Continuity of functions with a complex domain

To begin we describe a few <u>topological</u> concepts.

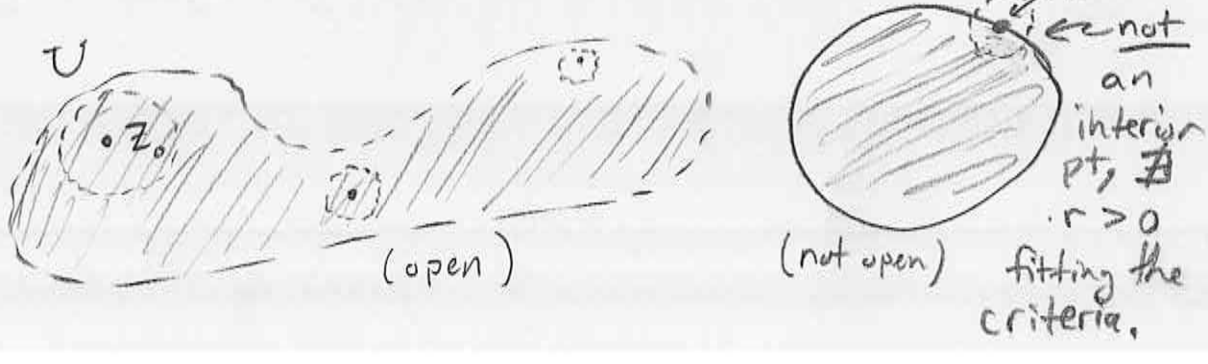
Def^n/ <u>Open</u> <u>Disk</u> of radius r centered at z_o,
$$D(z_o, r) = \{ z \in \mathbb{C} \mid |z - z_o| < r \}$$
The <u>deleted</u> <u>open</u> <u>disk</u> is $D(z_o, r)_o = D(z_o, r) - \{z_o\}$
The <u>closed</u> <u>disk</u> is $\overline{D(z_o, r)} = \{ z \in \mathbb{C} \mid |z - z_o| \leq r \}$.

<u>Remark</u>: you might worry we'll confuse the topological closure — with <u>conjugation</u> $\bar{z}$ but I think there is not much danger in context.

These open disks form a topological basis for the metric topology on $\mathbb{C}$. When you study topology properly you learn that topology is the general study of continuity and the concept of "open" and "closed" is quite subtle.

Def^n/ Let $U \subseteq \mathbb{C}$ and $z_o \in U$ then we say z_o is an <u>interior point</u> iff $\exists r > 0$ such that $D(z_o, r) \subseteq U$. If each point of U is interior then U is <u>open</u>.

E15



A point such as w_o in E15 is called a boundary point of U.

Defn/ $bd(U) = \{ z_o \in \mathbb{C} \mid D(z_o, r) \cap U \not\subseteq U \ \forall r \}$

A point z_o is on the boundary of U iff each open disk centered at z_o intersects points outside U.

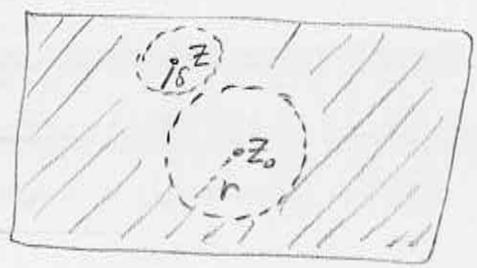
Defn/ $\overline{U} = U \cup bd(U)$ is the <u>closure</u> of U

I suppose, I should define <u>close</u> <u>sets</u>,

Defn/ $V \subseteq \mathbb{C}$ is closed iff $\mathbb{C} - V$ is open.

You can show that the <u>closure</u> of a set forms a closed set. Also, $\overline{D(z_o, r)}$ is closed. Why?

E16 $\mathbb{C} - \overline{D}(z_o, r)$ is pictured



note any point z is interior to the complement of $D(z_o, r)$ therefore $\mathbb{C} - \overline{D(z_o, r)}$ is open which proves $\overline{D(z_o, r)}$ is closed.

▶ Some texts go on to discuss limits, continuity etc... at this point, but I'll finish our big-picture tour of topology of $\mathbb{C}$.

<u>Connected?</u> What does this word mean? Perhaps <u>disconnected</u> is easier, a set $U \subseteq \mathbb{C}$ is called <u>disconnected</u> if $\exists$ nonempty U_1, U_2 that $U_1 \cap U_2 = \phi$ and $U_1 \cup U_2 = U$. The sets U_1 and U_2 <u>separate</u> U, they're a "separation"

Def^{n}/ $U \subseteq \mathbb{C}$ is <u>connected</u> iff $\nexists$ a separation of U.

That said, we have more use for the following,

Def^{n}/ $U \subseteq \mathbb{C}$ is <u>path</u> <u>connected</u> iff $\forall\, a, b \in U$ there is a continuous map $\gamma : [0,1] \longrightarrow \mathbb{C}$ with $\gamma(0) = a$ and $\gamma(1) = b$. This γ is called the path joining a and b

E17 $\mathbb{C}$ is path connected. Proof is simple, let $a, b \in \mathbb{C}$

$$\gamma(t) = a + t(b-a)$$

is continuous with $\gamma(0) = a$ and $\gamma(1) = b$.

<u>oh noes</u>: I just used "continuous" w/o meaning. We should fix that. I'll get back to it on (33).

<u>Remark</u>: the criteria of γ continuous could be replaced with some other type of curve. For example, you might take γ as a polygonal path to avoid

subtlety with continuity. This would be another kind of connectedness. Or you could use smooth curves...

Th^{m}/ If $f(x,y) = u(x,y) + i v(x,y)$ and $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial x} = 0$ for all points in an <u>open</u> <u>connected</u> set D then $f(x,y) = c_o$ $\forall\, (x,y) \in D$.

<u>Proof</u>: probably a homework. Notice Freitag <u>uses</u> the criteria to define <u>connected</u> on pg. 53. We'll discuss this in lecture a little later on (after CR – eq^{ns}.) //

Limit of a Complex Function

Def^{n}/ If $f : \text{dom}(f) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ then we say $\lim_{z \to z_0} f(z) = L$ or $f(z) \longrightarrow L$ as $z \longrightarrow z_0$ iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $z \in \mathbb{C}$ with $0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \epsilon$.

Notice this is equivalent to $\lim_{\vec{r} \to \vec{a}} f(\vec{r}) = L$ for a function $f : \text{dom}(f) \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. The limiting process approaches from <u>all</u> directions. We should keep in mind the calculus III fact:

- If $f(z_1(t)) \longrightarrow L_1$ as $f(z_2(t)) \longrightarrow L_2$ for z_1, z_2 paths going to $z_1(0) = z_2(0) = z_0$ and if $L_1 \neq L_2$ then $\lim_{z \to z_0} f(z)$ <u>d.n.e.</u> as a complex value.

Many properties are the same:

<u>Properties of lim for $\mathbb{C} \xrightarrow{f} \mathbb{C}$</u>

Given $\lim_{z \to z_0} f(z) = L_f$ and $\lim_{z \to z_0} g(z) = L_g$

1.) $\lim_{z \to z_0} (f(z) + g(z)) = L_f + L_g$

2.) $\lim_{z \to z_0} (f(z)\, g(z)) = L_f L_g$

3.) $\lim_{z \to z_0} (c) = c$

4.) $\lim_{z \to z_0} \left(\frac{f(z)}{g(z)}\right) = \frac{L_f}{L_g}$ provided $L_g \neq 0$

5.) If $\lim_{z \to z_0} g(z) = w_0$ and $\lim_{w \to w_0} f(w) = L$ then $\lim_{z \to z_0} f(g(z)) = L$.

<u>Proof</u>: see (34) and (35)

Continuity of $\mathbb{C} \xrightarrow{f} \mathbb{C}$

Def^{n}/ f is continuous at $z_0 \in dom(f)$ iff $\lim_{z \to z_0} f(z) = f(z_0)$. If f is continuous at each $z_0 \in U$ then f is continuous on U. If f is continuous on $dom(f)$ then f is continuous.

Thanks to the properties of limit on f with $\mathbb{C} \xrightarrow{f} \mathbb{C}$ we have the following:

Th^{m}/ If f, g are continuous at z_0 then $f \pm g$, fg and $\frac{f}{g}$ are also continuous at z_0 (assuming $g(z_0) \neq 0$ for $\frac{f}{g}$). Moreover, if f is continuous at $g(z_0)$ and g is continuous at z_0 then $f \circ g$ is cont. at z_0

Up to here this mirrors theory for $\mathbb{R} \xrightarrow{f} \mathbb{R}$, however a complex function $f : dom(f) \subseteq \mathbb{C} \rightarrow \mathbb{C}$ can be written as $f = Re(f) + i Im(f)$ where our usual notation is $f = u + iv$, and $u, v : dom(f) \subseteq \mathbb{C} \rightarrow \mathbb{R}$.

Th^{m}/ $\lim_{z \to z_0} (u + iv) = a + ib \iff \lim_{z \to z_0} u = a$ and $\lim_{z \to z_0} v = b$

<u>Proof</u>: see (36) //

<u>Notice</u>: the limit of a complex function is given by the limits of it's real & imaginary component functions.

Proofs of limit laws for $\mathbb{C} \xrightarrow{f} \mathbb{C}$

3.) **Claim:** $\lim_{z \to z_o} (c) = c.$

Proof: Let $\epsilon > 0$ and choose $\delta = 42$. If $z \in \mathbb{C}$ and $0 < |z - z_o| < \delta$ then $|c - c| = 0 < \epsilon$. Thus $\lim_{z \to z_o} c = c.$ //

1.) **Claim:** Given $\lim_{z \to z_o} f(z) = L_f$ and $\lim_{z \to z_o} g(z) = L_g$ it follows $\lim_{z \to z_o} (f(z) + g(z)) = L_f + L_g$

Proof: Let $\epsilon > 0$ and, by the given limits, choose $\delta_f, \delta_g > 0$ such that $0 < |z - z_o| < \delta_f \Rightarrow |f(z) - L_f| < \epsilon/2$ and $0 < |z - z_o| < \delta_g \Rightarrow |g(z) - L_g| < \epsilon/2$. Let $\delta = \min(\delta_f, \delta_g)$ and suppose $z \in \mathbb{C}$ such that $0 < |z - z_o| < \delta \leq \delta_f, \delta_g$. Observe, by triangle ineq.

$$\begin{aligned}|f(z) + g(z) - (L_f + L_g)| &\leq |f(z) - L_f| + |g(z) - L_g| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon.\end{aligned}$$

Thus $\lim_{z \to z_o} (f(z) + g(z)) = L_f + L_g$. //

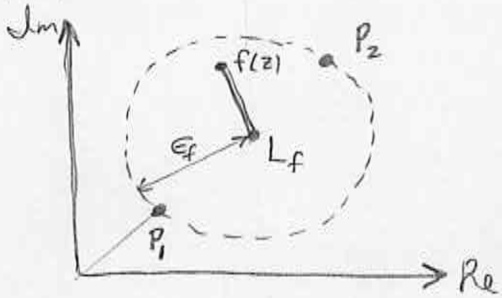
2.) **Claim:** If $\lim_{z \to z_o} f(z) = L_f$ and $\lim_{z \to z_o} g(z) = L_g$ then $\lim_{z \to z_o} f(z)g(z) = L_f L_g$

preparing for the proof, we wish to make $|f(z)g(z) - L_f L_g| < \epsilon$ by control of $|f(z) - L_f| < \epsilon_f$ and $|g(z) - L_g| < \epsilon_g$. We can choose δ_f and δ_g to make ϵ_f and ϵ_g as small as we like. Then $\delta = \min(\delta_f, \delta_g)$ will force all the conditions to apply at once. Consider, adding zero ↘

$$\begin{aligned}|f(z)g(z) - L_f L_g| &= |f(z)(g(z) - L_g) + (f(z) - L_f)L_g| \\ &\leq |f(z)||g(z) - L_g| + |f(z) - L_f||L_g| \\ &< \underbrace{|f(z)|}_{\text{we need to control this.}}\epsilon_g + \epsilon_f |L_g|\end{aligned}$$

Proofs of limit laws continued

Given $z \in \mathbb{C}$ with $|f(z) - L_f| < \epsilon_f$ what M bounds $|f(z)|$?
Picture this:



Let $\hat{L}_f = e^{i\arg(L_f)}$ the geometrically it's clear that:

$$\underbrace{|L_f - \epsilon_f \hat{L}_f|}_{P_1} < |f(z)| < \underbrace{|L_f + \epsilon_f \hat{L}_f|}_{P_2}$$

we should be able to prove this.

Consider, $|A| - |B| \leq |A - B| \Rightarrow |f(z)| - |L_f| \leq |f(z) - L_f| < \epsilon_f$ hence, $|f(z)| < \epsilon_f + |L_f|$. This is not as sharp as the geometry suggests since $|P_2| \leq |L_f| + |\epsilon_f \hat{L}_f| = |L_f| + \epsilon_f$, BUT, let's use it. We have, recalling our work on (34), that

$$|f(z)g(z) - L_f L_g| < |f(z)|\epsilon_g + \epsilon_f |L_g|$$

$$< (\epsilon_f + |L_f|)\epsilon_g + \epsilon_f |L_g| \leq \epsilon$$

pick ϵ_f, ϵ_g to make it happen.

A choice → $\epsilon_f = \dfrac{\epsilon}{2|L_g|}$ and $\epsilon_g = \dfrac{\epsilon}{2(\epsilon_f + |L_f|)}$. I invite the reader to complete the proof. Clearly $L_g = 0$ needs separate argument.

4.) **Claim:** If $\lim_{z \to z_0} f(z) = L_f$ and $\lim_{z \to z_0} g(z) = L_g \neq 0$ then $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L_f}{L_g}$.

Proof: Show $\lim_{z \to z_0} \dfrac{1}{z} = \dfrac{1}{z_0}$ then apply (5.) to see $\lim_{z \to z_0} \left(\dfrac{1}{g(z)}\right) = \dfrac{1}{L_g}$.

Finally $\dfrac{f(z)}{g(z)} = f(z) \cdot \dfrac{1}{g(z)} \longrightarrow L_f \cdot \left(\dfrac{1}{L_g}\right) = \dfrac{L_f}{L_g}$ by 2). I leave it to the reader to show $\dfrac{1}{z} \to \dfrac{1}{z_0}$ as $z \to z_0 \neq 0$.

5.) **Claim:** $\lim_{z \to z_0} g(z) = w_0$ and $\lim_{w \to w_0} f(w) = L \Rightarrow \lim_{z \to z_0} f(g(z)) = L$

Proof: Let $\epsilon > 0$. Further, let $\delta_f > 0$ such that $0 < |w - w_0| < \delta_f \Rightarrow |f(w) - L| < \epsilon$. We know δ_f exists as we are given $\lim_{w \to w_0} f(w) = L$.

Choose $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |g(z) - w_0| < \delta_f$.
We can choose such a δ since $\lim_{z \to z_0} g(z) = w_0$ is given.
Let $z \in \mathbb{C}$ with $0 < |z - z_0| < \delta$ and note by construction of δ,

$$|f(g(z)) - L| < \epsilon \quad \therefore \lim_{z \to z_0} (f(g(z))) = L. //$$

Proof of $\lim_{z\to z_0}(u+iv)=a+ib \Leftrightarrow \lim_{z\to z_0} u=a$ AND $\lim_{z\to z_0} v=b$.

<u>Proof</u>: We begin with $\Leftarrow$]. Assume $a,b\in\mathbb{R}$ and $\mathbb{C}\xrightarrow{u,v}\mathbb{R}$ where $\lim_{z\to z_0} u(z)=a$ and $\lim_{z\to z_0} v(z)=b$. This data indicates we can choose $\delta>0$ such that $0<|z-z_0|<\delta \Rightarrow |u(z)-a|<\epsilon_u$ and $|v(z)-b|<\epsilon_b$. We have <u>control</u> over ϵ_u and ϵ_b. Consider:

$$|u(z)+iv(z)-(a+ib)| \leq |u(z)-a|+|i(v(z)-b)|$$
$$= |u(z)-a|+|v(z)-b|. \quad (\text{I.})$$

Let $\epsilon>0$ and choose $\delta>0$ such that $0<|z-z_0|<\delta \Rightarrow |u(z)-a|<\epsilon/2$ and $|v(z)-b|<\epsilon/2$. Then for $z\in\mathbb{C}$ with $0<|z-z_0|<\delta$ and inequality (I.) we find $|(u+iv)(z)-(a+ib)|<\epsilon$. Thus $\lim_{z\to z_0}(u+iv)=a+ib$. //

$\Rightarrow$] Assume $\lim_{z\to z_0}(u+iv)=a+ib$. Let $\epsilon>0$ and choose $\delta>0$ s.t. $0<|z-z_0|<\delta \Rightarrow |u(z)+iv(z)-(a+ib)|<\epsilon$. Consider: as $|\text{Re}(w)|\leq|w|$ it follows

$$|u(z)-a| \leq |u(z)+iv(z)-a-ib| < \epsilon.$$

Thus $\lim_{z\to z_0} u(z)=a$. Likewise, as $|\text{Im}(w)|\leq|w|$

$$|v(z)-b| \leq |u(z)+iv(z)-a-ib| < \epsilon$$

Thus $\lim_{z\to z_0} v(z)=b$. //

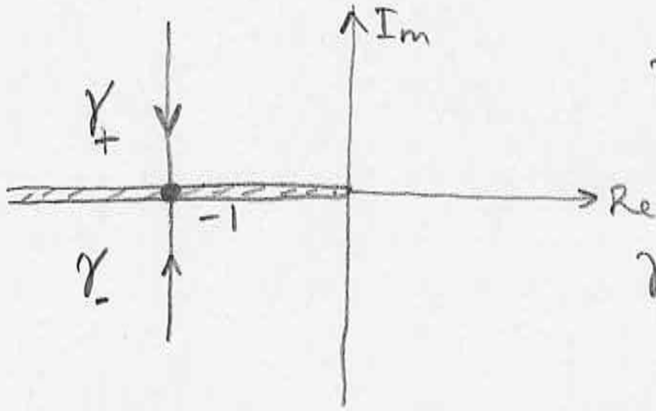
limit of vector is vector of limits. See Advanced Calculus for more...

<u>Remark</u>: to prove $\lim_{\vec{x}\to\vec{a}}\vec{f}(\vec{x})=\vec{L} \Leftrightarrow \lim_{\vec{x}\to\vec{a}} f^i(\vec{x})=L^i$ for all $i=1,2,\dots,n$ we use almost the same argument. Except, for $\Leftarrow$] use ϵ/n and for $\Rightarrow$] use $|L^i|\leq\|\vec{L}\|$.

E18 An important example of <u>discontinuity</u>.

Let $f(z) = \mathrm{Arg}(z)$ (principle argument).

We argue that $\lim_{z \to -1} f(z)$ d.n.e.



$\gamma_+(t) = -1 - it$ for $t \geq 0$

$\gamma_-(t) = -1 + it$ for $t \geq 0$

Path $\gamma_+(t) \longrightarrow -1$ and $\gamma_-(t) \longrightarrow -1$ as $t \to 0^+$

However, $\mathrm{Arg}(\gamma_+(t)) = \mathrm{Arg}(-1-it) = \pi - \tan^{-1}(t)$

whereas $\mathrm{Arg}(\gamma_-(t)) = \mathrm{Arg}(-1+it) = -\pi - \tan^{-1}(t)$

for $t \geq 0$. It follows,

$$\lim_{t \to 0^+} \mathrm{Arg}(\gamma_+(t)) = \pi$$

Whereas

$$\lim_{t \to 0^+} \mathrm{Arg}(\gamma_-(t)) = -\pi$$

Thus, $\lim_{z \to -1} \mathrm{Arg}(z)$ d.n.e. By the same argument with a replacing -1 we can show

$$\lim_{z \to a} \mathrm{Arg}(z) \text{ d.n.e. for all } a \in [-\infty, 0].$$

Therefore, $f(z) = \mathrm{Arg}(z)$ is discontinuous on negative real axis.

<u>Remark</u>: Freitag argued the same idea with <u>sequential limits</u>.

- At this point in lecture we discussed $\mathbb{C} \cup \{\infty\}$ and $z \longrightarrow \infty$. The essential ideas were:

$$D_\epsilon(\infty) = \{ w \in \mathbb{C} \mid |w| > 1/\epsilon \} \leftarrow \text{open nbhd. of } \infty$$

$$\lim_{z \to z_0} f(z) = \infty \text{ iff } \left[\begin{array}{l} \text{for } \epsilon > 0, \exists \delta > 0 \text{ such that} \\ 0 < |z - z_0| < \delta \Rightarrow |f(z)| > 1/\epsilon. \end{array} \right]$$

$$\lim_{z \to \infty} f(z) = w_0 \text{ iff } \left[\begin{array}{l} \text{for } \epsilon > 0, \exists \delta > 0 \text{ such that} \\ |z| > 1/\delta \Rightarrow |f(z) - w_0| < \epsilon. \end{array} \right.$$

Geometrically this <u>extended</u> <u>complex</u> <u>plane</u> $\mathbb{C} \cup \{\infty\}$ naturally corresponds to the Riemann Sphere through the Stereographic projections. I gave a handout from Saff & Snider.

- We looked over pages 39–40 of my previous complex notes (In Chapter 2 on topology and mappings, which has <u>much</u> overlap with our current endeavor)

One last item I overlooked on limits.

Thm/ If $\lim_{z \to z_0} g(z) = L_g$ and $\lim_{w \to L_g} f(w) = L_f$ then $\lim_{z \to z_0} (f(g(z))) = L_f$.

<u>Proof</u>: left to reader.

<u>Remark</u>: If $\lim_{w \to w_0} f(w) = f(w_0)$ and $w_0 = \lim_{z \to z_0} g(z)$ then $\lim_{z \to z_0} f(g(z)) = f(\lim_{z \to z_0} g(z))$. This is often how we use the Thm.