

Complex Exponential Function

(1)

Defⁿ/ Let $z = a + ib \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Then

$$e^z = \exp(z) = e^a (\cos(b) + i \sin(b))$$

This is a good definition because,

Proposition: Let $z, w \in \mathbb{C}$ then,

$$1.) e^z e^w = e^{z+w}$$

$$2.) e^{-z} = \frac{1}{e^z}$$

$$3.) e^{\theta+i0}_{\text{complex}} = 1$$

$$4.) e^z \neq 0 \text{ for any } z.$$

$$5.) e^a_{\text{complex}} = e^a_{\text{real}}$$

Items 3. & 5. I have added "complex" to draw attention to the fact we are stating things about the complex exponential, fortunately the notation is not confusing in the sense that the complex exp. function reduces to real exponential function when we restrict the domain to \mathbb{R} .

Proof: (1). Let $z = a + ib$ and $w = x + iy$.

$$\begin{aligned}
 e^z e^w &= e^{a+ib} e^{x+iy} \\
 &= e^a (\cos(b) + i \sin(b)) e^x (\cos(y) + i \sin(y)) \\
 &= e^a e^x [\cos(b) \cos(y) + i(\cos(b) \sin(y) + \sin(b) \cos(y)) + i^2 \sin(b) \sin(y)] \\
 &= e^{a+x} (\cos(b) \cos(y) - \sin(b) \sin(y) + i[\cos(b) \sin(y) + \sin(b) \cos(y)]) \\
 &= e^{a+x} (\cos(b+y) + i \sin(b+y)) \quad (\star) \\
 &= e^{a+x+i(b+y)} \\
 &= e^{z+w}
 \end{aligned}$$

I had to use the adding angle formulas in step (\star) . Otherwise the proof follows from the very definition of complex number addition and multiplication AND $e^a e^x = e^{a+x}$ the rule for real exponentials.

(2)

Proof of 2] Claim: $e^{-z} = \frac{1}{e^z}$. In other words we need that $e^z e^{-z} = 1$ but this is an easy consequence of part 1, NOTE:

$$\begin{aligned}
 e^z e^{-z} &= e^{z-z} : \text{by 1.} \\
 &= e^{0+i0} : \text{emphasizing } 0 \in \mathbb{C} \\
 &= e^0 (\cos(0) + i\sin(0)) : \text{using def}^n \text{ of complex exponential.} \\
 &= 1 (1) \\
 &= 1.
 \end{aligned}$$

We just proved 3.) in the calculation above; $e^0 = 1$.

Proof of 4.] Why $e^z \neq 0$,

$$e^z = e^a (\cos(b) + i\sin(b))$$

We need either ($e^a = 0$) or ($\cos b + i\sin b = 0$) if 4. were to be false. We know real exponentials are nonzero so $e^a \neq 0$. Can $\cos(b) + i\sin(b) = 0$? If so then,

$$\cos b + i\sin b = 0 \Rightarrow i = -\frac{\cos(b)}{\sin(b)} \Rightarrow \tan(b) = i$$

But the tangent is a real-valued function since $b \in \mathbb{R}$. thus $\cos(b) + i\sin(b)$ cannot be zero. Another way of seeing this is to observe that $\cos \theta$ and $\sin \theta$ are never both zero. In fact, if $\cos \theta = 0$ then $|\sin \theta| = 1$ and vice-versa.

Proof of 5]

$$\begin{aligned}
 e^{a+i(0)} &= e^a (\cos(0) + i\sin(0)) \\
 &= e^a (1) \\
 &= e^a
 \end{aligned}$$

thus, $e_{\text{complex}} /_{\mathbb{R}} = e_{\text{real}}$. The notation $/_{\mathbb{R}}$ means "restricted to \mathbb{R} ".

(3)

Remark: I have shown that the complex exponential enjoys many of the same nice algebraic properties as the real exponential. The heart of the proof of (1.) resides in the adding angles formulas.

Thus it is not surprising we can turn this around and use the complex exponential to remember or derive the adding angles formula (see pg. 7i of intro-to-complex.pdf)

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Euler's Identity:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

As I have ordered the material this is a simple consequence of taking $z = i\theta$ in the definition so $a = 0$ and $b = \theta$ hence $e^{0+i\theta} = e^0 (\cos \theta + i \sin \theta) = \cos \theta + i \sin \theta$.

Sine and Cosine from a complex variables viewpoint

Notice that since $\cos(-\theta) = \cos \theta$ and $\sin(-\theta) = -\sin \theta$,

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

Add or subtract these equations and divide by 2 or $2i$ to obtain the extremely useful formulas

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Look over pg. 5i, 6i, 7i, 8i of my intro-to-complex.pdf notes - to see how these can be used to derive most any