

~ GROUP ACTION ~

(5)

Defⁿ/ Let G be a Lie group and M a manifold to say σ is a left-action of G on M means that σ is a smooth map from $G \times M \rightarrow M$ such that

- i) $g \cdot x \equiv \sigma(g, x)$
- ii) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \quad \forall g_1, g_2 \in G, \forall x \in M \iff \sigma(g_1 g_2, x) = \sigma(g_1, \sigma(g_2, x))$
- iii) $e \cdot x = x \quad \text{where } e = \text{id}_G \text{ and } \forall x \in M.$

Ex/ $G \times G/H \rightarrow G/H$ Dirac Monopole ...

$$\sigma(g_1, g_2 H) = g_1 g_2 H$$

≈ 1906
Continuous Groups
by Eisenhart.
good local book.

Remark: the right action σ_R is defined analogously on $M \times G \rightarrow M$.

Remark: If σ is a ^{left} right action we can define a function $\hat{\sigma} : G \rightarrow \text{Diff}(M)$ such that $\hat{\sigma}(g) : M \rightarrow M$ is smooth mapping

$$\hat{\sigma}(g)(x) = \sigma(g, x)$$

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$$\begin{aligned}\hat{\sigma}(g_1 g_2)(x) &= \sigma(g_1 g_2, x) \\ &= g_1 \cdot (g_2 \cdot x) \\ &= \hat{\sigma}(g_1)(g_2 \cdot x) \\ &= [\hat{\sigma}(g_1) \circ \hat{\sigma}(g_2)](x)\end{aligned}$$

likewise $\hat{\sigma}(e) = \text{id}_M$ we'll see more of this later. An action on a manifold is a homeomorphism from

Example (1)

$T \in T_q^p \mathbb{R}^n$

$$\text{Def: } T: \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{p} \times (\mathbb{R}^n)^* \times \cdots \times (\mathbb{R}^n)^* \xrightarrow{q} \mathbb{R} \quad \text{multilinear map.}$$

$$T(x_1, x_2, \dots, x_p, \alpha_1, \alpha_2, \dots, \alpha_q) \in \mathbb{R}$$

$$T_{j_1 j_2 \dots j_p}^{i_1 i_2 \dots i_q} = T(e_{j_1}, e_{j_2}, \dots, e_{j_p}, e^{i_1}, e^{i_2}, \dots, e^{i_q})$$

Define an action of $\mathrm{GL}(n)$ on $T_q^p \mathbb{R}^n$ via the

$$(g \cdot T)(x_1, x_2, \dots, \alpha_1, \dots, \alpha_q) = T(g^{-1}x_1, g^{-1}x_2, \dots, g^{-1}x_p, g^{-1}\alpha_1, \dots, g^{-1}\alpha_q)$$

$$(g \cdot \alpha)(x) = \alpha(g^{-1}x)$$

$$\begin{aligned} [(g_1 g_2) \alpha](x) &= \alpha((g_1 g_2)^{-1} x) \\ &= \alpha(g_2^{-1} g_1^{-1} x) \\ &= (g_2 \alpha)(g_1^{-1} x) \\ &= g_1 \circ (g_2 \cdot \alpha)(x) \end{aligned}$$

- Basic Action on \mathbb{R}^n by $\mathrm{GL}(n)$ via matrix multiplication naturally lifts to the tensor bundle.

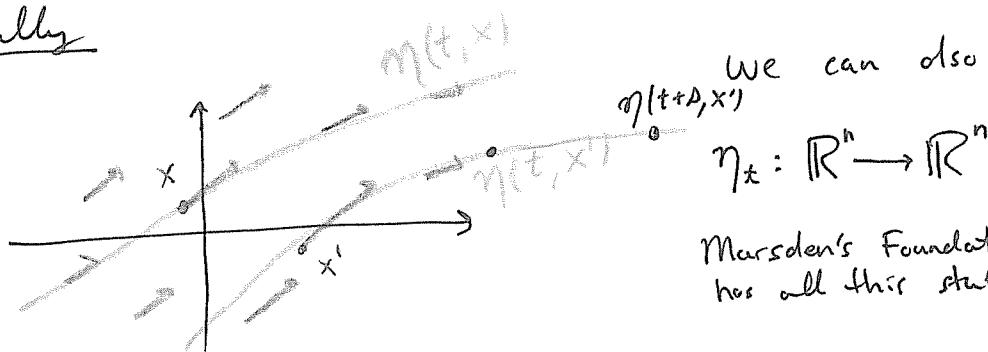
Example

Let X be a vector field on \mathbb{R}^n such that for each solⁿ of the diff eq⁼ $\gamma'(t) = X(\gamma(t))$ exists $\forall t \in \mathbb{R}$. (X complete)
 (If M is compact then every v.f. X is complete)
 the map $\eta : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$\frac{d}{dt}(\eta(t, x)) = X(\eta(t, x))$$

$$\eta(0, x) = x$$

Pictorially



we can also think of
 $\eta_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$

Marsden's Foundation of Mechanics
 has all this stuff.

Show via uniqueness of DEq's that

$$\eta(t+s, x) = \eta(s, \eta(t, x))$$

$$\eta_{t+s}(x) = \eta_s(\eta_t(x))$$

$$\eta_{t+s} = \eta_s \circ \eta_t$$

Action from \mathbb{R} $\xrightarrow[\text{addition.}]{} \text{Diff}(\mathbb{R}^n)$ composition.

$$(t+s) \circ x = s \circ (t \circ x)$$

$$0 \circ x = 0$$

Remark: Compact groups are well behaved. Use p.d. metric
 we get orthogonal group. Use η get non-compact isometry
 group nice ness no more.

(3)

Defn If a lie group G acts on a manifold M then the isotropy subgroup of $x \in G$ is (little group)

$$G_x = \{g \in G \mid g \cdot x = x\}$$

And also the orbit of x is in fact:

$$Gx = \{g \cdot x \mid g \in G\}$$

Here $Gx \subseteq M$, Whereas $G_x \subseteq G$.

Remark: for last example $O_x \equiv \mathbb{R}^n x$

$$\begin{aligned} O_x &= \{t \cdot x \mid t \in \mathbb{R}\} \\ &= \{\eta(t, x) \mid t \in \mathbb{R}\} \end{aligned}$$

If we take a multiparameter group then its a more interesting orbits.

Exercise: Isotropy subgroup is

Example: Let $G = SO(3)$ then G acts on \mathbb{R}^3 via multiplication, aka rotations

$$(A, x) \longrightarrow A \cdot x \leftarrow \text{matrix multiplication}$$

Notice if $x=0$ then $O_{x=0} = \{0\}$ the little group

of $SO(3)$ at $x=0$ is in fact everything. On the other hand $x \neq 0$

$$SO(3) \cdot x = \{A \cdot x \mid A \in SO(3)\}$$

You can prove that $Ax = x \Rightarrow \{y \mid y \perp x\}$ then $Ay \perp Ax$ then its $SO(2)$ and can calculate its a rot around axis..

$$SO(3) \cdot x = \text{sphere of radius length } x$$

The space fills with spheres. Note that the little group

$$SO(3)_x = SO(2)$$

So little groups & orbits relate in some way.

Th^m/ If G acts transitively on a manifold M and $x \in M$ then the mapping $\beta: G/G_x \rightarrow M$ defined by

$$\beta(gG_x) = g \cdot x$$

is a smooth injection, If the orbit $G \cdot X$ is a submanifold then $\beta: G/G_x \rightarrow G \cdot X$ is a diffeomorphism.

Remark: need $G \cdot X$ is a submanifold due to weird Torus.

If in addition G acts transitively; $G \cdot X = M$
then we get $G/G_x \cong G \cdot X$

e.g/ $\frac{SO(3)}{SO(2)} = S^2$ of various radii depending on x we pick.

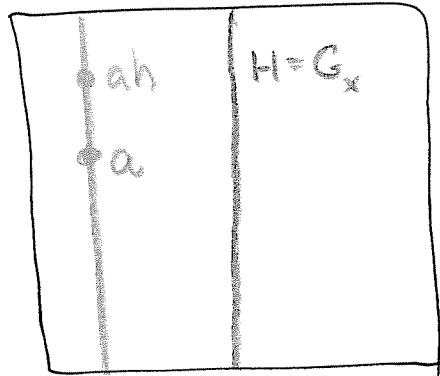
Pf/ Define $\gamma: G \rightarrow M$ (and fix an $x \in M$ before) by

$$\gamma(g) = g \cdot x = \sigma(g, x)$$

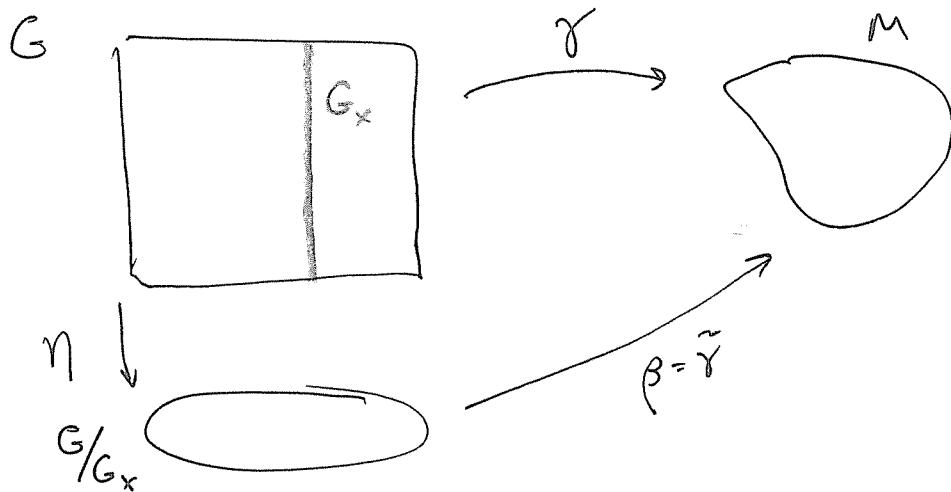
Now γ is smooth. Notice that if $a \in G$ and $h \in G_x$ then since h is in isotrop. subgroup $h \cdot x = x$, hence,

$$\gamma(ah) = ah \cdot x = a(h \cdot x) = a \cdot x$$

G



$\gamma(h) = \gamma(ah)$ its a smooth mapping which is constant on each coset $\therefore 3!$ mapping $\tilde{\gamma}: G/G_x \rightarrow M$ via the Th^m from before. ~~test the~~ Call $\tilde{\gamma} = \beta$



Call it β and again $\beta(\eta(y)) = \gamma(y)$
 $\therefore \boxed{\beta(yG_x) = y \cdot x}$

Additionally β is injective,

$$\begin{aligned} \beta(y_1 G_x) = \beta(y_2 G_x) &\Rightarrow \gamma(y_1) = \gamma(y_2) \\ &\Rightarrow y_1 \cdot x = y_2 \cdot x \\ &\Rightarrow y_2^{-1} y_1 \cdot x = x \\ &\Rightarrow y_2^{-1} y_1 \in G_x \\ &\Rightarrow y_1 \in y_2 G_x \\ &\Rightarrow y_1 G_x = y_2 G_x \quad \therefore \beta \text{ is 1-1.} \end{aligned}$$

Clearly $\beta(G/G_x) \subseteq G \cdot X$ since $\beta(yG_x) = y \cdot x \in G \cdot X$.

But by 1st exercise we have manifold map which goes
 into submanifold \therefore the map is a diffeomorphism. // (this proof was in notes)

We'll use this to do symmetry breaking
 with the Higgs Field...

Def^{ly} If G is a Lie Group and acts on a manifold M , The action is effective if

$$g \cdot x = x \quad \forall x \in M \Rightarrow g = e$$

The action is free iff whenever

$$g \cdot x = x \quad \text{for even one } x \in M \\ \text{it follows that } g = e$$

$$\sigma: G \times M \rightarrow M$$

$$\sigma_g: M \rightarrow M \quad g \in G$$

$$\sigma_g(x) = \sigma(g, x)$$

$$\sigma_g \in \text{Diff}(M)$$

$$\varphi(g) = \sigma_g$$

$\varphi: G \rightarrow \text{Diff}(M)$ is a homomorphism

effective is a way of saying φ is 1-1 homomorphism
that is $\sigma_g(x) = x \quad \forall x \Rightarrow g = e$

$$\varphi: G \rightarrow \text{Diff}(M)$$

$$\begin{array}{c} \uparrow \\ \Gamma_{\text{inv}} \end{array} \quad \mathcal{X}(M) \cong \Gamma(M)$$

Homomorphism,

$$d_e \varphi: T_e G \rightarrow T_{\text{id}_M} \text{Diff}(M) = \Gamma(M)$$

effectives means you can embed it...

free means if G has 1-fixed points

Group Actions

Definition Let G be a group and S any set.

To say that σ is a left-action of G on S means that σ is a function from $G \times S$ to S such that $\sigma(g_1 g_2, x) = \sigma(g_1, g_2, x)$ and $\sigma(e, x) = x$ for all $x \in S$ and $g_1, g_2 \in G$. Similarly, σ is a right-action of G on S if σ is a function from $S \times G$ to S such that $\sigma(x, g_1 g_2) = \sigma(\sigma(x, g_1), g_2)$ and $\sigma(x, e) = x$ for $x \in S$, $g_1, g_2 \in G$. When σ is a left-action $\sigma(g, x)$ is denoted $g \cdot x$ but when it is a right-action $\sigma(x, g) = x \cdot g$. In this notation one has:

$$g \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x, \quad e \cdot x = x \quad (\text{left-action})$$

$$x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2, \quad x \cdot e = x \quad (\text{right-action})$$

In the case that S is a manifold and G is a Lie-group it is required that σ be smooth though this is often emphasised by saying that σ is a smooth action. Often the mapping σ itself is suppressed when $\sigma(g, x)$ is denoted by $g \cdot x$. In such a case the notation used for left and right translations in a Lie group is extended to actions of the Lie group on a manifold. Thus, for a given left-action we define $l_g : S \rightarrow S$ by $l_g(x) = g \cdot x$ for $g \in G, x \in S$. Similarly, for a given right-action we define $r_g : S \rightarrow S$ by $r_g(x) = x \cdot g$ for $x \in S, g \in G$.

(2) Let $G = \mathrm{GL}(n, \mathbb{R})$ and $M = T_2(\mathbb{R}^n)$. For $g \in G$, $b \in M$ define $g \cdot b$ by

$$(g \cdot b)(x, y) = b(g^{-1}x, g^{-1}y)$$

for $x, y \in \mathbb{R}^n$. It is easy to show that the mapping $\sigma : G \times M \rightarrow M$ defined by $\sigma(g, b) = g \cdot b$ is a left-action of G on M .

(3) A generalization of the last action may be obtained as follows. Recall that a tensor $T \in T_q^p \mathbb{R}^n$ is a multilinear mapping from $(\mathbb{R}^n \times \dots \times \mathbb{R}^n) \times (\mathbb{R}^n)^* \times \dots \times (\mathbb{R}^n)^*$ into \mathbb{R} where there are p factors of \mathbb{R}^n and q factors of $(\mathbb{R}^n)^*$. For $g \in \mathrm{GL}(n, \mathbb{R})$ and $T \in T_q^p(\mathbb{R}^n)$ define $g \cdot T$ by

$$(g \cdot T)(x_1, x_2, \dots, x_p, \alpha_1, \dots, \alpha_q) = T(g \cdot x_1, \dots, g \cdot x_p, g \cdot \alpha_1, \dots, g \cdot \alpha_q)$$

where $x_i \in \mathbb{R}^n$, $\alpha_j \in (\mathbb{R}^n)^*$, $1 \leq i \leq p$, $1 \leq j \leq q$

and where for $\lambda \in (\mathbb{R}^n)^*$, $x \in \mathbb{R}^n$

$$(g \cdot \lambda)(x) = \lambda(g^{-1}x).$$

It is easy to show that $\sigma : G \times T_q^p \mathbb{R}^n \rightarrow T_q^p \mathbb{R}^n$ defined by $\sigma(g, T) = g \cdot T$ is a left action.

(4) Let \underline{X} be a vector field on \mathbb{R}^n such that each solution of the differential equation $\dot{\gamma}'(t) = \underline{X}(\gamma(t))$ exists for all $t \in \mathbb{R}$. If

η is the flow of \underline{X} , meaning that

$\eta : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the smooth mapping such that $\frac{d}{dt}(\eta(t, x)) = \underline{X}(\eta(t, x))$, $\eta(0, x) = x$

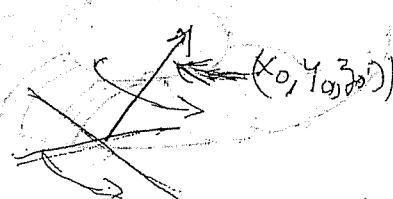
for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, then η defines an

the point (x_0, y_0) through the angle t we see that
 $R_{(x_0, y_0)} = \{t \in \mathbb{R} \mid t = 2\pi n, n \in \mathbb{Z}\} = 2\pi \mathbb{Z}$ is
clearly a subgroup of $(\mathbb{R}, +)$. Notice that the orbit
through (x_0, y_0) is a circle and $\mathbb{R}/2\pi \mathbb{Z}$ may be
identified with this circle. This illustrates a general
fact stated below namely that the orbit through
a point may be identified with the group modulo
its isotropy subgroup (provided the orbit is actually
a manifold as it is here).

Example 2 The last example may be generalized.

$SO(3)$ acts on \mathbb{R}^3 via matrix multiplication.

If $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $(x_0, y_0, z_0) \neq (0, 0, 0)$ then the
orbit of (x_0, y_0, z_0) is the set of all vectors in \mathbb{R}^3
obtained by rotating (x_0, y_0, z_0) , via a matrix $A \in SO(3)$.
The set of all such vectors is a sphere of radius
 $\sqrt{x_0^2 + y_0^2 + z_0^2}$. The only matrices in $SO(3)$ which fix
 (x_0, y_0, z_0) are those which rotate each vector about
the line through (x_0, y_0, z_0) . This set of rotations



may be identified with the rotations of the plane orthogonal
to (x_0, y_0, z_0) and so may be identified with $SO(2)$.
So the isotropy subgroup of (x_0, y_0, z_0) is $SO(2)$. Notice
that the orbit of $(0, 0, 0)$ is a single point, namely
 $(0, 0, 0)$. The isotropy subgroup of $(0, 0, 0)$ is all of $SO(3)$.

Theorem : If H is a closed subgroup of a Lie group G , then

(1) H is a Lie subgroup of G , and

(2) there is one and only one manifold structure on the set G/H of coset of H in G , for which the following conditions hold :

(i) the natural mapping from G to G/H is a C^∞ mapping,

(ii) $\eta: G \rightarrow G/H$ is a fiber bundle in the sense that at each point p of G/H there exists a smooth local section of η defined on an open subset of G/H containing p .

Theorem : If a Lie group G acts transitively on a manifold M and $x \in M$ then the mapping $\beta: G/G_x \rightarrow G \cdot x = M$ defined by $\beta(gG_x) = g \cdot x$ is a diffeomorphism. The result holds for both left and right actions.

$$\left. \begin{array}{l} \beta: G \rightarrow M, \beta(g) = g \cdot x \\ \beta \text{ constant on cosets} \end{array} \right\}$$

Remark. Observe that G_x is indeed a Lie subgroup of G as it is easy to show G_x is closed in G and the result follows from part (1) of the first theorem quoted above. Also part (2) of the first theorem also guarantees that G/G_x is a well-defined manifold. Finally note that by Example 3 above $G \cdot x$ generally is not a manifold or at least not a submanifold of G . This is the rationale behind the assumption that G acts transitively on M which is presumed to be a manifold. etc

So each curve through the identity of $\text{Diff}(M)$ defines a vector field X_g . With more care one can show that the set ΓM of all vector fields on M is the Lie algebra of $\text{Diff}(M)$, i.e. if G is an ordinary Lie group acting on M , if $\varphi : M \times G \rightarrow M$ then for each $g \in G$ we have

a mapping $\varphi_g : M \rightarrow M$ defined by

$$\varphi_g(x) = x \cdot g = \varphi(x, g).$$

Left action

Observe that φ_g is smooth and invertible with $(\varphi_g)^{-1} = \varphi_{g^{-1}}$ for each $g \in G$. So $\varphi_g \in \text{Diff}(M)$ for $g \in G$.

Let $\hat{\varphi} : G \rightarrow \text{Diff}(M)$ be defined by $\hat{\varphi}(g) = \varphi_g$.

Then $\hat{\varphi}$ is a group homomorphism. With a more

careful treatment of the manifold structure on $\text{Diff}(M)$, $\hat{\varphi}$ is generically smooth. Note that

$g \in \ker \hat{\varphi}$ iff $\hat{\varphi}(g) = \varphi_g$ is the identity mapping id_M and so $g \in \ker \hat{\varphi}$ iff $x \cdot g = x$ for all $x \in M$.

Thus the kernel of $\hat{\varphi}$ is trivial iff the action φ is an effective action. In such a case

G is embedded as a Lie subgroup of $\text{Diff}(M)$.

All of these statements are true under appropriate hypothesis, but caution should be observed in general.

We see that in case G is a Lie subgroup of

$\text{Diff}(M)$ the Lie algebra of G should be describable as a sub-Lie-algebra of $\text{Diff}(M)$, i.e. as a subalgebra of ΓM . This is in fact possible and is one motivation for the following definition.

Let M be any manifold and X and Y vector fields on M . If $\{\eta_t\}$ is the flow of X then the Lie derivative of Y with respect to X is defined to be the vector field $L_X Y$ where

$$(L_X Y)_x = \lim_{t \rightarrow 0} \frac{[\eta_{-t}(Y(\eta_t x)) - Y(x)]}{t}$$

Observe that one has a curve $t \mapsto \eta_{-t}(Y(\eta_t x))$ of tangent vectors all tangent to M at x and $(L_X Y)_x$ is the derivative of this curve at $t=0$.

O'Neil
Lie deriv

Lemma 2 Let M be a manifold and X and Y vector fields on M . Then $L_X Y = [X, Y]$.

Proof Notice that if $f \in C_c^\infty(M)$ then one has a real-valued function defined in an interval about 0 given by $t \mapsto \eta_{-t}(Y(\eta_t x))(f) = Y_{\eta_t x}(f \circ \eta_{-t})$.

For each y we can expand the mapping $t \mapsto f(\eta_{-t}(y))$ using the Taylor formula to obtain

$$f(\eta_{-t}(y)) = f(y) + t \left. \frac{d}{dt} (f(\eta_{-t}(y))) \right|_{t=0} + \sum_{n=2}^{\infty} \left. \frac{d^n}{dt^n} (f(\eta_{-t}(y))) \right|_{t=0} t^n$$

for some $t \in \mathbb{R}$. Let $g(y) = \left. \frac{d^2}{dt^2} (f(\eta_{-t}(y))) \right|_{t=t_k} = \left. \frac{\partial^2}{\partial t^2} (f(\eta_{-t}(y))) \right|_{t=t_k}$ and observe that g is smooth in a neighborhood of x . We have

$$(f \circ \eta_{-t})(y) = f(y) + t \left. df \left(-\frac{d}{dt} (\eta_t(y)) \right) \right|_{t=0} + t^2 g(y)$$

or $f \circ \eta_{-t} = f = t X(f) + t^2 g$.

We have

Theorem. Assume that G is a Lie group which acts on the right of the manifold M . Then the infinitesimal generator $S: \mathfrak{g} \rightarrow \Gamma M$ of the action is a Lie algebra homomorphism which is injective when the group action is effective. More generally, when the group action is a free action one has that S_A never vanishes for each $A \in \mathfrak{g} \setminus \{0\}$.

Proof. To show that S is linear we reformulate its definition slightly. For $x \in M$ let $\delta_x: G \rightarrow M$ be defined by $\delta_x(g) = x \cdot g$. Then $\delta_x(\exp(tA)) = x \cdot \exp(tA)$ and

$$d\delta_x(A) = \left. \frac{d}{dt} [\delta_x(\exp(tA))] \right|_{t=0} = \left. \frac{d}{dt} [x \cdot \exp(tA)] \right|_{t=0} = S_A(x).$$

It follows that $S_{A+B}(x) = d\delta_x(A+B) = d\delta_x(A) + d\delta_x(B)$

$$= (S_A + S_B)(x). \text{ Similarly, } S_{cA}(x) = d\delta_x(cA) = (cS_A)(x),$$

So S is a linear mapping from \mathfrak{g} into ΓM .

We show that S is injective when the action is effective.

In case $S_A = 0$ for $A \in \mathfrak{g}$ we argue as follows. See that $\frac{d}{dt}(\eta(t, x)) = 0$ where η is the flow of

case A . Thus $t \mapsto \eta(t, x) = x \cdot \exp(tA)$ is constant

for each $x \in M$ and consequently $x \cdot \exp(tA) = x$ for all x and all t . Thus $\exp(tA) = e$ for all t

But $\exp(tA) = \exp(0A)$ implies $tA = 0$ for t

sufficiently small, thus $A = 0$. Finally assume the action is a free action. Assume $S_A(x) = 0$

for some $x \in M$. Then $d\delta_x(\exp(tA)) = 0$ for all $t \in \mathbb{R}$

and $\left. \frac{d}{dt} [\eta(t, x) \cdot \exp(tA)] \right|_{t=0} = \left. \frac{d}{dt} x \cdot \exp(tA) \right|_{t=0} = S_A(x) = 0$.

$$\text{Thus } \frac{d}{ds} (x \cdot \exp(sA)) = \left. \frac{d}{dt} (x \cdot \exp(t+A)A) \right|_{t=0}$$

$$= \left. \frac{d}{dt} [r_{\exp(tA)}(x \cdot \exp(tA))] \right|_{t=0}$$

Thus s_A and $\exp(sA) = x$ for all s . It follows that $A = 0$
 if $A \neq 0$. Let remaine to show that for $A, B \in \mathfrak{g}$

$$\delta([A, B]) = [\delta(A), \delta(B)].$$

First observe that since $r_t(x) = \exp(tA)$ is the flow of A

$$[A, B]_e = (L_A B)_e = \left. \frac{d}{dt} [dr_{\exp(tA)}(B \exp(tA))] \right|_{t=0}$$

$$= \left. \frac{d}{dt} [dr_{\exp(-tA)}(dl_{\exp(tA)}(B_e))] \right|_{t=0}$$

$$= \left. \frac{d}{dt} [dr_{\exp(-tA)}(dl_{\exp(tA)}(\left. \frac{d}{ds} (\exp(sB)) \right|_{s=0}))] \right|_{t=0}$$

$$= \left. \frac{d}{dt} \frac{d}{ds} [r_{\exp(-tA)}(l_{\exp(tA)}(\exp(sB)))] \right|_{s=0, t=0}$$

$$= \left. \frac{d}{dt} \frac{d}{ds} [\exp(tA) \exp(sB) \exp(-tA)] \right|_{s=0, t=0}$$

Let $\{n_t\}$ be the flow of B so that $n_t(u) = u \cdot \exp(tA)$.

$$\text{But } [\delta_A, \delta_B]_u = L_{\delta_A}(\delta_B)_u$$

$$= \left. \frac{d}{dt} [dr_{-t}(\delta_B(n_t, u))] \right|_{t=0}$$

$$= \left. \frac{d}{dt} [dr_{\exp(-tA)}(do_{n_t, u}(B_e))] \right|_{t=0}$$

$$= \left. \frac{d}{dt} [dr_{\exp(-tA)}\left(\frac{d}{ds} [r_{n_t, u}(\exp(sB))] \right|_{s=0})] \right|_{t=0}$$

$$= \left. \frac{d}{dt} \frac{d}{ds} [r_{\exp(-tA)}(u \cdot \exp(tA) \exp(sB))] \right|_{s=0, t=0}$$

Thus the orbit $x \cdot SO(n+1)$ is precisely the n -sphere

$$S^n = \{ y \in \mathbb{R}^{n+1} \mid \|y\| = \|x\| \}.$$

Note that A belongs to the isotropy subgroup of x iff $x \cdot A = x$. Again choose an orthonormal basis $\{x, e_1, e_2, \dots, e_m\}$ of \mathbb{R}^{n+1} . Then $\langle e_i A, e_j A \rangle = \langle e_i, e_j \rangle = \delta_{ij}$ and $\langle x, e_i A \rangle = \langle x A, e_i A \rangle = \langle x, e_i \rangle = 0$.

Thus

$$A = \begin{pmatrix} 1 & 0 \\ 0 & A_\perp \end{pmatrix}$$

where A_\perp is the linear map from \mathbb{R}^n to \mathbb{R}^n such that $e_i A_\perp = e_i A$ for all i . So $A_\perp \in SO(n)$ and $\det A_\perp = \det A = 1$. Thus $A \in SO(n+1)_x$ iff it is of the form $1 \oplus A_\perp$ where $A_\perp \in SO(n)$. We have

$$\frac{SO(n+1)}{SO(n)} \cong S^n$$

is diffeomorphic to the n -sphere. It follows that the mapping $\eta: SO(n+1) \rightarrow S^n$ defined by $\eta(A) = x \cdot A$ is a fiber bundle with fiber $SO(n)$.

Limits & Brackets def^k of Lie Derivative now linked 9/27/04 ①

\bar{X}, \bar{Y} vector fields on M

○ η is the flow of \bar{X}

$$(L_{\bar{X}} \bar{Y})_x = \frac{d}{dt} (d\eta_{-t}) (\bar{Y}(\eta(t, x))) \Big|_{t=0} \stackrel{?}{=} [\bar{X}, \bar{Y}]$$

Taylor Expansion:

$$(f \circ \eta_{-t})(y) = f(y) + t df \left(-\frac{d}{dt} (\eta_t(y)) \Big|_{t=0} \right) + t^2 g(y)$$

Then via the remainder formula...

$$f \circ \eta_{-t} = f - t \bar{X}(f) + t^2 g \quad (*)$$

$$\text{since } \frac{d}{dt} (\eta_t(y)) \Big|_{t=0} = \bar{X}(\eta(t, y)) \quad \& \quad \bar{X}(y)(f) = \bar{X}(f)(y)$$

Then act on $(*)$ by $\bar{Y}_{\eta(t, x)}$ to get

○ ~~•~~ $\bar{Y}_{\eta(t, x)} (f \circ \eta_{-t}) = \bar{Y}_{\eta(t, x)}(f) - t \bar{Y}_{\eta(t, x)}(\bar{X}(f)) + t^2 (\bar{Y}_{\eta(t, x)}(g))$

Take derivative piece by piece,

$$\begin{aligned} \frac{d}{dt} (\bar{Y}_{\eta(t, x)}(f)) &= \frac{d}{dt} [\bar{Y}(f)(\eta(t, x))] \quad \text{use chain on } \mathbb{R} \text{ fact.} \\ &= d_{\eta(t, x)}(\bar{Y}(f)) \left[\frac{d}{dt} (\eta(t, x)) \right] \\ &= d_{\eta(t, x)}(\bar{Y}(f)) (\bar{X}(\eta(t, x))) \quad \text{①} \end{aligned}$$

Next we find:

$$\frac{d}{dt} (t \bar{Y}_{\eta(t, x)}(\bar{X}(f))) = t \frac{d}{dt} \bar{Y}_{\eta(t, x)}(\bar{X}(f)) + \bar{Y}_{\eta(t, x)}(\bar{X}(f)) \quad \text{②}$$

at $t=0$ the 1st term vanishes, leaving $\bar{Y}_x(\bar{X}(f))$

Finally notice:

$$\frac{d}{dt} \left[t^2 (\Upsilon_{\eta(t,x)} g) \right] = t \left[\frac{d}{dt} t \Upsilon_{\eta(t,x)} g \right] + t (\Upsilon_{\eta(t,x)} g) \quad \textcircled{3}$$

which is clearly zero when $t=0$

$$\begin{aligned} (\mathcal{L}_X \Upsilon)_x(f) &= d_x f \left((\mathcal{L}_X \Upsilon)_x \right) \\ &= d_x f \left(\frac{d}{dt} \left[d\eta_{-t} (\Upsilon(\eta(t,x))) \right] \Big|_{t=0} \right) \\ &= \frac{d}{dt} \left[d_x f (d\eta_{-t} (\Upsilon(\eta(t,x)))) \right] \Big|_{t=0} \\ &= \frac{d}{dt} \left[d(f \circ \eta_{-t}) (\Upsilon(\eta(t,x))) \right] \Big|_{t=0} \\ &= \frac{d}{dt} [\Upsilon(\eta(t,x)) (f \circ \eta_{-t})] \Big|_{t=0} \\ &= d_x (\Upsilon(f)) (\Xi(x)) - \Upsilon_x (\Xi(f)) \\ &= \Xi_x (\Upsilon(f)) - \Upsilon_x (\Xi(f)) \\ &= [\Xi, \Upsilon]_x(f) \quad // \end{aligned}$$

On page 218 & 219, well there should be
the 3 Lemmas

① the flow of Ξ is just $\Xi = \exp^{tA}$

② $(\mathcal{L}_X \Upsilon)_x(f) = [\Xi, \Upsilon]_x(f)$

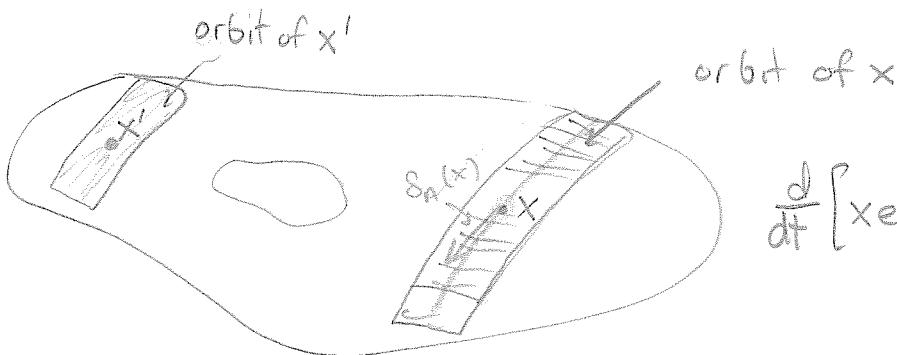
③ If $A \in \mathfrak{g} = \mathfrak{t}(G)$ and $x \in M$ such that
the infinitesimal v.f. generated by A at x is
zero then $x \exp(sA) = 0$ for all s .
 $(S_A(x) = 0)$

Lemma 3: If $A \in \mathfrak{g} = \mathfrak{g}(G)$ and $x \in M$ such that $S_A(x) = 0$ then $x \exp(tA) = x$ for all t . If G acts on M freely then $S_A(x) = 0$ for some $x \in M \Rightarrow A = 0$.

If G acts effectively then $S_A(x) = 0$ $\forall x \in M$ ($S_A = 0$) then $\Rightarrow A = 0$. Effective needs $S_A(x) = 0$ for every $x \in M$.

Pf/

$$S_A(x) \equiv \left. \frac{d}{dt} [x \exp(tA)] \right|_{t=0}$$



$$\left. \frac{d}{dt} [x \exp(tA)] \right|_{t=0} = S_A(x)$$

So if $S_A(x) = 0 = \left. \frac{d}{dt} (x \exp(tA)) \right|_{t=0}$

Now in these notes,

$$M \times G \xrightarrow{\sigma} M$$

$$R_g : M \rightarrow M$$

$$R_g(x) = \sigma(x, g) = x \cdot g$$

In the groups

$$r_g : G \rightarrow G$$

$$r_g(x) = x \cdot g$$

Apply $d_x R_{\exp(tA)}$ on both sides for arbitrary t .

$$\frac{d}{dt} (f(g(t))) = df(g(t))$$

$$d_x R_{\exp(tA)} \left(\left. \frac{d}{dt} (x \exp(tA)) \right|_{t=0} \right) = 0$$

Proof of Lemma 3

$$\delta_A(x) = 0 \Rightarrow d_x R_{\exp(sA)} \left(\frac{d}{dt} (x \exp(tA)) \Big|_{t=0} \right) = 0 \quad \forall s$$

$$\Rightarrow \frac{d}{dt} [R_{\exp(sA)} R_{\exp(tA)}(x)] \Big|_{t=0} = 0$$

$$\Rightarrow \frac{d}{dt} [R_{\exp((s+t)A)}(x)] \Big|_{t=0} = 0$$

Let $u = s+t$ then we get

$$\frac{d}{du} [R_{\exp(uA)}(x)] \Big|_{u=s} = 0 \quad \forall s.$$

$$\frac{d}{ds} [R_{\exp(sA)}(x)] = 0 \quad \forall s$$

Thus $R_{\exp(sA)}(x) = \text{constant}$. Hence its value at any s is the same as $s=0$

$$R_{\exp(0 \cdot A)}(x) = R_I(x) = x //$$

Next we'll finish this lemma

& the th?

(1)

Def^b/ Assume a Lie group G acts on the right of a manifold M . For $A \in \mathfrak{l}(G)$ define a vector field S_A of M by

$$S_A(x) = \left. \frac{d}{dt} [x \cdot \exp(tA)] \right|_{t=0}$$

The mapping $A \rightarrow S_A$ is the infinitesimal action of $\mathfrak{l}(G)$ on M . Beware in older literature the algebra of the terminal group were confused.

$$S : \mathfrak{l}(G) \rightarrow \Gamma(M)$$

Example $SO(3)$ acts on $\mathbb{R}^3 \setminus \{0\}$. Let $A \in \mathfrak{so}(3) = \mathfrak{l}(SO(3))$

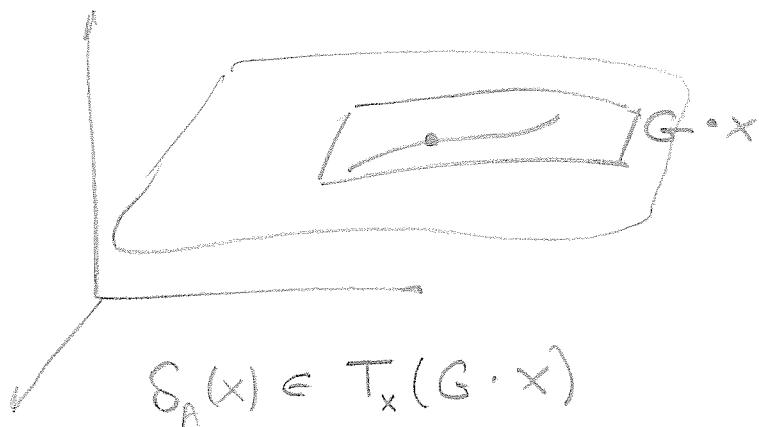
S_A is a vector field on $\mathbb{R}^3 \setminus \{0\}$

Since $\mathfrak{so}(3) = \mathfrak{l}(SO(3)) \subset \mathfrak{gl}(n)$ we have

$$\exp(tA) = e^{tA}$$

$$\begin{aligned} S_A(x) &= \left. \frac{d}{dt} (x e^{tA}) \right|_{t=0} \\ &= \left. \frac{d}{dt} (x + x t A + \frac{1}{2} x t^2 A^2 + \dots) \right|_{t=0} \\ &= \dots \end{aligned}$$

Notice that $x \cdot \exp(\cdot) \in G \cdot x$



Lemma 1 :

Assume that G acts on the right of a manifold M and let $A \in \mathfrak{g} = \mathfrak{L}(G)$. Define $\eta : \mathbb{R} \times M \rightarrow M$ by $\eta(t, x) = x \cdot \exp(tA)$. Then $\{\eta_t\}$ is the flow of S_A

Proof

$$\eta(0, x) = x \cdot \text{id} = x$$

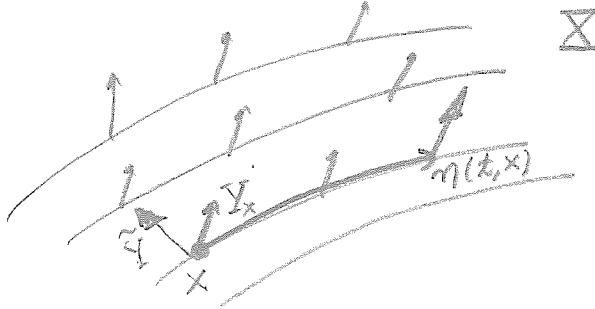
$$\begin{aligned} S_A(\eta(s, x)) &= \frac{d}{dt} [\eta(s, x) \cdot \exp(tA)] \Big|_{t=0} \\ &= \frac{d}{dt} [x \cdot \exp(sA) \cdot \exp(tA)] \Big|_{t=0} \\ &= \frac{d}{dt} [x \cdot \exp((s+t)A)] \Big|_{t=0} \quad \text{let } \tau = t+s \\ &= \frac{d}{d\tau} [x \cdot \exp(\tau A)] \Big|_{\tau=s} \\ &= \frac{d}{ds} [x \cdot \exp(sA)] \\ &= \frac{d}{ds} [\eta(s, x)] \end{aligned}$$

Hence η_t is indeed a flow of S_A //

3

Def/ If M be a manifold, \bar{X}, \bar{Y} are vector fields on M .
 If $\{\eta_t\}$ is the flow of \bar{X} then $L_{\bar{X}} \bar{Y}$ is defined by

$$(L_{\bar{X}} \bar{Y})_x = \lim_{t \rightarrow 0} \left(\frac{(\eta_{-t})(\bar{Y}(\eta(t, x))) - \bar{Y}_x}{t} \right)$$



the arrows are \bar{Y}

$\bar{Y}(\eta(t, x))$ is vector at
 $\eta(t, x)$

$$d\eta_{-t} : T_{\eta(t, x)} M \rightarrow T_x M$$

$$\tilde{Y} = d\eta_{-t}(\bar{Y}(\eta(t, x))) \in T_x M$$

So it's sensible.

Remark notice that: (Convinient than limit form.)

$$(L_{\bar{X}} \bar{Y})_x = \frac{d}{dt} \left[\eta_{-t}(\bar{Y}(\eta(t, x))) \right] \Big|_{t=0}$$

Def/ for α a differential form has Lie derivative along \bar{X}

$$L_{\bar{X}} \alpha = \lim_{t \rightarrow 0} \frac{(\eta_t^* \alpha - \alpha)}{t}$$

Lemma 2

Let M be a manifold with $\mathbf{X}, \mathbf{Y} \in \Gamma(M)$ then $L_{\mathbf{X}} \mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$

Proof: A nice proof probably from O'Neil. Let $f \in C_{loc}^\infty(x)$, $x \in M$

Consider that:

$$t \longrightarrow d\eta_{-t}(\mathbf{Y}(\eta(t, x)))(f) \in \mathbb{R}$$

Thus we can expand in Taylor formulae, For each y expand

$f(\eta_{-t}(y))$ as a funct. of t .

$$f(\eta_{-t}(y)) = f(y) + t \frac{d}{dt} [f(\eta_{-t}(y))] \Big|_{t=0} + \frac{1}{2} t^2 \frac{d^2}{dt^2} [f(\eta_{-t}(y))] \Big|_{t=t_0}$$

remainder of
Taylor series, can
use Lagrange
form & its
clear then
it's smooth
as y changes

Then let $g(y) = \frac{1}{2} \frac{d^2}{dt^2} [f(\eta_{-t}(y))] \Big|_{t=t_x}$

is in fact smooth in nbhd of x . Minus comes from chain ($-t$)

$$\begin{aligned} (f \circ \eta_{-t})(y) &= f(y) - t df \left(\frac{d}{dt} [\eta_{-t}(y)] \right) \Big|_{t=0} + t^2 g(y) \\ &= f(y) - t \sum_y (f) + t^2 g(y) \end{aligned}$$

As y varies for appropriate nbhd. of x

①

Action Function of Electromagnetic Field

$$S = S_m + S_{mf} + S_f$$

particles particle
 in field field

$$S_m = \sum_{\text{particles}} \int m c ds$$

$$S_{mf} = - \sum \int \frac{e}{c} A_k dx^k$$

$$S_f = ? \quad \text{Let's follow Landau}$$

$$(F^{ik})_{\text{tot}} = \sum_{a=\text{particles}} (F^{ik})^a$$

- superposition of forces, exp. fact.
- likewise 1st order deg⁺, satisfies superposition principle

$$\frac{d\vec{x}_i}{dt} = 0 \quad , \quad \vec{x}_i = (x_1, x_2, \dots, x_n)_i$$

$$\frac{d(\sum \vec{x}_i)}{dt} = 0$$

In order to obtain linear deg⁺ of 1st order from action principle \Rightarrow take quadratic action so construct invariant quadratic form

$$S_f = \frac{-1}{16\pi} \iint F_{ik} F^{ik} dV dt \quad (\text{Gaussian Units})$$

Hence from homework

$$S_f = \frac{1}{8\pi} \iint (E^2 - H^2) dV dt$$

Lagrangian: $L = \frac{1}{8\pi} \int (E^2 - H^2) dV$

Then the total action is simply:

$$S = - \sum_{\text{a-particles}} \int m_a c ds - \underbrace{\sum_a \left[\frac{e_a}{c} A_k dx^k - \frac{1}{16\pi c} \int F_{ik} F^{ik} d\Omega \right]}_{(*)}$$

Where we have used

$$\vec{E} = - \frac{\partial \vec{A}}{\partial t} - \nabla \phi$$

Now lets derive what (*) connects to. We always approximate nature with continuous charge distribution

$$e = \sum_a e_a \quad \text{Charge or Point}$$

$$e = \int \rho(\vec{r}, t) dV \quad \begin{matrix} \text{Passing to continuous limit,} \\ \text{where the point charges} \\ \text{are given} \end{matrix}$$

$$\rho(\vec{r}, t) = \sum_a e_a \delta(\vec{r} - \vec{r}_a)$$

Remark: Charge is a invariant hence ρdV is an invariant.

4- Current

$$de dx^i = \rho dV dx^i = \frac{1}{c} \rho \frac{dx^i}{dt} cdtdV = \frac{1}{c} \left(\rho \frac{dx^i}{dt} \right) (cdtdV)$$

$$\vec{j}^i \equiv \rho \frac{dx^i}{dt} = (\rho c, \vec{\gamma})$$

where $\vec{\gamma} = \rho \vec{v} \leftarrow \text{current density}$

$$\sum_a e_a dx^k \rightarrow \int \rho dV dx^k = \frac{1}{c} \int (cdtdV) \rho \frac{dx^k}{dt}$$

$$= \frac{1}{c} \int d\Omega \vec{j}^k$$

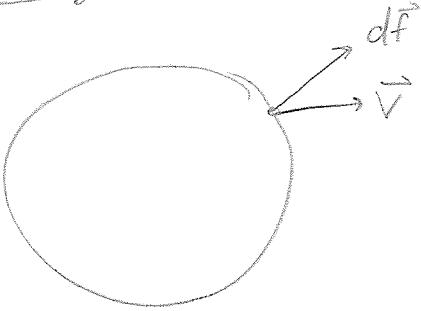
Consequently

$$S = - \int mc ds - \frac{1}{c^2} \int A_k \vec{j}^k d\Omega - \frac{1}{16\pi c} \int F_{ik} F^{ik} d\Omega$$

another integration
for the mass distribution

(3)

Continuity of Charge



$$\frac{\partial}{\partial t} \left(\int \rho dV \right) = - \left(\oint \vec{J} \cdot d\vec{f} \right) \\ = - \int (\nabla \cdot \vec{J}) dV$$

Hence $\int \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} \right) dV = 0$ holds

for arbitrary volume $\therefore \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$

In terms of the 4-current $\boxed{\partial_\mu j^\mu = 0}$
Covariant Formulation of Charge Conservation.

$$Q = \int \rho dV = \frac{1}{c} \int_{x^0=\text{constant}} j^0 dS_0 \quad dS_0 = dx dy dz$$

$$= \frac{1}{c} \int_{x^0=\text{constant}} j^0 dS_n \quad (\text{charge invariance manifest.})$$

Fix a time then the # of charges is fixed. We can actually count this on space-like hyper surface,

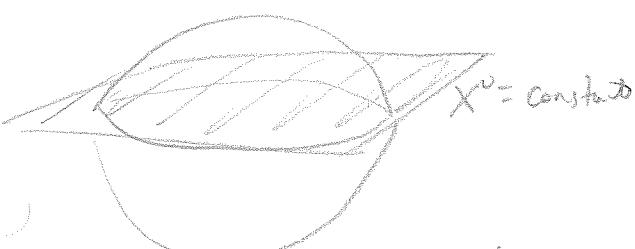
$$\oint j^0 dS_n = \int \partial_\mu j^\mu d\Omega$$

Generalized Stokes

↑
contains 2
hyper-surfaces

$$\int j^0 dS_n - \int j^0 dS_n = 0$$

hyp. surface



can get integration
in volume above or
below the surface

$$\int j^0 dS_n = \int j^0 dS_n = Q$$

hypersurface $x^0 = \text{constant}$

Physically, there are many ways to count the charge.
(Think this is clear from inv. of Q)

(4)

If $\partial_k A^k = 0$ then

$$\int_{\text{hypersurface}} A^k dS_k = \int A^0 dS_0$$

$x^0 = \text{constant}$

//

S Second Pair of Maxwell Eq's, Energy Density & Flux, ...

Can think of Max Eq's as eq's of motion for field obtained from least action principle,

$$S = - \sum \int m c ds - \frac{1}{c^2} \int A_k j^k d\Omega - \frac{1}{16\pi} \int F_{ik} F^{ik} d\Omega$$

Assumption the particles follow these paths, ignore S_m
What happens as the

$$\begin{aligned} S_S &= - \frac{1}{c^2} \int j^k \delta A_k d\Omega - \frac{1}{8\pi c} \int F^{ik} \delta F_{ik} d\Omega \\ &= - \frac{1}{c^2} \int j^k \delta A_k d\Omega - \frac{1}{8\pi c} \int F^{ik} \left(\frac{\partial \delta A_k}{\partial x^i} - F^{ik} \frac{\partial \delta A_i}{\partial x^k} \right) d\Omega \\ &= - \frac{1}{c^2} \int j^k \delta A_k d\Omega - \frac{1}{4\pi c} \int_{\text{a volume}} F^{ik} \frac{\partial \delta A_i}{\partial x^k} d\Omega \\ &= - \frac{1}{c^2} \int j^k \delta A_k d\Omega - \frac{1}{4\pi c} \int_{\text{hyper}} F^{ik} \delta A_i dS_i \\ &\quad \curvearrowleft \int \frac{\partial}{\partial x^k} (F^{ik} \delta A_i) d\Omega \\ &= - \frac{1}{c} \int \left(\frac{1}{c} j^i \delta A_i + \frac{1}{4\pi} \frac{\partial F^{ik}}{\partial x^k} \delta A_i \right) d\Omega \end{aligned}$$

$$\Rightarrow \boxed{\frac{\partial F^{ik}}{\partial x^k} = - \frac{4\pi}{c} j^i}$$

$$\begin{aligned} i=0 &\rightarrow \nabla \cdot \vec{E} = 4\pi \rho \\ i=1,2,3 &\rightarrow \nabla \times \vec{H} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{j} \end{aligned}$$

(5)

Maxwell's Eq's

1st Pair

$$\nabla \cdot \vec{H} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}$$

$$\partial_k F_{ik} + \partial_i F_{ki} + \partial_k F_{li} = 0$$

$$[\partial_k (F^*)^{ik} = 0]$$

2nd Pair

$$\nabla \cdot \vec{E} = 4\pi P$$

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\partial_k F^{ik} = -\frac{4\pi}{c} j^i$$

Conservation of Charge follows naturally

$$\partial^k j_p - \frac{c}{4\pi} \partial^k \partial_k F^{ik} = 0$$

Lemma:

If $A \in \mathbb{D}$ and $x \in M$ such that $S_A(x) = 0$, then $x \cdot \exp(sA) = x$ for all s . If G acts freely on M then $S_A(x) = 0$ for some $x \Rightarrow A = 0$. If G acts effectively on M then $S_A(x) = 0 \forall x \Rightarrow A = 0$.

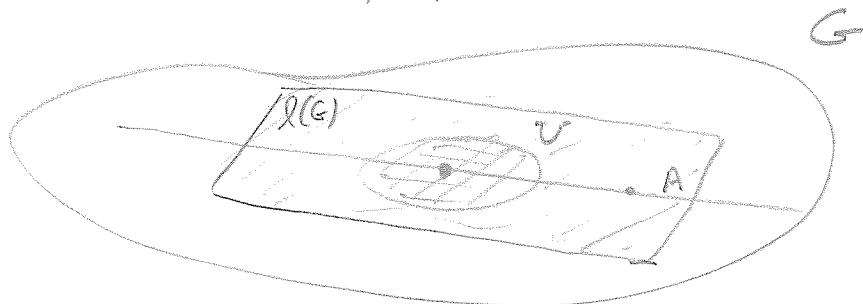
Proof: Last time we did the beginning now $\mathbb{F} \subseteq G$ acts freely & that $x \in M$ such that $S_A(x) = 0$, then

$$x \cdot \exp(sA) = x$$

$$\Rightarrow \exp(sA) = e$$

$$\Rightarrow \exp(sA) = \exp(0)$$

But $\exp|_U$ for some $U \subseteq \mathbb{L}(G)$ is a diffeomorphism in particular it's 1-1 so if s is small enough so that $s \cdot A \in U$, $\|sA\| = |s| \|A\|$



For such s then

$$\exp_U(sA) = \exp_U(0) \Rightarrow sA = 0 \Rightarrow \underline{\underline{A = 0}}$$

#

Assume G acts effectively and $S_A(x) = 0 \forall x$. Then following the same argument, now $x \cdot g = x$

$$x \cdot \exp(sA) = x \quad \forall x$$

Since action is effective

$$\exp(sA) = e \quad \forall s$$

Then by same argument $sA = 0 \Rightarrow \underline{\underline{A = 0}}$.

Corollary to Lemma: The mapping $A \rightarrow \delta_A$ is one-one.

Th^m/ Assume G is a Lie group that acts on the right of a manifold M . Then the infinitesimal generator

$\delta: \mathfrak{g} \rightarrow \Gamma M$ of the action is a

Lie algebra homomorphism.

If the action is effective then δ is injective.

If the action is free then δ is injective and δ_A is non-vanishing for $A \in \mathfrak{g} \setminus \{0\}$.

Remark:

If a free action of a Lie Algebra on the 2-sphere. The δ_A 's would be a vector field, non-vanishing on 2-sphere.

Proof: $\delta_A(x) = \left. \frac{d}{dt} [x \exp(tA)] \right|_{t=0}$

For $x \in M$ let $\sigma_x: G \rightarrow M$ defined by $\sigma_x(g) = \sigma(x, g) = x \cdot g$

$$\sigma_x(\exp(tA)) = x \cdot \exp(tA)$$

$$d_e \sigma_x \left(\left. \frac{d}{dt} (\exp(tA)) \right|_{t=0} \right) = \delta_A(x)$$

$$\delta_A(x) = d_e \sigma_x(A)$$

$$\text{Hence } \delta_{A+cB}(x) = d_e \sigma_x(A+cB) = d_e \sigma_x(A) + c d_e \sigma_x(B) = \delta_A(x) + c \delta_B(x)$$

So its a linear transformation, next we'll show
its a Lie algebra Homomorphism change notation

$$\delta_A(x) = \delta(A)(x)$$

(3)

Proof: We show that $S([A, B]) = [SA, SB]$ (*)

Continued.

First notice that since $\eta(t, x) = x \cdot \exp(tA)$

$$= \eta \exp(tA)(x) : \text{Right multiply by group elements}$$

By Lemma 1 on pg. 215 this is the flow of A :

$$[A, B]_e = (L_A B)_e \quad \text{from pg. 216 above Lemma 2;}$$

$$= \frac{d}{dt} \left(d\eta_{\exp(-tA)}(B(\exp(tA))) \right) \Big|_{t=0}$$

$$= \frac{d}{dt} \left(d\eta_{\exp(-tA)}(d\eta_{\exp(tA)}(B_e)) \right) \Big|_{t=0}$$

$$= \frac{d}{dt} \left(d\eta_{\exp(-tA)} \left(d\eta_{\exp(tA)} \left(\frac{d}{ds} (\exp(sB)) \Big|_{s=0} \right) \right) \right) \Big|_{t=0}$$

$$= \frac{d}{dt} \frac{d}{ds} \left(\eta_{\exp(-tA)} \left(\eta_{\exp(tA)} (\exp(sB)) \right) \right) \Big|_{t=0, s=0}$$

$$= \frac{d}{dt} \frac{d}{ds} \left(\exp(tA) \exp(sB) \exp(-tA) \right) \Big|_{\substack{t=0 \\ s=0}}$$

Now lets work on the other side of (*)

$$[SA, SB]_u = L_{SA}(S_B)$$

$$= \frac{d}{dt} \left[d\eta_{-t} (S_B(\eta(t, u))) \right] \Big|_{t=0} \xrightarrow{\frac{d}{ds} (\exp(sB))} \Big|_{s=0}$$

$$= \frac{d}{dt} \left[dR_{\exp(-tA)} (d\sigma_{u \exp(tA)}(B_e)) \right] \Big|_{t=0} \xrightarrow{\frac{d}{ds} (\exp(sB))} \Big|_{s=0}$$

$$= \frac{d}{dt} \frac{d}{ds} \left[R_{\exp(-tA)} (\sigma_{u \exp(tA)} (\exp(sB))) \right] \Big|_{\substack{t=0 \\ s=0}}$$

$$= \frac{d}{dt} \frac{d}{ds} \left[R_{\exp(-tA)} (u \exp(tA) \exp(sB)) \right] \Big|_{\substack{t=0 \\ s=0}}$$

$$= \frac{d}{dt} \frac{d}{ds} \left[\sigma_u (\exp(tA) \exp(sB) \exp(-tA)) \right] \Big|_{\substack{t=0 \\ s=0}}$$

$$\boxed{(ug_1)g_2 = u(g_1g_2)}$$

$$\sigma_{ug_1}(g_2) = \sigma_u(g_1g_2)$$

⑨

$$\begin{aligned}
 [\delta A, \delta B]_u &= \frac{d}{dt} \left(d\sigma_u \left(\frac{d}{ds} [\exp(tA) \exp(sB) \exp(-tA)] \right) \right) \Big|_{\substack{s=0 \\ t=0}} \\
 &= d\sigma_u \left(\frac{d}{dt} \frac{d}{ds} [\exp(tA) \exp(sB) \exp(-tA)] \right) \Big|_{\substack{s=0 \\ t=0}} \\
 &= d_e \sigma_u ([A, B]_e) \\
 &= \delta ([A, B]_e)(u)
 \end{aligned}$$

Even true for top. groups.

Theorem: Assume that $\varphi: G \rightarrow H$ is a continuous homomorphism from a topological group G into a connected topological group H . If there is an open subset O of H containing the identity e of H such that $O \subseteq \varphi(G) \subseteq H$, then φ is surjective.

Proof: Since the mapping from H to H defined by $x \mapsto x^{-1}$ is a homeomorphism,

$$O^{-1} = \{x^{-1} \mid x \in O\}$$

is open in H . Thus $W = O \cup O^{-1}$ is also open.

Since the mapping from H to H defined by $x \mapsto ax$ is a homeomorphism for each $a \in H$

we see that $W^2 = \bigcup_{a \in W} (aW)$ is open. Inductively

we see that for $n \geq 2$, $W^n = \bigcup_{a \in W} aW^{n-1}$ is open.

Now the subgroup of H generated by O consists of all products

$$x_1^{i_1} x_2^{i_2} \cdots x_R^{i_R}$$

such that $x_j \in O$ and $i_j \in \{1, -1\}$. Let K

denote this subgroup. Notice that K is open

since $K = \bigcup_n W^n$. Since $K \subseteq H$ is

open so is aK for $a \in H$. Observe that

$G - K = \bigcup_{a \notin K} (aK)$ is open. Thus K is

both open and closed in a connected space H .

Thus $K = H$. But every element of K

is of the form $x_1^{i_1} \cdots x_R^{i_R}$ with $x_j \in O \subseteq \varphi(G)$ as above. If $y_j \in G$ such that $\varphi(y_j) = x_j$

it is connected. We show that $d\varphi : T_e \mathrm{SU}(2) \rightarrow T_e \mathrm{SO}(3)$ is invertible. To do this it suffices to show that $d\tilde{\varphi} : T_e \mathrm{SU}(2) \rightarrow T_e(\mathrm{SO}(\mathrm{su}_2))$ is invertible. We first show that $d\tilde{\varphi} = \mathrm{ad}$. Let $\lambda \mapsto A_\lambda$ be a curve in $\mathrm{SU}(2)$ through the identity $e=I$ and let $B = \frac{d}{d\lambda}(A_\lambda)|_{\lambda=0} \in T_e \mathrm{SU}(2)$. Then

$$\begin{aligned} d\tilde{\varphi}(B) &= \left. \frac{d}{d\lambda} [\tilde{\varphi}(A_\lambda)] \right|_{\lambda=0} \text{ and for } \hat{x} \in \mathrm{su}_2, \\ d\tilde{\varphi}(B)(\hat{x}) &= \left. \frac{d}{d\lambda} [\tilde{\varphi}(A_\lambda)](\hat{x}) \right|_{\lambda=0} = \left. \frac{d}{d\lambda} [A_\lambda \hat{x} A_\lambda^{-1}] \right|_{\lambda=0} \\ &= \left. \frac{d}{d\lambda} (A_\lambda \hat{x} \bar{A}_\lambda^t) \right|_{\lambda=0} \\ &= B \hat{x} + \hat{x} \bar{B}^t = B \hat{x} - \hat{x} B = [B, \hat{x}] = \mathrm{ad}_B(\hat{x}) \end{aligned}$$

So $d\tilde{\varphi}(B) = \mathrm{ad}_B$. It follows that $B \in \mathrm{ker} d\tilde{\varphi}$ iff $[B, \hat{x}] = 0$ for all $\hat{x} \in \mathrm{su}(2)$. Thus

$B = \begin{pmatrix} i x_3 & -x_2 + i x_1 \\ x_2 + i x_1 & -i x_3 \end{pmatrix}$ where $x_1, x_2, x_3 \in \mathbb{R}$, and $[B, \hat{e}_i] = 0$ for $i=1, 2, 3$. Here $\{\hat{e}_i\}$ is the standard basis of \mathbb{R}^3 . Recall that $\{\hat{e}_i\}$ is a basis of su_2 where

$$\hat{e}_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\hat{e}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{e}_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

By direct computation we have that

$$i \begin{pmatrix} -\bar{\beta} & \bar{\alpha} \\ \alpha & \beta \end{pmatrix} = i \begin{pmatrix} \beta & \alpha \\ \bar{\alpha} & -\bar{\beta} \end{pmatrix} \Rightarrow \boxed{\alpha = \bar{\alpha}, \beta = -\bar{\beta}}$$

$$(2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \bar{\beta} & -\bar{\alpha} \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ \bar{\alpha} & -\bar{\beta} \end{pmatrix}$$

$$\Rightarrow \boxed{\alpha = \bar{\alpha} \quad \beta = -\bar{\beta}}$$

From (1) and (2) we see that α is real and $\beta = 0$

Thus $B = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$, $|\alpha|^2 = 1$, $\alpha = \bar{\alpha}$ and

$$B = \pm I.$$