

## Complex Exponential:

(8)

Our text gives a different def<sup>n</sup> which I'll discuss later.  
For now I offer this little argument from the  
text of Nagle and Salt. (not precisely, I changed it a bit)

Discussion: the def<sup>n</sup> of  $e^z$  ought to have  $e^{z_1+z_2} = e^{z_1}e^{z_2}$   
thus  $e^{x+iy} = e^x e^{iy}$ . Moreover,  $e^x$  ought to  
match its usual meaning. Assume the chain rule  
 $\frac{d}{dy}(e^{iy}) = ie^{iy}$  then  $\frac{d}{dy}\frac{d}{dy}(e^{iy}) = i\frac{d}{dy}(e^{iy}) = i^2 e^{iy}$

hence  $e^{iy}$  solves  $z'' = -z$  or  $z'' + z = 0$ .

It's well-known from DEq's  $z = A \cos y + B \sin y$

To find  $A$  &  $B$  note  $z(0) = e^{i(0)} = e^0 = 1 = A \cos 0 = A$   
and  $z'(0) = ie^{i(0)} = i = -A \sin(0) + B \cos(0) = B$

Hence  $z(y) = e^{iy} = \cos y + i \sin y$ .

$$\text{Def}^n / \text{cis}(\theta) = e^{i\theta} = \cos \theta + i \sin \theta$$

Clearly  $e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$ .

Many beautiful results flow from this profound  
notation. For example:

1.)  $e^{z_1+z_2} = e^{z_1}e^{z_2}$  

2.)  $(e^z)^n = e^{nz}$  — de Moivre's f-la.

3.)  $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$  &  $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

• derive all manner of trig. f-la's.

Remark: we  
defer the  
proper def<sup>n</sup>  
for next week.  
This is  
cheating a  
bit although  
the results  
are spot  
on.

Trigonometry:

$$\begin{aligned}\cos(\theta + \beta) &= \cos\theta \cos\beta - \sin\theta \sin\beta && \text{adding } \angle's \\ \sin(\theta + \beta) &= \sin\theta \cos\beta + \cos\theta \sin\beta && \text{f-lab. Can derive from basic geometry.}\end{aligned}$$

Notice that

$$\begin{aligned}\cos(\theta + \beta) + i\sin(\theta + \beta) &= \cos\theta \cos\beta - \sin\theta \sin\beta + i(\sin\theta \cos\beta + \cos\theta \sin\beta) \\ &= (\cos\theta + i\sin\theta)\cos\beta + \sin\beta(-\sin\theta + i\cos\theta) \\ &= (\cos\theta + i\sin\theta)\cos\beta + (\cos\theta + i\sin\theta)i\sin\beta \\ &= (\cos\theta + i\sin\theta)(\cos\beta + i\sin\beta).\end{aligned}$$

Therefore,

$$\boxed{\text{cis}(\theta + \beta) = \text{cis}(\theta) \text{ cis}(\beta)} \quad \star$$

Now, in the imaginary exponential notation the identity  $(\star)$  simply reads:

$$\boxed{e^{i(\theta + \beta)} = e^{i\theta} e^{i\beta}} \quad \tilde{\star}$$

We assumed this on ⑧ but the calculation above shows that if we define  $e^z = e^x e^{iy}$  then  $e^{z_1 + z_2} = e^{z_1} e^{z_2}$ ,

$$\begin{aligned}e^{z_1 + z_2} &= e^{x_1 + iy_1 + x_2 + iy_2} \\ &= e^{x_1 + x_2 + i(y_1 + y_2)} \\ &= e^{x_1 + x_2} e^{i(y_1 + y_2)} \\ &= e^{x_1} e^{x_2} e^{iy_1} e^{iy_2} \\ &= e^{x_1 + iy_1} e^{x_2 + iy_2} \\ &= e^{z_1} e^{z_2}.\end{aligned}$$

by Laws of exponents  
in  $\mathbb{R}$  and  $\tilde{\star}$

## Properties of The Complex Exponential

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1.)  $e^{x+iy} = e^x e^{iy}$  where  $e^{iy} = \text{cis}(y) = \cos(y) + i\sin(y)$

2.)  $e^{z+\pi i} = -e^z$

3.)  $e^{z+2\pi k i} = e^z$

4.)  $e^{z+w} = e^z e^w$

5.)  $e^{-z} = (e^z)^{-1} = \frac{1}{e^z}$  ( $\text{Def}^n / w^{-1} = \frac{1}{w}$ )

6.)  $e^{0+0i} = 1$

7.)  $|e^{x+iy}| = |e^x|$

8.) If  $z = re^{i\theta}$  for  $r \in (0, \infty)$  then  $\exp(\ln(r) + i\theta) = z$

9.)  $z(t) = z_0 + Re^{it}$  for  $0 \leq t \leq 2\pi$ ,  $R > 0$   
parametrizes a circle centered at  $z_0$  with  
radius  $R$ .

10.)  $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$  &  $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

11.)  $(e^z)^n = e^{nz}$  for  $n \in \mathbb{Z}$ .

12.) De Moivre's Th<sup>m</sup>/  $\text{cis}(\theta)^n = \text{cis}(n\theta)$ .

Proof: we've shown 4 on ⑨, 1 is the current def<sup>n</sup> of  $e^z$ .  
item 2 is hawk. Item 11.) is also homework. Proof of 3.) follows  
if from observing  $e^{z+2\pi k i} = e^{x+i(2\pi k+y)} = e^x (\text{cis}(2\pi k+y))$

$$\begin{aligned}
 &= e^x (\cos(2\pi k+y) + i\sin(2\pi k+y)) \\
 &= e^x (\cos(y) + i\sin(y)) \quad \text{sine} \\
 &= e^x e^{iy} \quad \text{def}^n \\
 &= e^{x+iy} \quad \text{def}^n \\
 &= e^z. // \quad \text{cosine periodic}
 \end{aligned}$$

Clearly  $e^{0+0i} = e^0 \cos 0 = 1$  hence 6. is true. 5. follows  
from  $e^0 = e^{z-z} = e^z e^{-z} = 1.$

proof continued

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To prove 7. Recall  $|zw| = |z||w|$  hence

$$|e^{x+iy}| = |e^x e^{iy}| = |e^x| |e^{iy}| = e^x |e^{iy}|.$$

But,  $e^{iy} = \cos y + i \sin y$  has  $|e^{iy}| = \sqrt{\cos^2 y + \sin^2 y} = 1$ .

Therefore,  $|e^{x+iy}| = e^x$ . To establish 8 just calculate,  $\exp(\ln(r) + i\theta) = e^{\ln(r)} e^{i\theta} = re^{i\theta}$ . To see 9 we can argue geometrically:

$$|z(t) - z_0| = |Re^{it}| = R |e^{it}| = R$$

thus the distance from  $z(t)$  to  $z_0$  is constantly  $R$ . Item 10 is found from a short calculation once again,

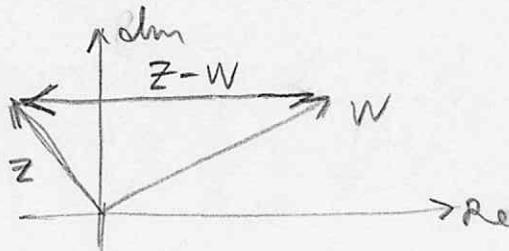
$$\pm \begin{pmatrix} e^{i\theta} &= \cos \theta + i \sin \theta \\ e^{-i\theta} &= \cos \theta - i \sin \theta \end{pmatrix} \leftarrow \begin{matrix} \text{used } \cos(-\theta) = \cos \theta \\ \sin(-\theta) = -\sin \theta \end{matrix}$$

$$e^{i\theta} + e^{-i\theta} = 2 \cos \theta \quad \text{and} \quad e^{i\theta} - e^{-i\theta} = 2i \sin \theta.$$

Finally, item 12 is a simple consequence of 11.  
in the case  $z = i\theta$ . □

Remark: something I should have emphasized more already,

$$d(z, w) = |z - w| = \text{distance from } z \text{ to } w$$



(this distance function makes ① a metric space)  
 It satisfies  
 the metric  
 axioms

$\left\{ \begin{array}{l} 1.) d(z, w) = d(w, z) \\ 2.) d(z, w) \geq 0 \text{ and } d(z, z) = 0 \text{ iff } z = 0. \\ 3.) d(a, b) \leq d(a, c) + d(c, b) \end{array} \right.$	1.) $d(z, w) = d(w, z)$
	2.) $d(z, w) \geq 0$ and $d(z, z) = 0$ iff $z = 0$ .
	3.) $d(a, b) \leq d(a, c) + d(c, b)$

E3 Calculate  $(1+i)^{24}$ .

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$$\text{Observe } 1+i = \sqrt{2} \left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = \sqrt{2} e^{i\pi/4}$$

$$\text{thus } (1+i)^{24} = (\sqrt{2} e^{i\pi/4})^{24} = (\sqrt{2})^{24} e^{\frac{24\pi i}{4}} = 2^{12} e^{6\pi i}$$

but  $e^{6\pi i} = e^0 = 1$  by property 3. of the exp. from (10).

$$\text{Thus } (1+i)^{24} = 2^{12}.$$

### Roots of Unity

These play a special role in complex analysis and algebra.

Consider  $z \in \mathbb{C}$  such that  $\underbrace{z^n}_{} = 1$  for some  $n \in \mathbb{N}$ .

$$|z^n| = 1 \Rightarrow |z|^n = 1 \Rightarrow |z| = 1.$$

Recall  $z = |z|e^{i\theta}$  hence  $z = e^{i\theta}$ . We find

$$(e^{i\theta})^n = e^{ni\theta} = 1$$

$$\Rightarrow \cos(n\theta) + i\sin(n\theta) = 1$$

$$\underbrace{\cos(n\theta)}_{} = 1 \quad \underbrace{\sin(n\theta)}_{} = 0$$

$$\text{I } \exists k \in \mathbb{Z}, n\theta = 2\pi k \quad \text{II } \exists j \in \mathbb{Z} \text{ s.t. } n\theta = j\pi$$

Observe I  $\Rightarrow$  II hence we choose sol's  $\theta = \frac{2\pi k}{n}$  for  $k \in \mathbb{Z}$ .

We find sol's of  $z^n = 1$  have the form;

$$z = \exp\left(\frac{2\pi ik}{n}\right) \text{ for } k \in \mathbb{Z}$$

How many distinct sol's do we find? Suppose  $\exists j, k \in \mathbb{Z}$  such that  $\exp\left(\frac{2\pi ik}{n}\right) = \exp\left(\frac{2\pi ij}{n}\right)$

$$\Rightarrow \exp\left(i\left[\frac{2\pi}{n}(k-j)\right]\right) = \exp(0)$$

$$\Rightarrow \frac{2\pi}{n}(k-j) = 2\pi l \text{ for some } l \in \mathbb{Z}.$$

$$\Rightarrow k-j = nl \therefore k \equiv_n j$$

Or,  $\bar{k} = \bar{j}$  in  $\mathbb{Z}_n$ . We find a connection with modular arithmetic.

## Roots of Unity Continued

We just proved  $e^{\frac{2\pi i k}{n}} = e^{\frac{2\pi i j}{n}}$  iff  $k \equiv_j$ .

It follows there are just  $n$ -distinct roots of unity:  $z^n = 1$  has sol's

$$\underbrace{e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, e^{\frac{6\pi i}{n}}, \dots, e^{\frac{2\pi i(n-1)}{n}}, e^{\frac{2\pi i n}{n}}}_{} = 1$$

$w_n$  ← notation in many texts.

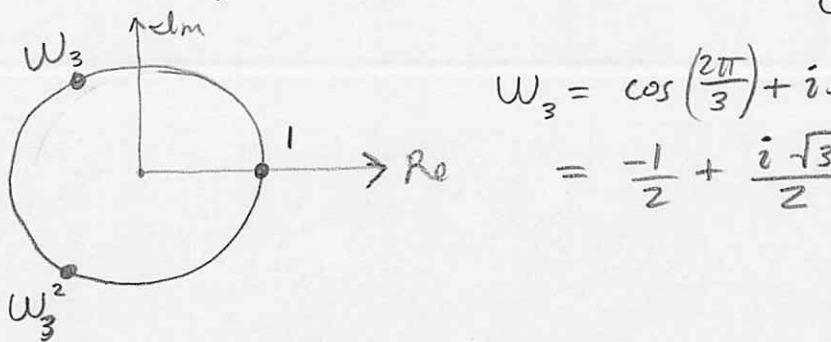
Freitag uses  $\zeta_1 = \zeta_{1,n}$  and  $\zeta_{v,n} = e^{\frac{2\pi i v}{n}}$  for  $v = 0, 1, 2, \dots, n-1$ .

(I'll use  $w_n$  since my  $\zeta$  is ugly.)

Th<sup>m</sup>/  $\exists$   $n$ -unique sol's  $w_n, w_n^2, w_n^3, \dots, w_n^{n-1}$  to the eq<sup>n</sup>  $z^n = 1$ . Moreover,  $w_n = e^{\frac{2\pi i}{n}}$  and  $w_n^j = e^{\frac{2\pi i j}{n}}$ .

E4  $z^3 = 1$  has  $\Theta = \frac{2\pi i}{3} = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

we have sol's  $1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$ . Graphically:



$$w_3 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \\ = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

The factor theorem states  $f(r) = 0 \Rightarrow f(z) = (z-r)g(z)$ .

we can prove this assertion for  $f \in \mathbb{C}[z]$ .

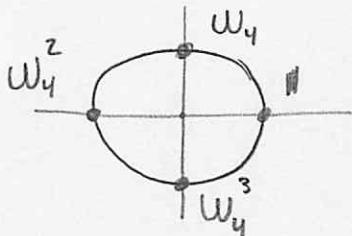
For E4 we obtain a factorization: complex coeff. poly's.

$$\underbrace{z^3 - 1}_{\in \mathbb{R}[z]} = (z-1)(z + \frac{1}{2} - \frac{i\sqrt{3}}{2})(z + \frac{1}{2} + \frac{i\sqrt{3}}{2})$$

Conjugate pair, no accident!

**[E5]** 4<sup>th</sup> roots of unity: solve  $z^4 = 1$

$$w_4 = e^{\frac{2\pi i}{4}}, e^{\frac{4\pi i}{4}}, e^{\frac{6\pi i}{4}}, e^{\frac{8\pi i}{4}} = 1$$



$$\begin{aligned} w_4 &= i \\ w_4^2 &= i^2 = -1 \\ w_4^3 &= i^3 = -i \\ w_4^4 &= 1 \end{aligned}$$

Observe the factoring,

$$\begin{aligned} z^4 - 1 &= (z^2 + 1)(z^2 - 1) \\ &= (z+i)(z-i)(z+1)(z-1). \end{aligned}$$

{ see pg. 18 of  
Frigg's for  $n^{th}$   
order comments  
and connection  
to Fund. Thm  
of Algebra }

Remark: once we've the sol's of  $z^n = 1$  it becomes a relatively simple matter to solve  $z^n = z_0$  where  $z_0 = |z_0|e^{i\theta_0}$ . Simply use  $\sqrt[n]{|z_0|}$  multiplied by the roots of unity,  
 $z^n = z_0$  has sol's  $(\sqrt[n]{|z_0|}) \exp\left(\frac{i(2\pi j + \theta_0)}{n}\right)$   $j = 0, 1, \dots, n-1$ .

**Def'n** If  $z_0 = |z_0|e^{i\theta_0}$  then we define for  $n \in \mathbb{N}$ ,

$$\begin{aligned} z_0^{1/n} &= \{ z \in \mathbb{C} \mid z^n = z_0 \} = \\ &= \left\{ \sqrt[n]{|z_0|} \exp\left[\frac{i(2\pi j + \theta_0)}{n}\right] \mid j = 1, 2, \dots, n-1 \right\} \end{aligned}$$

With this notation  $1^{1/n}$  = set of  $n$  roots of unity.

**[E6]**  $(1-i)^{1/2} = (\sqrt{2} e^{-i\pi/4})^{1/2} = \left\{ 2^{1/4} e^{-i\pi/8}, 2^{1/4} e^{7\pi i/8} \right\}$

Note:  $\frac{2\pi j - \frac{\pi}{4}}{2} = \frac{-\pi}{8}, \frac{7\pi}{8}$

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Th<sup>n</sup> If  $\gcd(m, n) = 1$  then  $(z^{1/n})^m = (z^m)^{1/n}$

Proof: left to reader (i). Notice that fractional exponents are nontrivial, we have to think about sets of values. Properties you might assume will fail.

$$1 = \sqrt[1]{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i\bar{i} = i^2 = -1 \leftarrow \text{bad.}$$

verses,

$$1^{1/2} = \{1, -1\} \leftarrow \text{good.}$$

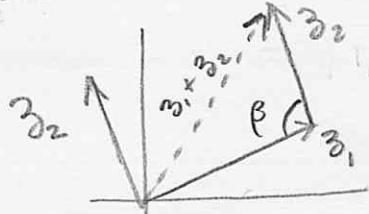
Remark:  $e^z$  is not  $(e^z)^z$ . As Freitag wisely mentions on p. 31, " $e^z = \exp(z)$  is one of the  $z^{\frac{1}{k}}$  powers of e"

We must be careful as we extend algebra we know over  $\mathbb{R}$  to the  $\mathbb{C}$  case. Factoring polynomials works better, however exponents and power laws require some elaboration. I have more to say, but I pause to work a few examples for breadth and depth.

E7 Show  $|z_1 + z_2| = |z_1| + |z_2|$  iff  $\arg(z_1) = \arg(z_2)$

$\Leftarrow$  If  $\arg(z_1) = \arg(z_2) \Rightarrow z_1 = |z_1|e^{i\theta}$  and  $z_2 = |z_2|e^{i\theta}$   
 hence  $z_1 + z_2 = (|z_1| + |z_2|)e^{i\theta}$  and it follows  
 that  $|z_1 + z_2| = ||z_1| + |z_2|| = |z_1| + |z_2|$ .

$\Rightarrow$  Geometrically clear, but requires some thought algebraically.



$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos\beta$$

$$\text{But, } |z_1 + z_2|^2 = (|z_1| + |z_2|)^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\text{Hence } -2|z_1||z_2|\cos\beta = 2|z_1||z_2| \text{ which yields } \beta = \pi. \text{ This indicates } z_1 \parallel z_2.$$

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E8) Show  $\exists k \in (0, \infty)$  such that  $\bar{z}_1 = k \bar{z}_2$   
 iff  $\arg(\bar{z}_1) = \arg(\bar{z}_2)$ . (assume  $\bar{z}_1, \bar{z}_2 \neq 0$ )

$\Rightarrow$  If  $\bar{z}_1 = k \bar{z}_2$  for  $k > 0$ . Let  $\bar{z}_1 = r_1 e^{i\theta_1}$   
 and  $\bar{z}_2 = r_2 e^{i\theta_2}$  where  $r_1 = |\bar{z}_1|$  and  $r_2 = |\bar{z}_2|$ .  
 Observe  $r_1 e^{i\theta_1} = k r_2 e^{i\theta_2} \Rightarrow \frac{k r_2}{r_1} \exp(i(\theta_2 - \theta_1)) = e^0$

It follows  $\theta_2 - \theta_1 = 2\pi k$  for some  $k \in \mathbb{Z}$ .

Thus elements of  $\arg(\bar{z}_1)$  and  $\arg(\bar{z}_2)$  are congruent  
 modulo  $2\pi$ . It follows  $\arg(\bar{z}_1) = \arg(\bar{z}_2)$ . //

Remark: I used this result to complete E7.

E9) As vectors,  $\bar{z}_1 \cdot \bar{z}_2 = \operatorname{Re}(\bar{z}_1 \bar{z}_2)$  and  
 $\bar{z}_1 \times \bar{z}_2 = \langle 0, 0, \operatorname{clm}(\bar{z}_1 \bar{z}_2) \rangle$

$$\bar{z}_1 \cdot \bar{z}_2 = \langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle = x_1 x_2 + y_1 y_2$$

$$\bar{z}_1 \times \bar{z}_2 = \langle x_1, y_1, 0 \rangle \times \langle x_2, y_2, 0 \rangle = \langle 0, 0, x_1 y_2 - x_2 y_1 \rangle$$

(viewing  $\bar{z}_1, \bar{z}_2$  as three-dim'l vectors for this  
 example and -- the next --)

$$\begin{aligned}\bar{z}_1 \bar{z}_2 &= (x_1 - i y_1)(x_2 + i y_2) \\ &= \underbrace{x_1 x_2 + y_1 y_2}_{\operatorname{Re}(\bar{z}_1 \bar{z}_2)} + i \underbrace{(x_1 y_2 - x_2 y_1)}_{\operatorname{clm}(\bar{z}_1 \bar{z}_2)}\end{aligned}$$

Note:  $H = R \oplus iR \oplus jR \oplus kR$  with  $i^2 = j^2 = k^2 = -1$   
 and  $ij = k$  are called Quaternions. Multiplication  
 encodes both dot and the full 3-dim'l cross-product.  
 Ask if interested...

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E10  $\vec{z}_1$  is parallel to  $\vec{z}_2$  iff  $\text{dm}(\vec{z}, \vec{z}_2) = 0$ .

$\Rightarrow$  Recall,  $\vec{z}_1 \parallel \vec{z}_2 \Rightarrow \vec{z}_1 \times \vec{z}_2 = \langle 0, 0, 0 \rangle$

But  $\vec{z}_1 \times \vec{z}_2 = \langle 0, 0, \text{dm}(\vec{z}, \vec{z}_2) \rangle$  hence  $\text{dm}(\vec{z}, \vec{z}_2) = 0$ .

$\Leftarrow$  essentially same argument. //

E11 Prove  $\left| \sum_{k=1}^n \vec{z}_k \right| \leq \sum_{k=1}^n |\vec{z}_k|$

We proceed by induction on  $n$ . For  $n=1$  this is true. Assume the proposition true for  $n \geq 1$  and consider,

$$\begin{aligned} \left| \sum_{k=1}^{n+1} \vec{z}_k \right| &= \left| \vec{z}_{n+1} + \sum_{k=1}^n \vec{z}_k \right| : \text{def}^{\text{b}} \text{ of finite sum.} \\ &\leq |\vec{z}_{n+1}| + \left| \sum_{k=1}^n \vec{z}_k \right| : \Delta\text{-inequality.} \\ &\leq |\vec{z}_{n+1}| + \sum_{k=1}^n |\vec{z}_k| : \text{induction hypothesis.} \\ &= \sum_{k=1}^{n+1} |\vec{z}_k| : \text{def}^{\text{b}} \text{ of finite sum.} \end{aligned}$$

Therefore, the proposition is true for  $n+1$  and we conclude the prop. is true for all  $n \in \mathbb{N}$  by induction. //

E12 Suppose  $m_1, m_2, m_3 > 0$  and  $|\vec{z}_1|, |\vec{z}_2|, |\vec{z}_3| \leq 1$

then show  $\left| \frac{m_1 \vec{z}_1 + m_2 \vec{z}_2 + m_3 \vec{z}_3}{m_1 + m_2 + m_3} \right| \leq 1$

$$\begin{aligned} \left| \frac{m_1 \vec{z}_1 + m_2 \vec{z}_2 + m_3 \vec{z}_3}{M} \right| &\leq \frac{1}{M} (m_1 |\vec{z}_1| + m_2 |\vec{z}_2| + m_3 |\vec{z}_3|) \\ &\leq \frac{1}{M} (m_1 + m_2 + m_3) = \frac{M}{M} = 1. // \end{aligned}$$

- The center of mass of  $m_1, m_2, m_3$  inside unit-disk is once again within the disk.

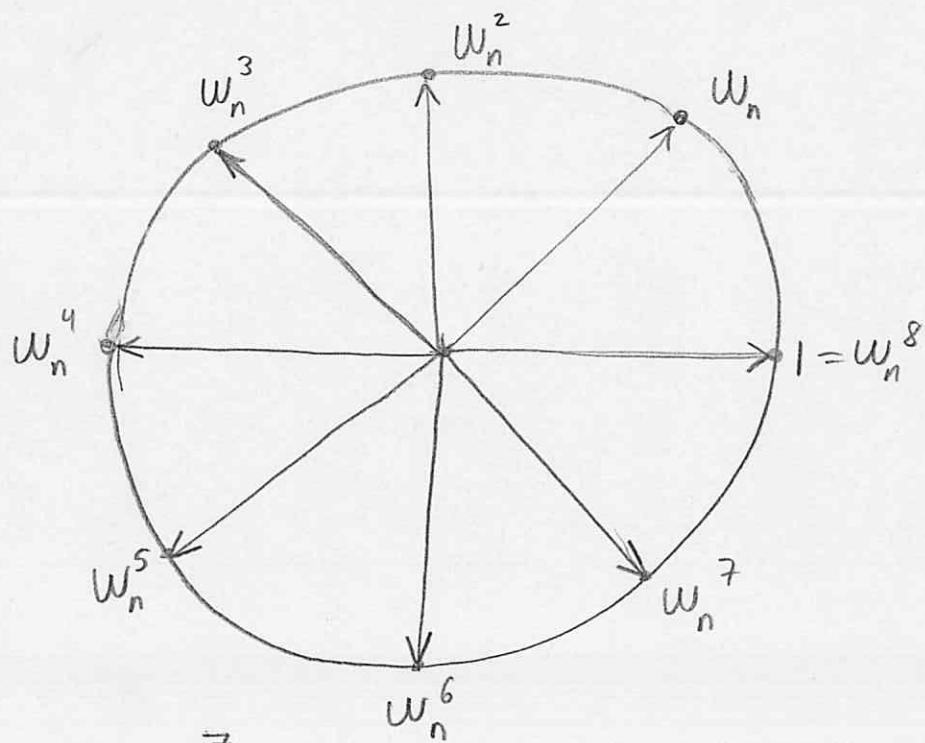
E13)  $1 + w_n + w_n^2 + \dots + w_n^{n-1} = 0$  (show it,  
for  $n \in \mathbb{N}$ .) (18)

By symmetry this is obvious if you look at how the roots of unity in  $\mathbb{C}^1$  are positioned around the unit-circle. Let's give an analytical argument, consider:

$$\begin{aligned} (w_n - 1)(1 + w_n + \dots + w_n^{n-1}) &= \\ &= w_n + w_n^2 + \dots + w_n^n - 1 - w_n - \dots - w_n^{n-1} \\ &= w_n^n - 1 \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

However,  $w_n - 1 \neq 0$  thus  $1 + w_n + \dots + w_n^{n-1} = 0$ . //

(note:  $ab = 0 \Rightarrow a=0$  or  $b=0$  for  $a, b \in \mathbb{C}$ )  
 $\mathbb{C}$  has no "zero-divisors".)



Sum of  $\sum_{j=0}^7 w_n^j = 0$ . (see how they cancel)

## (19)

### THE LOGARITHMIC FUNCTION & COMPLEX POWERS

What we build here is a natural extension of the algebra thus far considered.

$$\text{Def: } e^z = e^w \Rightarrow e^{\operatorname{Re}(z)} e^{i\operatorname{clm}(z)} = e^{\operatorname{Re}(w)} e^{i\operatorname{clm}(w)}$$

$$\Rightarrow \operatorname{Re}(z) = \operatorname{Re}(w) \text{ and } \underbrace{\operatorname{clm}(z)}_{\arg(z)} \equiv \underbrace{\operatorname{clm}(w)}_{\arg(w)}$$

$\Rightarrow f(z) = e^z$  where  $f: \mathbb{C} \rightarrow \mathbb{C}$   
is not injective on  $\mathbb{C}$ .

$\Rightarrow f^{-1}$  does not exist. However,  
we can construct local inverses  
on  $U \subseteq \mathbb{C}$  such that  $f|_U$  is 1-1.

The standard choice is

$$U = \mathbb{R} \times (-\pi, \pi]$$

$$\text{and } f^{-1}(z) = \operatorname{Log}(z)$$

Consider,  $z, w \in U = \mathbb{R} \times (-\pi, \pi]$  then suppose  $e^z = e^w$

$$\text{If } z = x+iy \text{ and } w = a+ib \Rightarrow e^x e^{iy} = e^a e^{ib}$$

thus  $e^x = e^a$  by modulus. Also  $y = b - 2\pi k$   
such that  $k \in \mathbb{Z}$ , but  $k \geq 1$  not possible since two  
values in  $(-\pi, \pi]$  are never  $2\pi$  separated. Thus  $k=0$   
and injectivity of  $\exp(z)$  on  $U$  is shown.

$$\boxed{\text{Def: } e^z = w \iff z = \operatorname{Log}(w) \quad ] a \text{ for } z \in U.}$$

$$\operatorname{Log}(w) = \operatorname{Log}|w| + i\operatorname{Arg}(w)$$

(20)

Let  $w = re^{i\theta}$  where  $r > 0$  and  $\theta \in (-\pi, \pi]$

then  $e^z = e^x e^{iy} = re^{i\theta} \Rightarrow e^x = r$  and  $e^{iy} = e^{i\theta}$   
 but, if  $y \in (-\pi, \pi]$  then  $y = \theta$  and of course  $x = \ln(r)$ .  
 We defined  $\bar{z} = \text{Log}(w) \Leftrightarrow e^{\bar{z}} = w$ ,  $\bar{z} \in \mathbb{R} \times (-\pi, \pi]$

hence  $\boxed{\text{Log}(re^{i\theta}) = \ln(r) + i\theta}$

Alternatively, capture the local inverse idea with the Arg notation,

$\boxed{\text{Log}(z) = \ln|z| + i\text{Arg}(z) \text{ for } z \neq 0}$

The choice of  $(-\pi, \pi]$  is just one convention. We can set-up a Logarithm for any half-open,  $2\pi$ -length interval. This choice is called the Principal Branch of log. Little  $\log(z)$  is not a function on  $\mathbb{C}$ , instead it is a relation or a "multiply-valued-function" defined by

$\boxed{\text{Defn } e^z = w \Leftrightarrow z \in \log(w)}$

We can calculate  $\log(z) = \ln|z| + i\arg(z)$   
 where addition of sets is defined in the natural manner.  
 $\alpha + \beta S = \{\alpha + \beta s \mid s \in S\}$  (here  $\alpha = \ln|z|$ ,  $\beta = i$ )

$\boxed{\text{E14}} \quad \log(2+3i) \cong \log(\sqrt{13} e^{0.9828i}) \quad \text{note } 0.9828 \in (-\pi, \pi]$   
 $\Rightarrow \underline{\log(2+3i)} = \ln \sqrt{13} + 0.9828i$ .

In contrast,

$$\log(2+3i) = \{\ln \sqrt{13} + (0.9828 + 2\pi k)i \mid k \in \mathbb{Z}\}$$

## Sequences and Series

Freitag pushes us to think about this now.  
I'll try to keep it brief and relegate some review to your homework. Crucial pts.

- A sequence is a function from  $\mathbb{N}$  to something.  
(or  $T \subseteq \mathbb{Z}$  such that  $T$  has a least member and the immediate successor property.)

a complex sequence can be written as

$$\{z_n\}_{n=n_0}^{\infty} = \{z_{n_0}, z_{n_0+1}, \dots\}$$

- The limit of a complex sequence is defined with the modulus in the same way as a real sequence with absolute-value.

Def/ We say  $\lim_{n \rightarrow \infty} z_n = L$  or  $z_n \rightarrow L$  as  $n \rightarrow \infty$   
iff for each  $\epsilon > 0$ ,  $\exists N \in \mathbb{Z}$  such that  
 $n > N \Rightarrow |z_n - L| < \epsilon$ .

- A series of complex numbers is defined just as in the real case. The series  $\sum_{k=0}^{\infty} z_k$   
converges iff the sequence of partial sums  $\{z_0, z_0 + z_1, z_0 + z_1 + z_2, \dots\} = \{\sum_{k=0}^n z_k\}_{n=1}^{\infty}$   
converges. Otherwise the series is said to diverge.

- A series  $\sum_{n=0}^{\infty} z_n$  is absolutely convergent  
iff  $\sum_{n=0}^{\infty} |z_n|$  converges. (again, just as in  $\mathbb{R}$ -case. Many of same Th's persist.)