Differentiation on $\mathbb{C}$

Suppose $f : \text{dom}(f) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ we also denote this by $\mathbb{C} \xrightarrow{f} \mathbb{C}$. We seek to understand differentiation with respect to a <u>complex</u> <u>variable</u>.

Defn/ Let $f'(z_o) = \lim\limits_{h \to 0} \left[\dfrac{f(z_o+h) - f(z_o)}{h} \right]$.
If $f'(z_o) \in \mathbb{C}$ then f is complex-differentiable at z_o.
Moreover, $z \longmapsto f'(z)$ for all $z \in \text{dom}(f)$ with $f'(z) \in \mathbb{C}$ defines a new function f' called the <u>derivative</u> of f.

<u>Remark</u>: Notice $\lim\limits_{z \to z_o} f(z) = L_1$ and $\lim\limits_{z \to z_o} f(z) = L_2$

$$\Rightarrow 0 = \lim_{z \to z_o} (f(z) - f(z)) = \lim_{z \to z_o} f(z) - \lim_{z \to z_o} f(z) = L_1 - L_2$$

hence $L_1 = L_2$ and the limit, when it exists, is <u>unique</u>. This suffices to show f' is in fact a function.

<u>Notation</u>: $f'(z_o) = \left.\dfrac{df}{dz}\right|_{z_o} = \left(\dfrac{df}{dz}\right)(z_o)$ and $f'(z) = \dfrac{df}{dz}$.
generally we continue to use the same notations as for $\mathbb{R} \xrightarrow{f} \mathbb{R}$. For example $f''(z) = \dfrac{d}{dz}\left(\dfrac{df}{dz}\right)$ etc...

$\boxed{\text{E19}}$ Let $f(z) = z$.

$$f'(z) = \lim_{h \to 0} \left(\frac{z + h - z}{h} \right) = \lim_{h \to 0} \left(\frac{h}{h} \right) = \lim_{h \to 0} (1) = 1. //$$

We've found

$$\frac{d}{dz}(z) = \frac{dz}{dz} = 1.$$

You can try $f(z) = z^2, z^3, 2z - 3$ etc... just like Math 131 month 1.

Thm/ If f, g are differentiable at z then

1.) $(f+g)'(z) = f'(z) + g'(z)$,

2.) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$,

3.) $\left(\frac{f}{g}\right)'(z) = \dfrac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$ (for $g(z) \neq 0$)

4.) $(cf)'(z) = cf'(z)$ for constant $c \in \mathbb{C}$.

Proof: Lemma: if f is differentiable at z_o then f is continuous at z_o.

Proof of Lemma: note $\lim_{z \to z_o} f(z) = f(z_o) \iff \lim_{h \to 0} f(z_o+h) = f(z_o)$ by substitution $h = z - z_o$. Consider then,

$$\lim_{h \to 0}\left(f(z_o+h) - f(z_o)\right) = \lim_{h \to 0}\left(\frac{f(z_o+h) - f(z_o)}{h} \cdot h\right)$$

$$= \lim_{h \to 0}\left(\frac{f(z_o+h) - f(z_o)}{h}\right) \lim_{h \to 0}(h)$$: note we need differentiable at z_o to apply limit law here.

$$= f'(z_o) \cdot (0) = 0.$$

Hence, $\lim_{h \to 0} f(z_o+h) = f(z_o)$, this proves the Lemma. Now we turn to the Thm,

1.) I leave for the students as an easy hwk ☺.

Proof of Th$^{\underline{m}}$ from 40 continued

(2) $$(fg)'(z) = \lim_{h\to 0}\left[\frac{f(z+h)g(z+h) - f(z)g(z)}{h}\right]$$

$$= \lim_{h\to 0}\left[\left(\frac{f(z+h)-f(z)}{h}\right)g(z+h) + f(z)\left(\frac{g(z+h)-g(z)}{h}\right)\right]$$

$$= \lim_{h\to 0}\left[\frac{f(z+h)-f(z)}{h}\right]\lim_{h\to 0} g(z+h) + f(z)\lim_{h\to 0}\left[\frac{g(z+h)-g(z)}{h}\right]$$

$$= f'(z)g(z) + f(z)g'(z)$$ by Lemma & def$^{\underline{n}}$ of f', g'. //

(3) Let $H(z) = \dfrac{f(z)}{g(z)}$ and consider $Hg = f$

thus $H'g + Hg' = f'$. Solve for H'

$$H' = \frac{f' - Hg'}{g} = \frac{f' - \frac{f}{g}g'}{g} = \frac{f'g - fg'}{g^2} \;//$$

(4) $$\frac{d}{dz}(c) = \lim_{h\to 0}\left(\frac{c-c}{h}\right) = \lim_{h\to 0}\left(\frac{0}{h}\right) = 0. //$$

<u>Corolary to Th$^{\underline{m}}$</u>: If $f(z) \in \mathbb{C}[z]$ or $f(z) = \dfrac{p(z)}{q(z)}$ where $p(z), q(z) \in \mathbb{C}[z]$ (= polynomials in z with coefficients from $\mathbb{C}$) then $f'(z)$ exists for $z \in \mathbb{C}$ with $q(z) \neq 0$.

- Summary of proof about to give on 42

$$\frac{d}{dz}\left(a_0 + a_1 z + \cdots + a_n z^n\right) = \underbrace{a_1 + 2a_2 z + \cdots + na_n z^{n-1}}$$

$$\frac{d}{dz}\left(\frac{p(z)}{q(z)}\right) = \frac{p'(z)q(z) - p(z)q'(z)}{q^2(z)}$$

we'll prove by induction on $n \in \mathbb{N}\cup\{0\}$.

Proof: Lemma: for $n \in \mathbb{N}$, $\frac{d}{dz}(z^n) = nz^{n-1}$.

Proof of Lemma: in $\boxed{E19}$ we proved $n=1$ case $\frac{dz}{dz} = 1$.

Assume $\frac{d}{dz}(z^n) = nz^{n-1}$ for some $n \geq 1$. Consider

$$\frac{d}{dz}(z^{n+1}) = \frac{d}{dz}(z^n z) \quad : \text{def}^{\underline{n}} \text{ of } z^{n+1}.$$

$$= \frac{d}{dz}(z^n)\, z + z^n \frac{dz}{dz} \quad : \text{product rule.}$$

$$= nz^{n-1}z + z^n \quad : \text{by induction hypothesis.}$$

$$= (n+1)z^n$$

$$= (n+1)z^{n+1-1}$$

Thus the induction hypothesis is true for $n+1$ and we conclude by induction on n that $\frac{d}{dz}(z^n) = nz^{n-1}$ for all $n \in \mathbb{N}$.//

[I'll let you show in homework that the sum rule for derivatives is inductively extended to n-summands in the natural way; $\frac{d}{dz}\sum_{j=1}^{n} f_j(z) = \sum_{j=1}^{n} \frac{df_j}{dz}$.

Using the above-mentioned hwk result, if $\sum_{j=0}^{n} a_j z^j$ polynomial

$$\frac{d}{dz}\left(\sum_{j=0}^{n} a_j z^j\right) = \sum_{j=0}^{n} \frac{d}{dz}(a_j z^j)$$

$$= \sum_{j=0}^{n} a_j \frac{dz^j}{dz}$$

$$= \sum_{j=1}^{n} j a_j z^{j-1}$$

note $j=0$ gives constant which diff. to zero.

Finally, the quotient rule completes the Proof of the Coro.//

$\boxed{E20}$ $\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{0(z)-1(1)}{z^2} = \frac{-1}{z^2}$

<u>Remark</u>: YES, we can extend the power rule $\frac{d}{dz}(z^n) = nz^{n-1}$ to $n \notin \mathbb{N}$, and we will, just not quite yet.

$\boxed{\underline{\text{Th}^{\underline{m}} \text{ (Chain Rule)}} \quad \frac{d}{dz}\left[f(g(z))\right] = f'(g(z))\,g'(z).}$

<u>Proof</u>: we begin with a Lemma know as the Th$^{\underline{m}}$ of Caratheodory:

$\boxed{\underline{\text{Lemma}}: f'(z_0) = l \in \mathbb{C} \text{ iff } \exists \varphi \text{ s.t. } f(z) = f(z_0) + \varphi(z)(z-z_0) \text{ where } \lim_{z\to z_0} \varphi(z) = l \in \mathbb{C}.}$

<u>Proof of Lemma</u>:

$\Leftarrow$ given φ with $f(z) = f(z_0) + \varphi(z)(z-z_0)$ and $\varphi(z) \to l$ as $z \to z_0$. Consider,

$$\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right) = \lim_{z\to z_0}\left(\varphi(z)\right) = l$$

hence we identify that $f'(z_0) = l \in \mathbb{C}$.

$\Rightarrow$ Given $f'(z_0)$ exists construct

$$\varphi(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{for } z \neq z_0 \\ f'(z_0) & \text{for } z = z_0 \end{cases}$$

Clearly $\lim_{z\to z_0} \varphi(z) = \lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right) = f'(z_0)$ by differentiability of f at z_0. But, this shows φ continuous at z_0 as desired. //

Proof of Chain Rule via Th$^{\underline{m}}$ of Caratheodory:

Assume $g'(z_0)$ and $f'(g(z_0))$ exist and define $w_0 = g(z_0)$ for convenience. By Th$^{\underline{m}}$ of Caratheodory $\exists\, \varphi, \psi$ such that

I $f(w) = f(w_0) + \psi(w)(w - w_0)$

with $\lim\limits_{w \to w_0} \psi(w) = f'(w_0)$

II $g(z) = g(z_0) + \varphi(z)(z - z_0)$

with $\lim\limits_{z \to z_0} \varphi(z) = g'(z_0)$.

Consider, let $w = g(z)$. III

$$\lim_{z \to z_0} \left(\frac{f(g(z)) - f(g(z_0))}{z - z_0} \right) = \lim_{z \to z_0} \left(\frac{\overbrace{f(w) - f(w_0)}}{z - z_0)} \right).$$

$$= \lim_{z \to z_0} \left[\frac{f(w_0) + \psi(w)(w - w_0) - f(w_0)}{z - z_0} \right]$$

$$= \lim_{z \to z_0} \left[\frac{1}{\cancel{z - z_0}} \Big(\psi\big(g(z_0) + \varphi(z)(z - z_0)\big) \Big) \underbrace{\varphi(z)(\cancel{z - z_0})}_{} \right]$$

this is $w - w_0$ given I and III

$$= \lim_{z \to z_0} \Big[\psi\big(g(z_0) + \varphi(z)(z - z_0)\big)\, \varphi(z) \Big]$$

$$= \psi(g(z_0) + 0)\, \varphi(z_0)$$ by limit law for composite limits. on pg. 38

$$= f'(g(z_0))\, g'(z_0). \;//$$

$\boxed{E21}$ Consider $f(z) = \bar{z}$

$$f'(z) = \lim_{h \to 0}\left(\frac{\overline{z+h} - \bar{z}}{h}\right) = \lim_{h\to 0}\left(\frac{\bar{z}+\bar{h}-\bar{z}}{h}\right) = \lim_{h\to 0}\left(\frac{\bar{h}}{h}\right).$$

In $h = x+iy$ notation we face:

$$\lim_{x+iy \to 0}\left(\frac{x-iy}{x+iy}\right)$$

Consider $x+iy = t+it$ as $t \to 0$ we find

$$\frac{x-iy}{x+iy} = \frac{t-it}{t+it} = \frac{1-i}{1+i}.$$

However, if we approach 0 via $x+iy = t-it$ then

$$\frac{x-iy}{x+iy} = \frac{t+it}{t-it} = \frac{1+i}{1-i} \quad \text{as } t \to 0.$$

Thus two paths approaching the limit pt. $0=(0,0)$ are not equal hence $\lim_{h\to 0}\left(\frac{\bar{h}}{h}\right)$ d.n.e.

<u>Remark</u>: Saff and Snider characterize the existence of $f'(z)$ as indicating $f(x,y)$ can be expressed in terms of z alone.

$$f_1(x,y) = x^2-y^2+2ixy = (x+iy)^2 = z^2$$
$$f_2(x,y) = x^2-y^2-2ixy = (x-iy)^2 = \bar{z}^2$$

we can argue $f_1'(z)$ exists whereas $f_2'(z)$ d.n.e. for reasons very much like those given for $\boxed{E21}$.
Generically we could take $f(x,y) = u(x,y)+iv(x,y)$ and use $x = \frac{1}{2}(z+\bar{z})$ & $y = \frac{1}{2i}(z-\bar{z})$ to check if $f'(z)$ exists (a horrible calculation really, but gives some concept)

$$f(x,y) = x-iy = \tfrac{1}{2}(z+\bar{z}) - \tfrac{i}{2i}(z-\bar{z}) = \bar{z}$$ (see this is silly!)