Complex Numbers and their Properties

A group is a set G with an associative operation which is closed under inverses. Furthermore, a ring R is a commutative group under addition (identity called zero) paired with a multiplication which distributes across addition. If a ring R has a multiplicative identity 1 then every invertible element u such that $uv = 1$ is called a unit of R. A ring such that every nonzero element is a unit is called a field. (see Math 422 for more on this) We assume the reader has studied $\mathbb{R}$.

> **Def^n/** A field $\mathbb{F}$ for which
> 1. $\mathbb{R}$ is a subfield
> 2. the eq^n $z^2+1=0$ has precisely two sol^n's.
> 3. If i is one of the sol^n's in (2) then $-i$ is the other.
>
> is called a complex number system.

There are many sets which allow the above structure. For convenience we define $\mathbb{C} = \mathbb{R}^2$ with the following multiplication:

$$\boxed{\text{Def}^n/\ (a,b)*(c,d) = (ac-bd,\ ad+bc)}$$

We use the following notation

$$i = (0,1) \quad \text{and} \quad 1 = (1,0)$$

With this,
$$\begin{aligned}(a,b) &= (a,0)+(0,b)\\ &= a(1,0) + b(0,b)\\ &= a1 + bi\\ &= a+ib\end{aligned}$$

Remark: the notation $a+ib = (a,b)$ allows us to replace $*$ with juxtaposition. In particular:

$$(a,b)*(c,d) = (ac-bd,\ ad+bc)$$

$$\longrightarrow \underline{(a+ib)(c+id) = ac-bd + i(ad+bc)}.$$

$\star$

You can easily derive that $\star$ follows from assuming $i^2 = -1$ and $(a+ib)(c+id)$ multiplies in the same fashion as real numbers.

Note: $\mathbb{C} = \{a+ib \mid a,b \in \mathbb{R}\}$

1. $a = a + i(0) \in \mathbb{C}$ for each $a \in \mathbb{R}$ hence $\mathbb{R} \subseteq \mathbb{C}$ and it is infact a subfield.

2. Suppose $z = x+iy$ and $z^2+1=0$

$$z^2 = (x+iy)(x+iy) = x^2-y^2+2ixy$$

$$z^2+1=0 \Rightarrow \underbrace{x^2-y^2+1+2ixy = 0}_{(x^2-y^2+1,\ 2xy) = 0}$$

$$(x^2-y^2+1,\ 2xy) = 0 \begin{cases} \to x^2-y^2+1=0 \\ \to 2xy = 0 \end{cases}$$

If $y=0$ then $x^2+1=0$ but $\nexists\, x \in \mathbb{R}$ such that $x^2+1=0$.
Therefore, $x=0$ hence $-y^2+1=0 \Rightarrow (1-y)(1+y)=0$.
We find precisely two sol^{n}'s; $(0,\pm 1) = \pm i$.

3. the sol^{n}'s to $z^2+1=0$ came in a conjugate pair.

This shows $\mathbb{C}$ is indeed a complex number system.

<u>Notation</u>: If $z = x + iy \in \mathbb{C}$ then $x, y \in \mathbb{R}$ (convention)

Th^{m}/ Let $z_1, z_2, z_3 \in \mathbb{C}$ then

1.) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$,

2.) $z_1(z_2 z_3) = (z_1 z_2) z_3$,

3.) $z_1 + z_2 = z_2 + z_1$,

4.) $z_1 z_2 = z_2 z_1$,

5.) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

<u>Proof of 2</u>: Let $z_j = x_j + iy_j = (x_j, y_j)$ for $j = 1, 2, 3$.

$$
\begin{aligned}
z_1(z_2 z_3) &= (x_1, y_1) * [(x_2, y_2) * (x_3, y_3)] \\
&= (x_1, y_1) * (x_2 x_3 - y_2 y_3,\ x_2 y_3 + y_2 x_3) \\
&= \big(x_1(x_2 x_3 - y_2 y_3) - y_1(x_2 y_3 + y_2 x_3), \\
&\qquad x_1(x_2 y_3 + y_2 x_3) + y_1(x_2 x_3 - y_2 y_3) \big) \\
&= \big(x_1 x_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 - y_1 y_2 x_3, \\
&\qquad x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3 - y_1 y_2 y_3 \big) \quad 😬
\end{aligned}
$$

Next, calculate $(z_1 z_2) z_3$ by calculating

$[(x_1, y_1) * (x_2, y_2)] * (x_3, y_3) =$ 👻

and show 😬 = 👻. I leave the details to you in a homework problem ☺.

$$\boxed{\text{Th}^{m}/ \text{ Suppose } x + iy \neq 0 \text{ then } \frac{1}{x+iy} = \frac{x - iy}{x^2 + y^2}}$$

<u>Proof</u>: Show $(x+iy)\left(\frac{x - iy}{x^2 + y^2}\right) = 1$. //

Conjugation

$$\boxed{\text{Def}^{\underline{n}}/ \text{ If } z = x+iy \text{ then } \underbrace{\bar{z} = x-iy}_{\text{complex conjugate of } z.}}$$

Properties: for all $z, w \in \mathbb{C}$,

1.) $\overline{z+w} = \bar{z} + \bar{w}$
2.) $\overline{zw} = \bar{z}\bar{w}$
3.) $\bar{\bar{z}} = z$
4.) $z\bar{z} = x^2+y^2$
5.) $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$ for $z \neq 0$.

Proof: left to reader.

Discussion: Notice $z + \bar{z} = (x+iy)+(x-iy) = 2x$ and $z - \bar{z} = (x+iy)-(x-iy) = 2iy$. It follows that: if $z = x+iy$ then

$$x = \tfrac{1}{2}(z+\bar{z})$$

$$y = \frac{1}{2i}(z-\bar{z})$$

Def$^{\underline{n}}$/ $Re(z) = \frac{1}{2}(z+\bar{z})$ and $Im(z) = \frac{1}{2i}(z-\bar{z})$

If $z = x+i(0)$ then $Re(z) = z$ and z is pure real.

If $z = 0+iy$ then $Im(z) = z$ and z is $\underbrace{\text{pure imaginary}}_{\text{set of all such complex \#s called } i\mathbb{R}.}$

Observe:
$$\begin{aligned} z &= \tfrac{1}{2}(z+\bar{z}) + \tfrac{1}{2}(z-\bar{z}) \\ &= \tfrac{1}{2}(z+\bar{z}) + i\,\tfrac{1}{2i}(z-\bar{z}) \\ &= Re(z) + i\,Im(z) \end{aligned}$$

Notice $z \in \mathbb{R}$ iff $\bar{z} = z$ whereas $z \in i\mathbb{R}$ iff $\bar{z} = -z$. In total we've shown $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ (from linear algebra if you know it...)

Modulus aka "absolute value"

Unfortunately some words in our text are probably mistranslated from German. Although, some other authors use "absolute value" for modulus. I'll make a point of the following in my work:

$$x \in \mathbb{R} \quad \text{has} \quad \overbrace{|x|}^{\text{absolute value}} = \sqrt{x^2} = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

Def^n/ $z = x + iy \in \mathbb{C}$ has $\underbrace{|z|}_{\text{modulus of } z} = \sqrt{x^2+y^2}$

<u>Proposition</u>: if $z, w \in \mathbb{C}$ then

1.) $|z|^2 = z\bar{z}$
2.) $|z+w| \leq |z| + |w|$ ← <u>triangle inequality</u>
3.) $|z| - |w| \leq |z - w|$
4.) $|\bar{z}| = |z|$
5.) $|z| \geq |\mathrm{Re}(z)|$
6.) $|z| \geq |\mathrm{Im}(z)|$
7.) $|iz| = |-z| = |z|$
8.) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

(<u>Comment</u>: $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$.)

<u>Proof</u>: 1.) $|z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2 = z\bar{z}$ (recall 4.) from ④)

2.) homework

3.) homework

4.) $|\bar{z}| = |x - iy| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2+y^2} = |z|$.

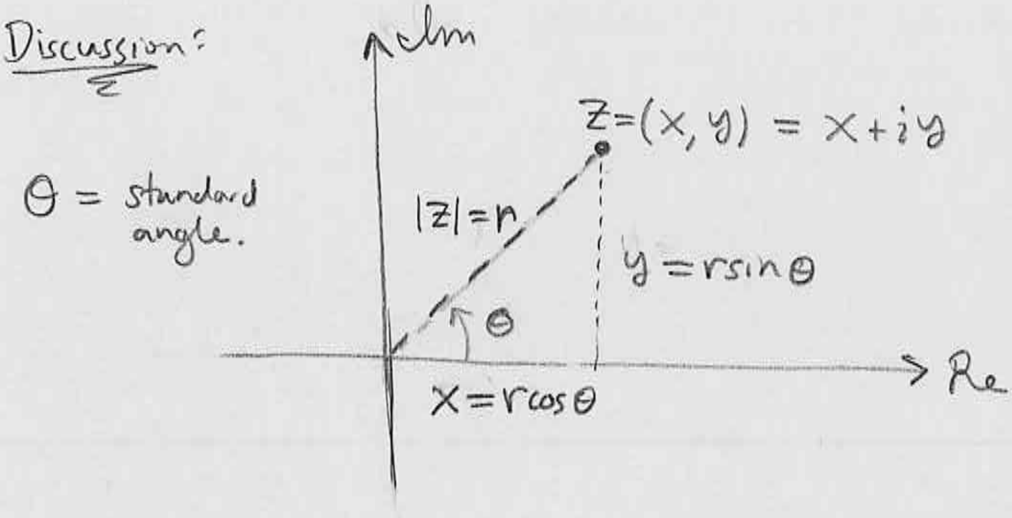
<u>OR</u> $|\bar{z}| = \sqrt{\bar{z}\bar{\bar{z}}} = \sqrt{\bar{z}z} = \sqrt{z\bar{z}} = |z|$.

5.) $|z| = \sqrt{x^2+y^2} \geq \sqrt{x^2} = |x| = |\mathrm{Re}(z)|$.

Proofs of 6 & 7 & 8 left to reader. //

Geometry of Complex Plane

Discussion:



Observe $z = r\cos\theta + i r\sin\theta = |z| \underbrace{(\cos\theta + i\sin\theta)}_{cis(\theta)}$ popular notation for this expression.

We find a complex number can be expressed (non-uniquely) by

Def^n/ $\boxed{z = |z|\, cis(\theta)}$ ← polar form of z

The angle θ is easily replaced with $\tilde{\theta} = \theta + 2\pi k$ for any integer k. We can remove this ambiguity by a <u>choice</u> (example of "branch-cut"...)

Def^n/
$$\arg(z) = \{\theta \in \mathbb{R} \mid z = |z|\, cis\,\theta\}$$
$$\text{Arg}(z) = \{\theta \in (-\pi, \pi] \mid z = |z|\, cis\,\theta\}$$