Continuity of functions with a complex domain

To begin we describe a few <u>topological</u> concepts.

Def$^{\underline{n}}$/ <u>Open</u> <u>Disk</u> of radius r centered at z_o,

$$D(z_o, r) = \{ z \in \mathbb{C} \mid |z - z_o| < r \}$$

The <u>deleted</u> <u>open</u> <u>disk</u> is $D(z_o, r)_o = D(z_o, r) - \{z_o\}$

The <u>closed</u> <u>disk</u> is $\overline{D(z_o, r)} = \{ z \in \mathbb{C} \mid |z - z_o| \leq r \}$.

$D(z_o, r)$ $\quad$ $D(z_o, r)_o$ $\quad$ $\overline{D(z_o, r)}$

<u>Remark</u>: you might worry we'll confuse the topological closure — with <u>conjugation</u> $\bar{z}$ but I think there is not much danger in context.

These open disks form a topological basis for the metric topology on $\mathbb{C}$. When you study topology properly you learn that topology is the general study of continuity and the concept of "open" and "closed" is quite subtle.

Def$^{\underline{n}}$/ Let $U \subseteq \mathbb{C}$ and $z_o \in U$ then we say z_o is an <u>interior point</u> iff $\exists r > 0$ such that $D(z_o, r) \subseteq U$. If each point of U is interior then U is <u>open</u>.

E15

(open) $\quad$ (not open)

← <u>not</u> an interior pt, $\nexists\, r > 0$ fitting the criteria.

A point such as W_0 in $\boxed{E15}$ is called a boundary point of U.

$\boxed{\text{Def}^{n}/ \ \text{bd}(U) = \{ z_0 \in \mathbb{C} \mid D(z_0, r) \cap U \not\subseteq U. \ \forall r \}}$

A point z_0 is on the boundary of U iff each open disk centered at z_0 intersects points outside U.

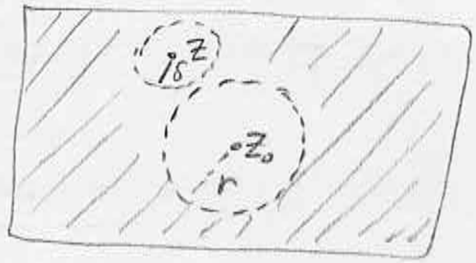
$\boxed{\text{Def}^{n}/ \ \overline{U} = U \cup \text{bd}(U) \text{ is the } \underline{\text{closure}} \text{ of } U}$

I suppose, I should define close sets,

$\boxed{\text{Def}^{n}/ \ V \subseteq \mathbb{C} \text{ is closed iff } \mathbb{C} - V \text{ is open.}}$

You can show that the closure of a set forms a closed set. Also, $\overline{D(z_0, r)}$ is closed. Why?

$\boxed{E16}$ $\mathbb{C} - \overline{D}(z_0, r)$ is pictured



note any point z is interior to the complement of $D(z_0, r)$ therefore $\mathbb{C} - \overline{D(z_0, r)}$ is open which proves $\overline{D(z_0, r)}$ is closed.

▶ Some texts go on to discuss limits, continuity etc... at this point, but I'll finish our big-picture tour of topology of $\mathbb{C}$.

Connected? What does this word mean? Perhaps disconnected is easier, a set $U \subseteq \mathbb{C}$ is called disconnected if $\exists$ nonempty U_1, U_2 that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = U$. The sets U_1 and U_2 separate U, they're a "separation"

$\boxed{\text{Def}^{n}/\ U \subseteq \mathbb{C} \text{ is } \underline{\text{connected}} \text{ iff } \nexists \text{ a separation of } U.}$

That said, we have more use for the following,

> Def^{n}/ $U \subseteq \mathbb{C}$ is <u>path</u> <u>connected</u> iff $\forall\ a, b \in U$ there is a continuous map $\gamma : [0,1] \longrightarrow \mathbb{C}$ with $\gamma(0) = a$ and $\gamma(1) = b$. This γ is called the path joining a and b

$\boxed{\text{E17}}$ $\mathbb{C}$ is path connected. Proof is simple, let $a, b \in \mathbb{C}$

$$\gamma(t) = a + t(b-a)$$

is continuous with $\gamma(0) = a$ and $\gamma(1) = b$.

<u>oh noes</u>: I just used "continuous" w/o meaning. We should fix that. I'll get back to it on 33.

<u>Remark</u>: the criteria of γ continuous could be replaced with some other type of curve. For example, you might take γ as a polygonal path to avoid

subtlety with continuity. This would be another kind of connectedness. Or you could use smooth curves...

> Th^{m}/ If $f(x,y) = u(x,y) + iv(x,y)$ and $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial x} = 0$ for all points in an <u>open</u> <u>connected</u> set D then $f(x,y) = C_0 \quad \forall (x,y) \in D$.

<u>Proof</u>: probably a homework. Notice Freitag <u>uses</u> the criteria to define <u>connected</u> on pg. 53. We'll discuss this in lecture a little later on (after CR-eq^{n}s.) //

Limit of a Complex Function

Def^n/ If $f : \text{dom}(f) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ then we say $\lim_{z \to z_0} f(z) = L$ or $f(z) \longrightarrow L$ as $z \longrightarrow z_0$ iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $z \in \mathbb{C}$ with $0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \epsilon$.

Notice this is equivalent to $\lim_{\vec{r} \to \vec{a}} f(\vec{r}) = L$ for a function $f : \text{dom}(f) \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. The limiting process approaches from <u>all</u> directions. We should keep in mind the calculus III fact:

- If $f(z_1(t)) \longrightarrow L_1$ as $f(z_2(t)) \longrightarrow L_2$ for z_1, z_2 paths going to $z_1(0) = z_2(0) = z_0$ and if $L_1 \neq L_2$ then $\lim_{z \to z_0} f(z)$ d.n.e. as a complex value.

Many properties are the same:

Properties of lim for $\mathbb{C} \xrightarrow{f} \mathbb{C}$

Given $\lim_{z \to z_0} f(z) = L_f$ and $\lim_{z \to z_0} g(z) = L_g$

1.) $\lim_{z \to z_0} (f(z) + g(z)) = L_f + L_g$

2.) $\lim_{z \to z_0} (f(z)\, g(z)) = L_f L_g$

3.) $\lim_{z \to z_0} (c) = c$

4.) $\lim_{z \to z_0} \left(\dfrac{f(z)}{g(z)}\right) = \dfrac{L_f}{L_g}$ provided $L_g \neq 0$

5.) If $\lim_{z \to z_0} g(z) = w_0$ and $\lim_{w \to w_0} f(w) = L$ then $\lim_{z \to z_0} f(g(z)) = L$.

Proof: see 34.

Continuity of $\mathbb{C} \xrightarrow{f} \mathbb{C}$

$\boxed{\text{Def}^{\underline{n}}/}$ f is continuous at $z_0 \in \text{dom}(f)$ iff $\lim_{z \to z_0} f(z) = f(z_0)$. If f is continuous at each $z_0 \in U$ then f is continuous on U. If f is continuous on $\text{dom}(f)$ then f is continuous.

Thanks to the properties of limit on f with $\mathbb{C} \xrightarrow{f} \mathbb{C}$ we have the following:

$\boxed{\text{Th}^{\underline{m}}/}$ If f, g are continuous at z_0 then $f \pm g$, fg and $\frac{f}{g}$ are also continuous at z_0 (assuming $g(z_0) \neq 0$ for $\frac{f}{g}$). Moreover, if f is continuous at $g(z_0)$ and g is continuous at z_0 then $f \circ g$ is cont. at z_0

Up to here this mirrors theory for $\mathbb{R} \xrightarrow{f} \mathbb{R}$, however a complex function $f : \text{dom}(f) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ can be written as $f = \text{Re}(f) + i\text{Im}(f)$ where our usual notation is $f = u + iv$, and $u, v : \text{dom}(f) \subseteq \mathbb{C} \longrightarrow \mathbb{R}$.

$\boxed{\text{Th}^{\underline{m}}/}$ $\lim_{z \to z_0} (u + iv) = a + ib \iff \lim_{z \to z_0} u = a$ and $\lim_{z \to z_0} v = b$

<u>Proof</u>: see 34 //

<u>Notice</u>: the limit of a complex function is given by the limits of it's real & imaginary component functions.