since $r = \sqrt{x^2 + y^2} \geq 0$ we see that the formulas $x = r\cos(\theta)$ and $y = \sin(\theta)$ reproduce the correct signs for the Cartesian coordinates x and y. My point here is simply that sine and cosine not only include basic geometric ratios about triangles, they also encode the signs of the Cartesian coordinates in all four quadrants.

Calculator Warning: Given the Cartesian coordinates of a point it is a common task to find the standard angle θ, we can solve $\tan(\theta) = y/x$ for θ by taking the inverse tangent to obtain $\theta = \tan^{-1}(y/x)$. Let me explain some of the dangers of this formula. Notice that $\tan(\theta) = \sin(\theta)/\cos(\theta)$ is positive in quadrants I and III and is negative in quadrants II and IV. If you try to solve for θ with a calculator it cannot detect the difference between I and III or II and IV. Let's see how the formula is ambiguous if you are not careful,

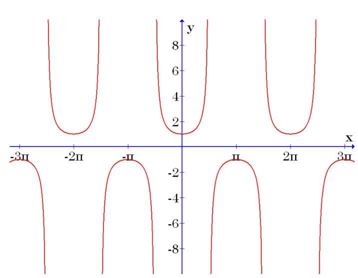
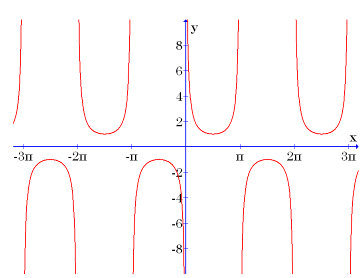
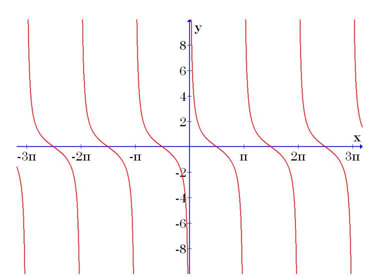
i.) Suppose $x = 1, y = 1$ then $tan(\theta) = 1/1 = 1$. We can solve for θ by taking the inverse tangent of both sides, $\tan^{-1}(\tan(\theta)) = \theta = \tan^{-1}(1)$ now most scientific calculators will calculate the inverse tangent, it gives $\tan^{-1}(1) = \pi/4$. In this case the calculator has not misled, the standard angle is $\pi/4$.

ii.) Suppose $x = -1, y = -1$ then $tan(\theta) = (-1)/(-1) = 1$. We can solve for θ by taking the inverse tangent of both sides, $\tan^{-1}(\tan(\theta)) = \theta = \tan^{-1}(1)$. Now the scientific calculator will again calculate that $\tan^{-1}(1) = \pi/4$. But in this case the calculator might mislead us, the standard angle is not $\pi/4$. In fact the standard angle here lies in quadrant III and so we have to add π to the angle the calculator found to get the correct angle; $\theta = 5\pi/4$.

6.) ***Reciprocal Trigonometric functions:*** these appear quite often in difficult integrations. ***Secant, cosecant*** and ***cotangent*** are defined to be one over the functions cosine, sine and tangent respectively. We use the notation,

$$\boxed{\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}}$$

We could say more about these, and we will later, but for now let me just show you the graphs of these functions.

Graph of $y = \sec(x)$	Graph of $y = \csc(x)$	Graph of $y = \cot(x)$
		

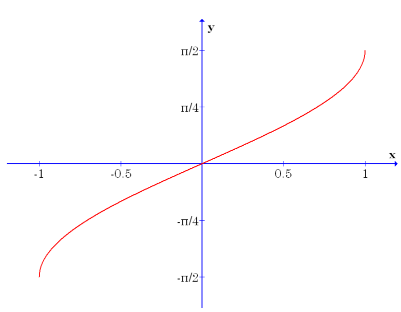
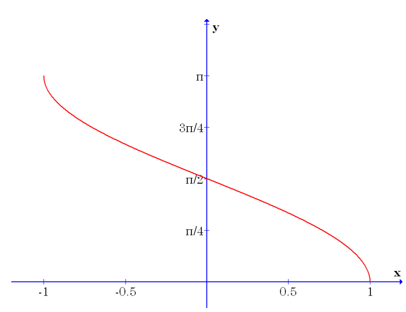
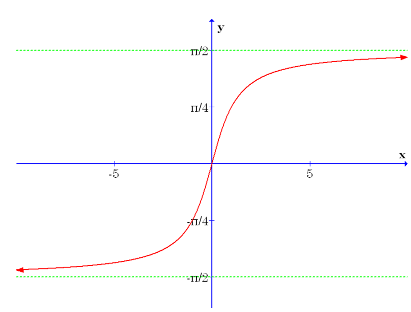
7.) ***Inverse Trigonometric functions:*** we should be careful to distinguish the inverse trigonometric functions from the reciprocal trig functions. The inverse trig functions are $\sin^{-1} = asin, \cos^{-1} = acos$ and $\tan^{-1} = atan$ which I refer to as "inverse sine", "inverse cosine" and "inverse tangent" respectively. They satisfy the equations,

$$\boxed{\sin^{-1}(\sin(x)) = x \qquad \cos^{-1}(\cos(x)) = x \qquad \tan^{-1}(\tan(x)) = x}$$

and,

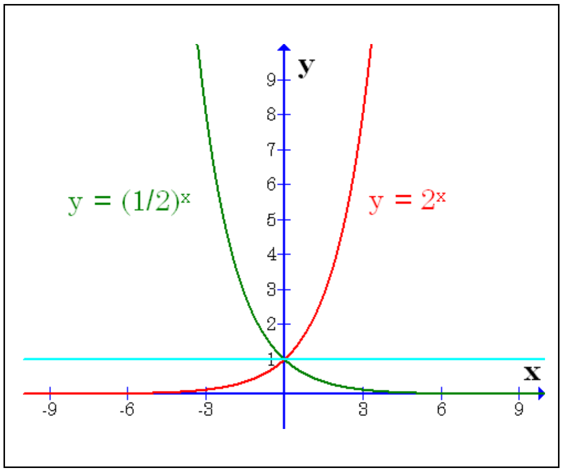
$$\boxed{\sin(\sin^{-1}(x)) = x \qquad \cos(\cos^{-1}(x)) = x \qquad \tan(\tan^{-1}(x)) = x}.$$

Let us collect the graphs of the inverse trig functions for future reference.

Graph of $y = \sin^{-1}(x)$	Graph of $y = \cos^{-1}(x)$	Graph of $y = \tan^{-1}(x)$
		

The green lines illustrate *horizontal asymptotes* of inverse tangent. The occur at $y = \pi/2$ and $y = -\pi/2$. *These are all local inverses, more on that in* §*2.6.*

8.) ***Exponential functions:*** let $a > 0$ then we say that f is an ***exponential function*** if $f(x) = a^x$. The fixed number a is called the *base* of the exponential function. Exponential functions are nonzero everywhere. The graph below shows the three shapes an exponential function may take.



If $a > 1$ then $f(x) = a^x$ gives us exponential growth. If $0 < a < 1$ then $f(x) = a^x$ gives us exponential decay. The graph appears to get to zero, but this is not the case, exponential functions never reach zero. We see that

$$\boxed{dom(a^x) = (-\infty, \infty) \qquad range(a^x) = (0, \infty)}$$

If $f(x) = e^x$ then this is ***the exponential function***, more often than not we will work with this particular base, the number $e = 2.718...$ is called Euler's number in honor of the famous mathematician Euler. It is a *transcendental number* which means it is defined by an equation which transcends simple algebra. We will discuss e^x further in later chapters.

9.) ***Logarithmic functions:*** these are the inverse functions of the exponential functions. We say that $f(x) = \log_a(x)$ is a ***logarithmic function***, we assume that $a > 1$ and that the *"log base a of x"* (this is how we verbalize the formula when we're talking out the math) $\log_a(x)$ satisfies the following equations,

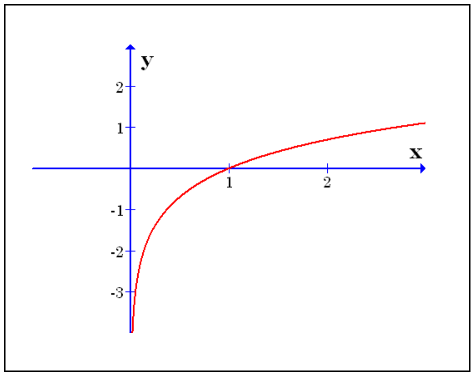
$$\boxed{\log_a(a^x) = x \qquad a^{\log_a(x)} = x}$$

In this sense the logarithm and exponential functions cancel. An equivalent way to define the logarithm is to say that if $y = a^x$ then $\log_a(y) = x$. Notice that the input of the logarithm must be positive since $a^{\log_a(x)}$ is positive; $dom(\log_a(x)) = (0, \infty)$.

$$\boxed{dom(\log_a(x)) = (0, \infty) \qquad range(\log_a(x)) = (-\infty, \infty)}$$

The ***natural log function*** is denoted $\ln(x)$, this the logarithmic function with base $e = 2.718...$ that simply means $\log_e(x) = \ln(x)$. This particular logarithmic function is so important that it gets its own notation. We will encounter it frequently in later chapters.

The graph of $y = \ln(x)$ shows that the natural log has one zero at $x = 1$.



We can see that $dom(\ln(x)) = (0, \infty)$ and the $range(ln(x)) = (-\infty, \infty)$.

PROPERTIES OF EXPONENTIALS AND LOGARITHMS:

We assume that $a, b > 0$ in the equations that follow. I assume that you know these formulas and how to use them. Technically there is no need for the equations in the bottom two squares since they are the same as the top two once we set $a = e$ .For your convenience I include them.

$a^{x+y} = a^x a^y.$	$\log_a(xy) = \log_a(x) + \log_a(y)$
$(a^x)^y = a^{xy}$	$\log_a(x^c) = c\log_a(x)$
$a^{-x} = \frac{1}{a^x}$	$\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
$a^{x-y} = \frac{a^x}{a^y}$	$\log_a(a) = 1$
$(ab)^x = a^x b^x$	$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$
$e^{x+y} = e^x e^y.$	$\ln(xy) = \ln(x) + \ln(y)$
$(e^x)^y = e^{xy}$	$\ln(x^c) = c\ln(x)$
$e^{-x} = \frac{1}{e^x}$	$\ln(\frac{x}{y}) = \ln(x) - \ln(y)$
$e^{x-y} = \frac{e^x}{e^y}$	$\ln(e^x) = x$
$e^{\ln(x)} = x$	$\log_a(x) = \frac{\ln(x)}{\ln(a)}$

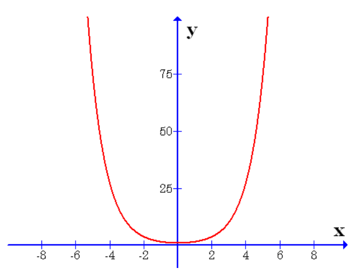
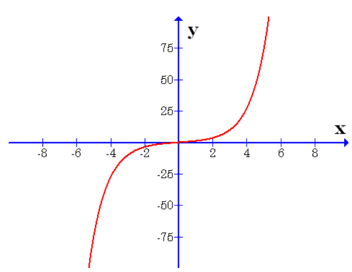
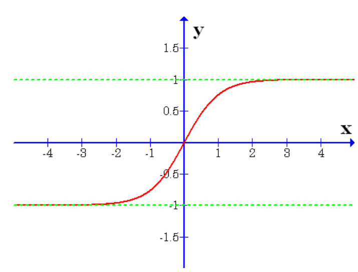
10.) ***Hyperbolic trigonometric functions:*** these are little less common then some of the other functions we have discussed so far, however they are useful both for certain questions of integration and also *special relativity (ask me if you are interested, in short, the hyperbolic angle is the rapidity…).*

name of function	defining formula
hyperbolic cosine	$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
hyperbolic sine	$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
hyperbolic tangent	$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

At first glance it is a little strange to call these "trigonometric", that label comes from an understanding of cosine and sine in terms of imaginary exponentials e^{ix} where $i = \sqrt{-1}$. We will discuss imaginary exponentials in section 4.11. For now just observe that

$$\boxed{\cosh^2(x) - \sinh^2(x) = 1.}$$

This is clearly similar to the corresponding identity $\cos^2(x) + \sin^2(x) = 1$. I remind you that $\sin^2(x) \equiv (\sin(x))^2$ and the same goes for the other functions.

graph of $y = \cosh(x)$	graph of $y = \sinh(x)$	graph of $y = \tanh(x)$
		

The ***inverse hyperbolic trigonometric functions*** are $\cosh^{-1}(x)$, $\sinh^{-1}(x)$ and $\tanh^{-1}(x)$. These satisfy the formulas,

$$\boxed{\cosh(\cosh^{-1}(x)) = x \qquad \sinh(\sinh^{-1}(x)) = x \qquad \tanh(\tanh^{-1}(x)) = x}$$

and,

$$\boxed{\cosh^{-1}(\cosh(x)) = x \qquad \sinh^{-1}(\sinh(x)) = x \qquad \tanh^{-1}(\tanh(x)) = x}$$

I include these *inverse hyperbolic trig functions* for the sake of completeness, I don't expect we will use them too much in this course.

2.5. WAYS TO COMBINE FUNCTIONS

Given two functions f and g we can create new functions by adding, subtracting, dividing or multiplying by a constant.

function	defining formula	domain of new function
$f+g$	$(f+g)(x) = f(x) + g(x)$	$dom(f) \cap dom(g)$
$f-g$	$(f-g)(x) = f(x) - g(x)$	$dom(f) \cap dom(g)$
fg	$(fg)(x) = f(x)g(x)$	$dom(f) \cap dom(g)$
f/g	$(f/g)(x) = f(x)/g(x)$	$dom(f) \cap dom(g) \cap \{x \mid g(x) \neq 0\}$
cf	$(cf)(x) = cf(x)$	$dom(f)$

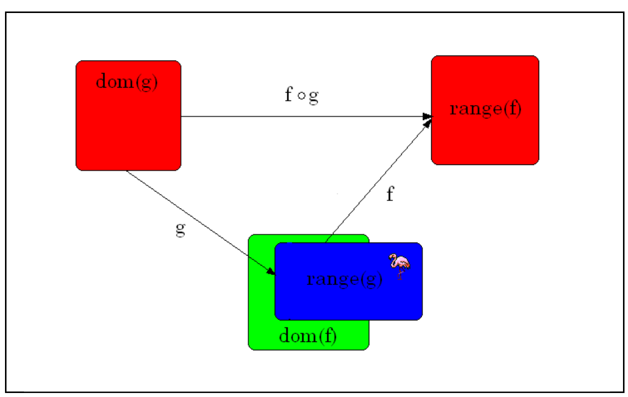
These formulas go to show that functions are a lot like numbers, we can add, subtract, multiply and even divide functions and the result will be a function. Functions are different than numbers of course, for example, I'm not sure what the analogue for the following would be in terms of numbers.

<u>*Definition 2.5.1:*</u> if f and g are functions then the composite of f with g is $f \circ g$ which is defined by

$$\boxed{(f \circ g)(x) = f(g(x))}$$

for all $x \in dom(g)$ such that $g(x) \in dom(f)$. In other words the domain of $f \circ g$ is the set of all numbers such that the formula $f(g(x))$ makes sense.

It is true that figuring out the domain of a composite function is little tricky. Fortunately, the focus of this course is formulas more than domains. I am just trying to be careful. Let me illustrate how a composite function works.



To start with we need $x \in dom(g)$ then even after that we must be sure that $g(x) \in dom(f)$, so for example the x that has $g(x) =$  is not allowed in the domain since the flamingo is outside the domain of f. Ok, enough about the domain of a composite function. The important thing is you be able to identify when a function is a composite, we will come back to this issue when we study the *chain rule* for derivatives. Let me give one example before we go on just to refresh your memory on how the composite works.

Example 2.5.1: Suppose that $f(x) = x^2$ and $g(x) = \frac{1}{x-1}$. Let us calculate the formulas for $f \circ g$ and also $g \circ f$.

$(f \circ g)(x) = f(g(x)) = f(\frac{1}{x-1}) = \left(\frac{1}{x-1}\right)^2.$

On the other hand,

$(g \circ f)(x) = g(f(x)) = g(x^2) = \frac{1}{x^2-1}.$

The domain for $f \circ g$ is completely determined by the domain of the inside function g since $dom(f) = \mathbb{R}$

$$dom(f \circ g) = (-\infty, 1) \cup (1, \infty).$$

On the other hand the domain of $g \circ f$ is not limited by the domain of the inside function. Its domain is narrowed by where the range of f happens to fall outside the domain of g; in particular $x = \pm 1$. So,

$$dom(g \circ f) = (-\infty, -1) \cup (-1, 1) \cup (1, \infty).$$

2.6. INVERSE FUNCTIONS

We will work primarily with ***local inverses*** for known functions.

Definition 2.6.1: We say that f has a local inverse f^{-1} on $I \subseteq dom(f)$ if it satisfies the following two equations,

$$f^{-1}(f(x)) = x$$

for each $x \in I$, and

$$f(f^{-1}(y) = y$$

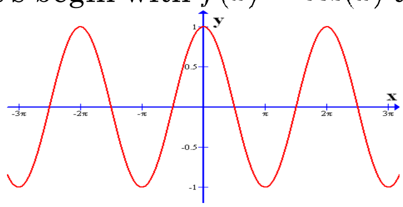
for each $y \in f(I)$. I should mention that $f(I) = \{y \mid y = f(x), x \in I\}$. If the set $I = dom(f)$ then we say that f^{-1} is the global inverse of f.

Some functions we have discussed so far have global inverses. For example, e^x has the global inverse $\ln(x)$. The theorem below helps guide us in our quest to find local inverses,

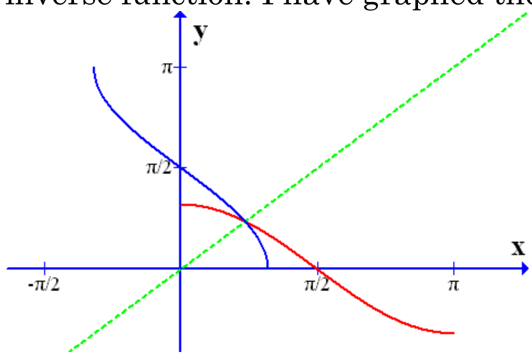
Theorem 2.6.1: If f is one to one over an interval I then f has a local inverse on I. Conversely, if f is not one to one on I then f has no local inverse on I.

Examples of local inverses are $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$. These cannot be global inverses since $\sin(x), \cos(x), \tan(x)$ all are cyclic, they repeat the same output values many many times. See the graphs in section 2.4.4. A one-one function only outputs a particular value just once.

Example 2.6.1: Let's begin with $f(x) = \cos(x)$ the graph looks like,

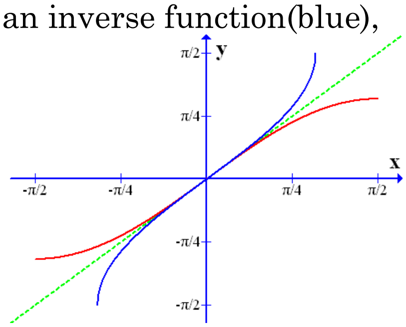


this cannot have a global inverse since it is not 1-1. However, if we reduce the domain to $[0, \pi]$ we obtain a 1-1 function on that interval, so we can find an inverse function. I have graphed the inverse in blue,



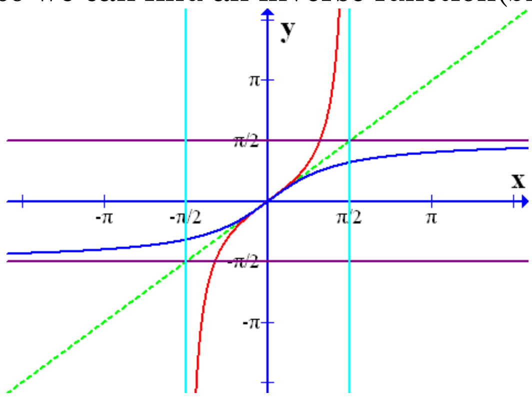
and you can see that the inverse is the reflection of the graph of cosine about the line $y = x$ (green). Let it be understood that when we speak of inverse cosine we intend the local inverse for cosine on the interval $[0, \pi]$. The domain of inverse cosine is $[-1, 1]$ and the range is $[0, \pi]$. In principle one could construct other inverses for cosine based on other intervals, the choice of $[0, \pi]$ is simply one of convention.

Example 2.6.2: Lets $f(x) = \sin(x)$ with $dom(f) = \mathbb{R}$. This is not 1-1 because sine oscillates just like cosine. However, if we reduce the domain to $[-\pi/2, \pi/2]$ we obtain a 1-1 function on that interval (red), so we can find an inverse function(blue),



and you can see that the inverse is the reflection of the graph of cosine about the line $y = x$ (green). The domain of inverse sine is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$. In principle one could construct other inverses for sine based on other intervals, the choice of $[-\pi/2, \pi/2]$ is simply one of convention.

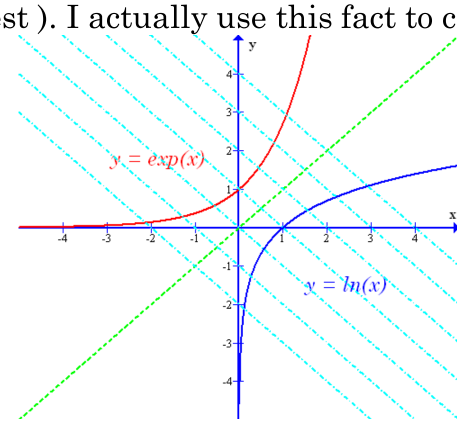
Example 2.6.2: Lets $f(x) = \tan(x)$ with $dom(f) = \mathbb{R}$. This is not 1-1 because tangent oscillates just like sine and cosine. However, if we reduce the domain to $[-\pi/2, \pi/2]$ we obtain a 1-1 function on that interval (red), so we can find an inverse function(blue),



and you can see that the inverse is the reflection of the graph of cosine about the line $y = x$ (green). The domain of inverse tangent is $(-\infty, \infty)$ and the range is $(-\pi/2, \pi/2)$.

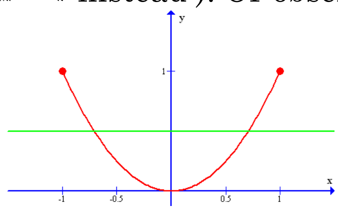
I have added the vertical asymptotes of tangent in cyan at $x = \pm\pi/2$ you can see that the inverse tangent has horizontal asymptotes at $y = \pm\pi/2$. This illustrates a general pattern, vertical asymptotes for a function will morph into horizontal asymptotes for the inverse function. We will make use of this example in later chapters. It helps us understand what the *limit of* $\tan(x)$ *is as* $x \to \infty$ (it's $\pi/2$).

By now you should have noticed that we can construct the inverse function's graph by reflection about the line $y = x$ (assuming that the function is 1-1 on the interval of interest). I actually use this fact to construct certain graphs.

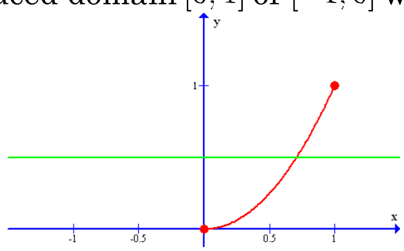


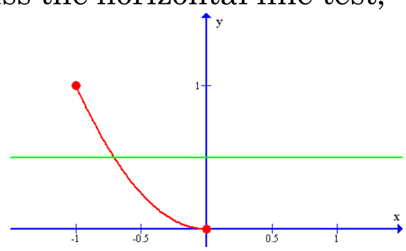
You can draw the graph $y = e^x$ (red) then draw the line $y = x$ (green) and a bunch of perpendicular bisectors (cyan) then the graph of the inverse function $y = \ln(x)$ follows. If we travel one unit from the red graph to the green line along the cyan line then the corresponding point on the blue graph is one unit further past the green line. That is the green line should intersect the cyan line at the midpoint between the intersection points of the red and blue graphs. Now, I should warn you that this advice is given for graphs with horizontal and vertical directions given the same scale. The cyan lines and the green line would take a different slant if x-axis and y-axis used a different scale (Example 2.6.1 is such a case).

<u>*Example 2.6.4:*</u> Consider $f(x) = x^2$ with $dom(f) = [-1, 1]$ we can argue algebraically that this function is not one-one since $a^2 = b^2$ implies $a = \pm b$ (we needed $a = b$ instead). Or observe that



it fails the horizontal line test. In contrast, the same formula with reduced domain $[0, 1]$ or $[-1, 0]$ will pass the horizontal line test,





So then what is the formula for the inverse functions? We need,

$$(i.)\ f^{-1}(f(x)) = f^{-1}(x^2) = x \qquad (ii.)\ f(f^{-1}(x)) = (f^{-1}(x))^2 = x$$

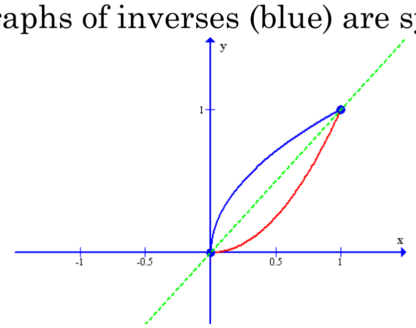
in the interest of making the calculation easier let's say that $f^{-1}(x) = y$, then (ii.) above becomes $y^2 = x$ which yields $y = \pm\sqrt{x}$. Two solutions! Which one to pick? Well remember we also need to solve (i.) which reads

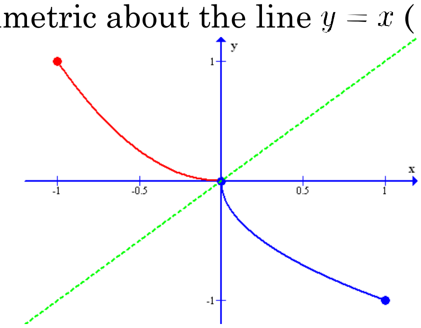
$$\pm\sqrt{x^2} = x$$

Interesting. This has two solutions,

- If $x \geq 0$ then $\sqrt{x^2} = x$ so we choose the + solution; $f^{-1}(x) = \sqrt{x}$
- If $x \leq 0$ then $\sqrt{x^2} = -x$ so we choose the - solution; $f^{-1}(x) = -\sqrt{x}$

We find that the inverse of $f(x) = x^2$ on $[0, 1]$ is $f^{-1}(x) = \sqrt{x}$ and the inverse of $f(x) = x^2$ on $[-1, 0]$ is $f^{-1}(x) = -\sqrt{x}$. Notice that the graphs of inverses (blue) are symmetric about the line $y = x$ (green)





Incidentally we just stumbled across a nice algebraic formula for the absolute value function; $|x| = \sqrt{x^2}$. For example, $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$.

Why does the reflection rule hold ?

We have seen in a number of cases that we can construct the graph of the inverse by reflection about the line $y = x$. Examples are not proof. They are evidence in favor of a proof, can we give a general argument as to why this trick works ? Notice that one way of characterizing a reflection about the $y = x$ line is to say that the reflected function is the same graph just with x and y switched. For example, $y = 2x$ is reflected to $x = 2y$ which is otherwise known as $y = x/2$. So the reflection of $y = \cos(x)$ is $x = cos(y)$. If you go back and look at our graphs you'll see the inverse is the same shape as the function, it's just run vertically instead of horizontally.

So the question reduces to why does the inverse function have the same graph as the function except with x and y reversed? Is it obvious from the definition of inverse ? Recall,

$$\boxed{f^{-1}(f(x)) = x \qquad f(f^{-1}(y)) = y.}$$

If the graph of the inverse has the roles of x and y reversed we ought to look at the vertical graph $x = f^{-1}(y)$. Using the boxed equation (which is the very definition of the inverse) we see $f(f^{-1}(y)) = f(x) = y$. We have shown that the graph of the inverse function is just the graph of the function itself with the roles of x and y reversed.

You may recall from your algebra course that the formula for the inverse function is found by taking the formula for the function and switching $x \leftrightarrows y$. For example, if $f(x) = e^{x+3}$ then we take $y = e^{x+3}$ and switch it to $x = e^{y+3}$ and solve for y: remember to remove exponentials we can take the natural log,

$$x = e^{y+3} \quad \Longrightarrow \ \ln(x) = \ln(e^{y+3}) = y + 3.$$

Thus, $y = \ln(x) - 3$. Hence, $f^{-1}(x) = \ln(x) - 3$.

That high school trick makes good sense in view of the arguments on this page.