

Fiber Bundles (missed typed notes)

Defⁿ/ A fiber bundle is a mapping $\pi: E \rightarrow M$ such that

(1.) π is smooth & surjective

(2.) there exists a manifold F called the fiber of π and an open cover of M along with diffeomorphisms ψ_v .

$$\psi_v: \pi^{-1}(U) \rightarrow U \times F$$

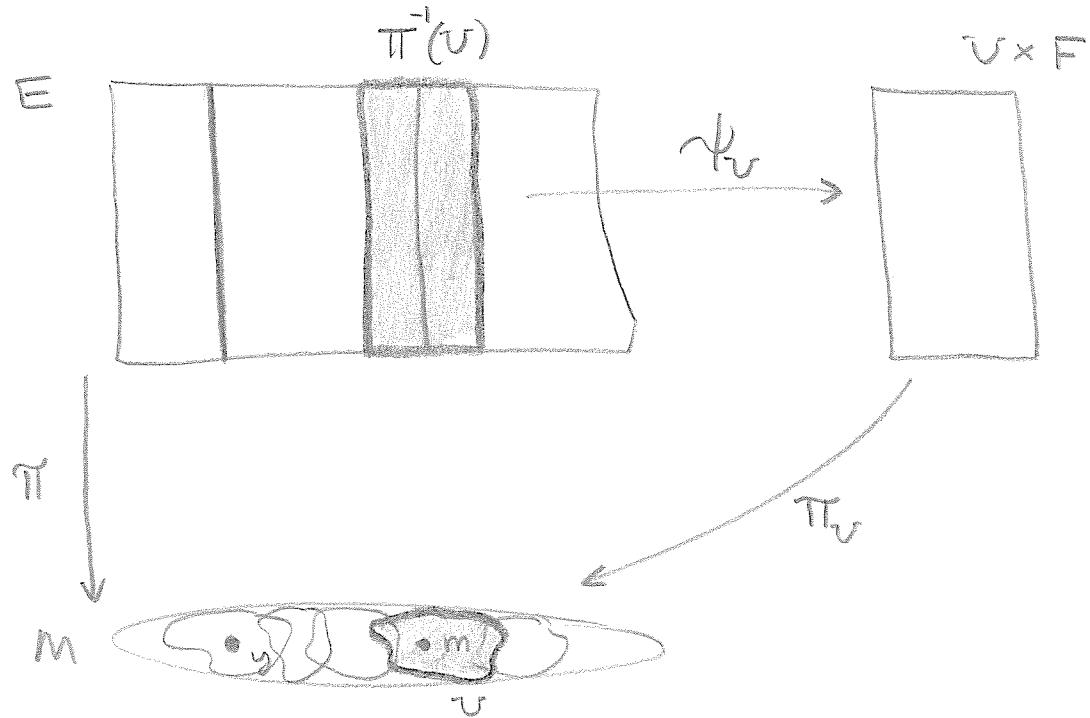
subject to the condition

$$\pi_v(\psi_v(y)) = \pi(y)$$

where $\pi_v: U \times F \rightarrow U$ is a projection.

Remark: Some people refer to the fiber bundle as (E, M, π, F) but really π necessitates these objects anyway as described above.

Pictorially



Notice for $m \in U$.

$$\psi_v(\pi^{-1}(m)) = \{m\} \times F \Rightarrow \pi^{-1}(m) \text{ diffeomorphic to } F$$

PF

$$y \in \pi^{-1}(m) \Rightarrow \pi(y) = m \Rightarrow \pi_v(\psi_v(y)) = m$$

Since π of that is $m \Rightarrow \psi_v(y) = (m, *)$ but $*$ must be in F

$$\text{Hence } y \in \pi^{-1}(m) \Rightarrow y \in \{m\} \times F \therefore \psi_v(\pi^{-1}(m)) \subseteq \{m\} \times F$$

Remark: there is a π for each U , we call these maps local trivialisations of the fiber bundle.

Additionally:

E = Bundle Space

M = Base Space

F = Fiber

Remark: For G/H we showed \exists local sections, ...

Defⁿ: If $\pi: E \rightarrow M$ is a fiber bundle then s is a local section if s is smooth map from $U \subseteq M$ into E such that

$$\pi \circ s = \text{id}_U$$

a local section is said to be global if $U = M$

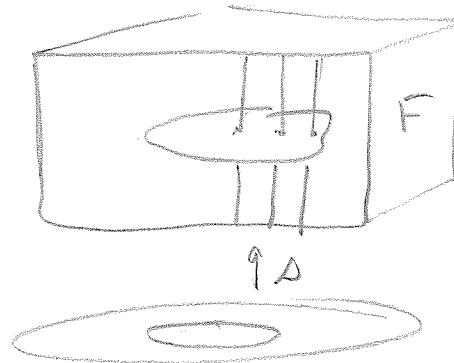
Discussion:

$$x \in U$$

$$\pi(s(x)) = \text{id}_U(x) = x$$

$$\pi(s(x)) = x$$

$$s(x) \in \pi^{-1}(x)$$



Claim: s maps only 1 point to $\pi^{-1}(x)$ if you take a y then $\pi^{-1}(y) \neq \pi^{-1}(x)$ due to this locally diff stuff.
 s picks one point of each fiber hence
 $s(U)$ is a submanifold which intersects each fiber at only one point.

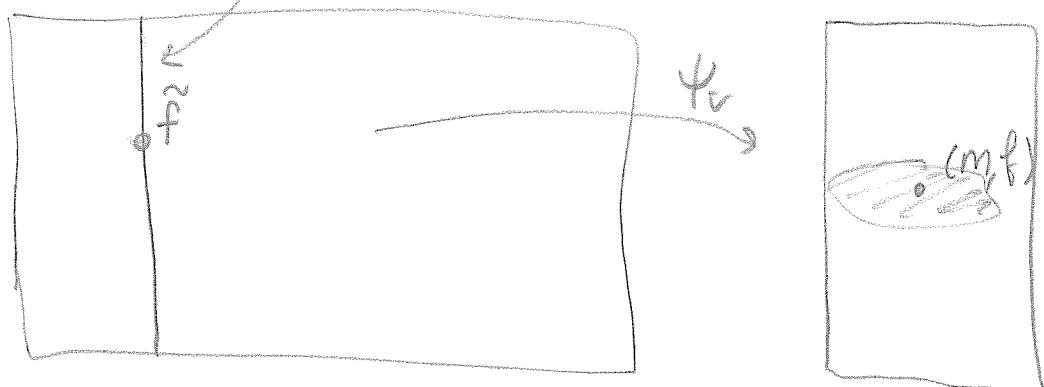
Let $m \in M$ and $\tilde{f} \in \pi^{-1}(m)$, $m \in U$
 for some $U \in \mathcal{U}$ ← open cover.

$$\psi_U(\tilde{f}) = (m, f) \in \{m\} \times F \quad (f \in F)$$

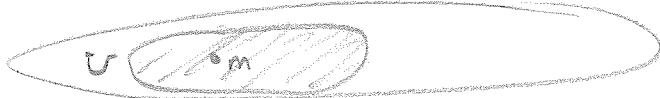
Then define $s : U \rightarrow \pi^{-1}(U) \subseteq E$ by

$$s(x) = \psi_U^{-1}(x, f)$$

$$\psi_U(\tilde{f}) = (m, f) \quad \text{😊}$$



$$\pi_U(u, f) = u$$



$$\pi(s(x)) = \pi(\psi_U^{-1}(x, f))$$

$$\pi_U \circ \psi_U = \pi \Rightarrow \pi_U = \pi \circ \psi_U^{-1}$$

$$\Rightarrow \pi(s(x)) = \pi_U(x, f) = x$$

thus it a section, more over

$$s(m) = \psi_U^{-1}(m, f) = \tilde{f} \quad \text{by 😊}$$

Can run a section through any point in the bundle space.
 Trouble is to find a global section!
Locally

Proved: A bundle with local trivialization map
it's true there exist local sections everywhere.

Conversely: We need the bundle to be Principal in order
for the existence of local sections $\Rightarrow \exists$ local trivializations.

Example on page 3

$$TM \xrightarrow{\pi} M$$



(U, x)

$$\pi^{-1}(U) = TU = \{ (m, v) \mid m \in U, v \in T_m M \}$$

$$dx : TU \rightarrow X(U) \times \mathbb{R}^n$$

$$dx(m, v) = (x(m), d_m x(v) n_i) \quad n_i = \text{standard basis} \\ (0, 0, \dots, 1, \dots 0)$$

Then (TU, dx) forms an atlas of the tangent bundle, but the local trivializing maps are a bit tricky

$$\phi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$$

$$\phi_U = (\bar{x}^i \times id_{\mathbb{R}^n}) \circ dx$$

Same as dx just drops the x .

$$\phi_U(m, v) = (m, d_m x(v) n_i)$$

Remark / $\pi: T^*M \rightarrow M$ forms a fiber bundle

$$\textcircled{1}, \quad \pi(\alpha_p) = p$$

If M is a m -dim'l it is a fiber-bundle with fiber $(\mathbb{R}^n)^*$

If (U, α) is a chart on M define

$$\psi_U: \pi^{-1}(U) \rightarrow U \times (\mathbb{R}^n)^*$$

$$\psi_U(p, \alpha_p) = (p, \alpha_p(\frac{\partial}{\partial x^i}|_p)) r^i \quad \text{where } r^i(r_j) = \delta_j^i$$

dual-basis on \mathbb{R}^n

You can check to see that it's smooth.

Charts of T^*M are of the form $(\pi^{-1}(U), p_x)$

where (U, α) is chart of M and

$$p_x(p, \alpha_p) = (\alpha(p), \alpha_p(\frac{\partial}{\partial x^i}|_p)) r^i$$

$$p_x: \pi^{-1}(U) \rightarrow \alpha(U) \times (\mathbb{R}^m)^*$$

but we ought not identify the trivializing space and the chart space right, or not

Local Coord. rep of ψ_U ; $(y \times \text{id}) \circ \psi_U \circ p_x^{-1}$ not so different than manifolds...

Ex. 3) The k^{th} Exterior Product of Cotangent bundle.

$$\Lambda^k M = \{(m, \alpha_m) \mid m \in M, \alpha_m \in \underbrace{\Lambda^k(T_m^*M)}_{\substack{\text{real linear span of} \\ \{\alpha_m^{i_1} \wedge \alpha_m^{i_2} \wedge \dots \wedge \alpha_m^{i_k}\}}}\}$$

$$\Lambda^k(T^*M) = \{\alpha : T_m M \times T_m M \times \dots \times T_m M \rightarrow \mathbb{R} \mid \begin{array}{l} \alpha \text{ is multilinear} \\ \alpha \text{ is skew} \end{array}\}$$

$$\pi : \Lambda^k M \rightarrow M$$

$$\pi(m, \alpha_m) = m$$

Charts pick a chart (U, x) of M

define then $\varphi_x : \pi^{-1}(U) \rightarrow X(U) \times \underbrace{\Lambda^k(\mathbb{R}^n)^*}_{\subset \mathbb{R}^n}$

$$\text{in fact } X(U) \times \Lambda^k(\mathbb{R}^n)^* \subseteq \mathbb{R}^n \times \Lambda^k(\mathbb{R}^n)^*$$

$$\varphi_x(m, \alpha_m) = (x(m), \alpha_m \left(\frac{\partial}{\partial x^{i_1}}|_m, \frac{\partial}{\partial x^{i_2}}|_m, \dots, \frac{\partial}{\partial x^{i_k}}|_m \right) (\alpha^{i_1 i_2 \dots i_k}))$$

This would define the chart. We should demonstrate

$$A_n = \{(\pi^{-1}(U), \varphi_x) \mid (U, x) \in \mathcal{A}_n\}$$

is in fact composed of compatible charts. Trivializing map $\pi_U : \pi^{-1}(U) \rightarrow U \times \Lambda^k(\mathbb{R}^n)^*$

$$\pi_U(m, \alpha_m) = (m, \alpha_m(\partial_{i_1}, \partial_{i_2}, \dots, \partial_{i_k}) (\alpha^{i_1 i_2 \dots i_k}))$$

We could do the same for tensor bundles.

$$\pi^{-1}(m) \cong \Lambda^k(\mathbb{R}^n)^* \leftarrow \text{fiber}$$

(4) The Frame Bundle : $\mathcal{F}M$

10/4/04 (3)

tetrad field is a global section only
non-compact manifold has spin iff its parallelizable.

$\text{Def}^b / \mathcal{F}M = \{(m, \{e_i\}) \mid m \in M \text{ and } \{e_i\} \text{ is a basis of } T_m M\}$

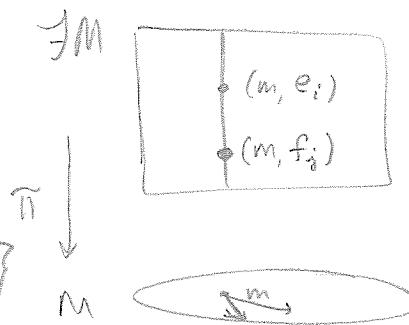
It's a point along with a frame. Given two frames we could find a non-singular transformation between the bases. We should identify the manifold which is the fiber; it's $GL(n)$ which is of course a manifold...

Remark: Up to now the fibers were vector spaces, now the fiber is more interesting.

$$\pi : \mathcal{F}M \rightarrow M$$

$$\pi(m, \{e_i\}) = m$$

$$\pi^{-1}(m) = \{(e_i) \mid \{e_i\} \text{ is basis of } T_m M\}$$



If $(m, \{e_i\})$ and $(m, \{f_i\})$ are frames at m

then $\exists! A \in GL(n)$, upper index row, lower index column

$$f_j = A_{ij} e_i \quad (\text{good convention for frame bundles})$$

Consider them assuming (U, x) is a chart on M then

$$(\mathcal{F}x)(m, \{e_i\}) = (x(m), [d_m x^i(e_i)])$$

Means the matrix; $A_{ij} = d_m x^i(e_i)$. And

$$\pi^{-1}(U) = \{(m, \{e_i\}) \in \mathcal{F}M \mid m \in U\} \subseteq \mathcal{F}M$$

Is it clear that A_{ij} was a non-singular matrix well yes thus $(x(m), [d_m x^i(e_i)]) \in X(U) \times GL(n) \subseteq \mathbb{R}^n \times gl(n)$

open dense subset of

$$\psi_U(m, \{e_i\}) = (m, [d_m x^i(e_i)]) = ((x^{-1} \times id_{GL(n)}) \circ \mathcal{F}x)(m, [d_m x^i(e_i)])$$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi_U} & U \times GL(n) \\ \mathcal{F}x \downarrow & & x \times id_{GL(n)} \\ X(U) \times GL(n) & \xrightarrow{id} & X(U) \times GL(n) \end{array}$$

Well the identity is what we need to check for smoothness. good.

T_h^M/M is a manifold then g is a metric on M if

$g_p: T_p M \times T_p M \rightarrow \mathbb{R}$ which is

- 1.) bilinear
- 2.) symmetric
- 3.) nondegenerate

Notice that $\hat{v}(x) = g_p(V, X)$ so \hat{v} is a covector at p and then $\psi(v) = \hat{v}$ provides a isomorphism of $T_p M \cong T_p^* M$

$$\psi: T_p M \rightarrow T_p^* M$$

Let $\{e_i\}$ be a basis of $T_p M$ then $G_{ij} = g_p(e_i, e_j)$ and $G \in \mathcal{M}(n = \dim(T_p M))$. More than that symmetry of $G = G^T$ says that $\exists f_i$ such that

$$g_p(f_i, f_j) = D_{ij} = \text{Diag}(d_1, d_2, \dots, d_n)$$

Then let $\hat{f}_i = \frac{1}{\sqrt{d_i}} f_i \Rightarrow g_p(\hat{f}_i, \hat{f}_j) = \text{Diag}(1, 1, \dots, 1, -1, \dots, -1)$

at each $p \in M$ you can choose a basis that forces the matrix of g_p to have this form of $\text{Diag}(1, 1, 1, \dots, 1, -1, \dots, -1)$ (zero off-diagonal).

* We can do this at a point, not in a region unless the manifold is flat. For if the metric is constant the Γ_{ij}^k 's are zero and the curvature vanishes!! rather we can't do it globally

* However we can say that the index which is the number of +1's and -1's is constant.

Th^m 1.1 on page 6
 $m_0 \in M$

10/6/04

(2)

$m_0 \in U$, x_1, x_2, \dots, x_n

$$g_m(x_i(m), x_j(m)) = G_{ij} \quad \forall m \in U$$

$$\text{These vary so } g_m(a_i^l \partial_{x_i}, a_j^l \partial_{x_j}) = G_{ij} \rightarrow g_m = (a^l)^T G a^l \text{ not constant}$$

Proof of 2.1 is proof of 1.1 and it takes 7-9, probably from Curtis & Miller. For positive def' metric you can just do Gram-Schmidt on $\frac{\partial}{\partial x_i} \rightarrow a_i^l \frac{\partial}{\partial y^l}$ are smooth

Hawking & Ellis do something for Minkowski which is also nice. [I should read that book!]

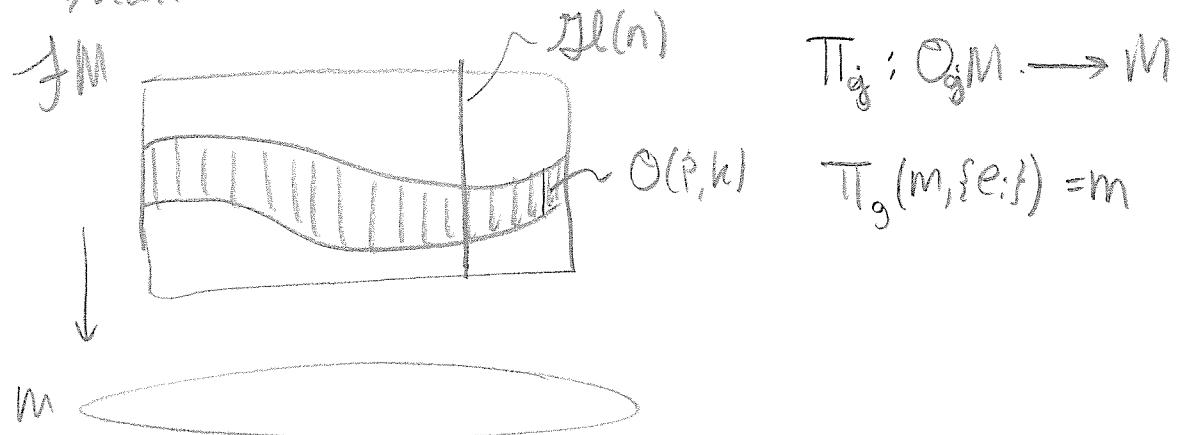
$$\mathcal{O}(p, k) = \{A \in \mathcal{M}(n) / A^T G A = G\}$$

$$G = \begin{pmatrix} I_p & 0 \\ 0 & -I_{k-p} \end{pmatrix}$$

$$\mathcal{O}_g(m) = \{(m, \{e_i\}) \in \mathcal{FM} / g_m(e_i, e_j) = G_{ij}\}$$

all those basis which diagonalize g_m

Claim: this is another fiber bundle with a smaller fiber than \mathcal{FM}



(3)

By Th² 1.1 there exists an open cover \mathcal{U} of M such that on each $U \in \mathcal{U} \exists$

x_1, x_2, \dots, x_n

on U such that $\{x_i(m)\}$ is a basis of $T_m M \forall m \in U$.

We can construct the local trivialization mapping $\chi_U : \pi_g^{-1}(U) \rightarrow U \times O(p, k)$

$$\chi_U(m, \{e_i\}) = (m, \chi)$$

Where χ is defined as follows

$$\chi_i^j = \alpha^i(e_j)$$

Where $\alpha^1, \alpha^2, \dots, \alpha^n$ are differential 1-forms on U such that for each j

$$\alpha_m^i(\chi_j(m)) = \delta_j^i$$

I can drop the m and see this is a well-defined family of differential forms.

$$\alpha_m^i(\sum a_j^i(m) \chi_j(m)) = a^i(m)$$

Well then $\alpha^i(e_j) = \chi_j^i$ so

$$\begin{aligned} G_{ij} &= g_m(e_i, e_j) = g_m(\chi_i^k x_k, \chi_j^l x_l) \\ &= \chi_i^k \chi_j^l g_m(x_k, x_l) \\ &= (\chi^k G \chi)_i^j \end{aligned}$$

$$\therefore \boxed{\chi^k G \chi = G \Leftrightarrow \chi \in O(p, k)}$$

$$\text{Gap } e_i = a_i^k x_k \Rightarrow \alpha^i(e_i) = a_i^k \alpha^k(x_k) = a_i^k = \chi_i^k$$

Last time cleaned up a bit, logically better ordered

10/11/04 ①

$$Q_g(M) = \{ (P, \{e_i\}) \mid g_p(e_i, e_j) = G_{ij} \} \text{ for fixed } G = \begin{pmatrix} I_p & C \\ C^T & I_k \end{pmatrix}$$

Choose an open cover \mathcal{U} of M such that $\forall U \in \mathcal{U}$
 \exists vector fields X_i on U with,

$$g_p(X_i(p), X_j(p)) = G_{ij}$$

Define α^k differential forms on U with $\alpha^k(X_i) = \delta_i^k$.

$$\text{Define } \gamma_U(P, \{e_i\}) = (P, (\alpha^k(e_i))) \quad P \in U, (P, \{e_i\}) \in Q_g M$$

Notice that if $e_i = b_i^k X_k$ then $\alpha^j(e_i) = b_i^k \alpha^j(X_k) = b_i^j$.

Thus if $\lambda = (\alpha^k(X_i))$ then

$$e_i = \lambda_i^k X_k$$

$$e_j = \lambda_j^l X_l$$

$$G_{ij} = g(e_i, e_j) = \lambda_i^k \lambda_j^l g(X_k, X_l)$$

$$= \lambda_i^k \lambda_j^l G_{kl}$$

$$\therefore \boxed{G = \lambda^t G \lambda} \quad (*)$$

And then $\lambda \in \mathrm{GL}(n)$ well even more $\lambda \in O(p, k)$. by *.

Remark: we're mapping into $O(p, k)$ so no global chart exists, see page 7.

Wlog we can assume $V \in \mathcal{U}$ is a chart domain

$$\psi_v : \pi_v^{-1}(V) \rightarrow V \times \mathcal{O}(p, k)$$

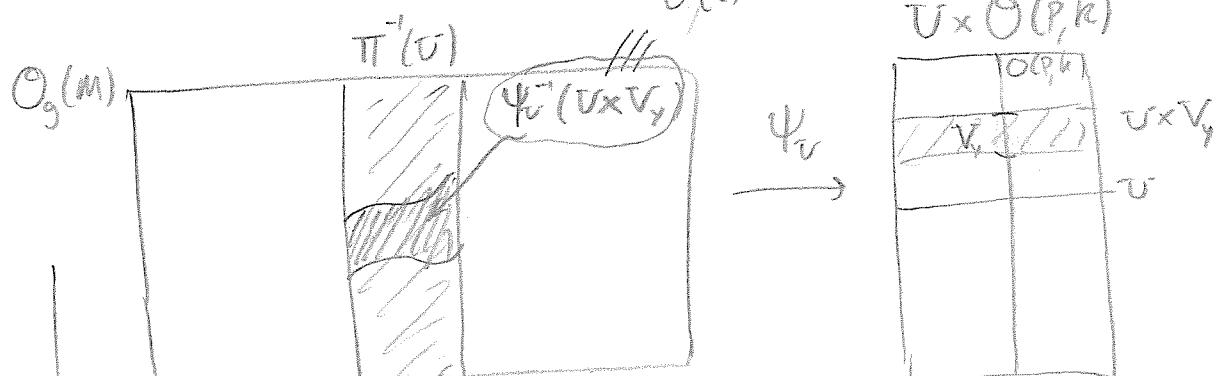
Let $\mathcal{Q}(p, k)$ be an atlas of $\mathcal{O}(p, k)$, recall that we proved $\mathcal{O}(p, k)$ was a submanifold of $GL(n)$, now $\forall V \in \mathcal{U}$ and x a chart of M on V , and each chart $(V_y, y) \in \mathcal{Q}(p, k)$ now let

$$U(y) = \psi_v^{-1}(V \times V_y)$$

and let

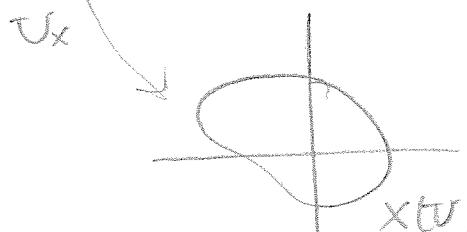
$$\eta_y : U(y) \rightarrow X(V) \times Y(V_y)$$

ψ_v^{-1} set as diffeomorphism.



then note that:

$$\eta_y = (X \times Y) \circ \psi_v$$



Note that $\eta_y^{-1} = \psi_v^{-1} \circ (X^{-1} \times Y^{-1})$ and clearly $\eta_y \circ \eta_y^{-1} = \text{id}$. To check compatibility just tilde allate everything and check $\tilde{\eta}_y \circ \eta_j^{-1}$ is smooth ...

Compatibility of charts in $\mathcal{O}(P, k)$ shouldn't be too difficult. It works because $x \& y$ are diffeomorphisms and we're forcing ψ to be ...

$$\begin{array}{ccc} U_y & \xrightarrow{\psi_y} & U \times V_y \\ \pi_y \downarrow & & \downarrow x \times y \\ x(U) \times y(V_y) & \xrightarrow{id} & x(U) \times y(V_y) \end{array}$$

Chicken or Egg. Anyways it's consistent.

Problem's: Focus on local trivializing maps, Ex. 1.2 and Ex. 1.3 to show if admissible coordinate system, need to use... Change the y to p. should read $p \in \underline{\quad}$ not $y \in \underline{\quad}$. Then Ex. 1.4 on page 14. (Part of Midterm)

Defⁿ

(1) $\exists \pi: M \rightarrow \mathbb{R}^n$

Defⁿ A fiber bundle $\pi: E \rightarrow M$ with fiber V is a vector bundle iff

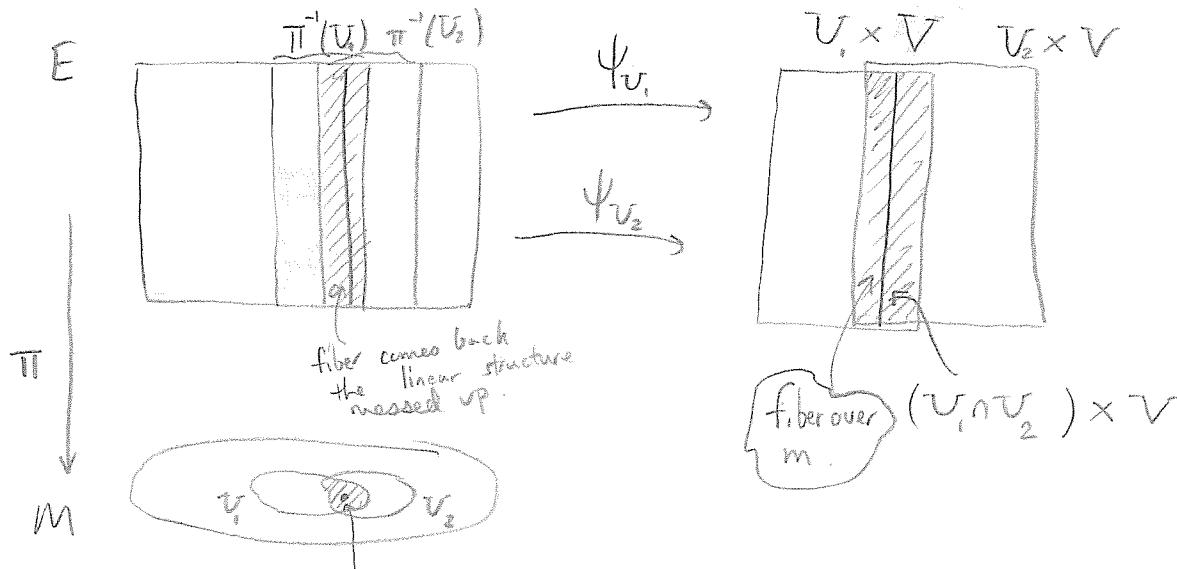
(1.) V is a vector space

(2.) \exists local trivializing maps $\psi_v: \pi^{-1}(U) \rightarrow U \times V$

$U \in \mathcal{U}$ such that $U_1, U_2 \in \mathcal{U}$ and $U_1 \cap U_2 \neq \emptyset$ then V $m \in U_1 \cap U_2$ the map from V to V defined by

$$x \mapsto \pi_V(\psi_{U_2}(\psi_{U_1}^{-1}(m, x)))$$

is a vector space isomorphism.



Well if $\psi_{U_1}^{-1}$ and ψ_{U_2} are such that their composition as above produces a linear transformation, well \exists such a ψ_{U_1} and $\psi_{U_2}^{-1}$ is really the question, and the answer is yes for the vector bundles.

Remark: $+_p: \pi^{-1}(p) \times \pi^{-1}(p) \rightarrow \pi^{-1}(p)$ is continuous. Addition pointwise.

Exercise 1.2: Modify it to the following:

① Show T^*M is a vector bundle, find charts etc...

② Show that if g is a metric on M , then the vector bundles TM and T^*M are vector bundle isomorphic.

* $\Rightarrow p \in M$, $g_p: T_p M \times T_p M \rightarrow \mathbb{R}$ with g_p a symmetric, nondegenerate map meaning $v \in T_p M$ we have $\hat{v}(x) = g_p(x, v)$ is then an isomorphism $v \mapsto \hat{v}$.

$$\int_0^\infty e^{+ikx} dx = \pi S(k)$$

Definition 1.5 A fiber bundle $\pi : E \rightarrow M$ is called a vector bundle iff

1. the fiber of the bundle is a vector space V ,
2. there is a family of local trivializing mappings $\psi_U : \pi^{-1}(U) \rightarrow U \times V$, $U \in \mathcal{U}$ such that if $U_1, U_2 \in \mathcal{U}$ and $U_1 \cap U_2 \neq \emptyset$, then for each $m \in U_1 \cap U_2$ the mapping from V to V defined by

$$x \mapsto \pi_V(\psi_{U_2}(\psi_{U_1}^{-1}(m, x)))$$

is a vector space isomorphism.

Observe that in this case there exist well-defined continuous operations $+$ and \cdot on each fiber $\pi^{-1}(m)$, $m \in M$. These operations are defined by

$$\begin{aligned} v + w &= \psi_U^{-1}(m, \pi_V(\psi_U(v)) + \pi_V(\psi_U(w))) \\ cv &= \psi_U^{-1}(m, c \cdot \pi_V(\psi_U(v))). \end{aligned}$$

Exercise 1.2 Show that $TM, T^*M, \Lambda^k M$ are vector bundles.

modified

Exercise 1.3 (a) Show that T^*M is a vector bundle. This involves the following steps: exhibit the charts of T^*M explicitly, the local trivializing mappings explicitly, and show that the local trivializing mappings have the correct linearity properties. ← not to long

(b) Assume the existence of a metric on M . Recall that a metric has the property that if $v \in T_p M$, then the mapping $v \rightarrow \hat{v}$ is injective where $\hat{v} \in T_p^* M$ is defined by $\hat{v}(x) = g_p(x, v)$. Show that TM is isomorphic to T^*M as a vector bundle. The definition of vector bundle isomorphism is given below.

Definition 1.6 Two vector bundles (E_1, M_1, π_1) and (E_2, M_2, π_2) are vector bundle isomorphic iff there exists a fiber bundle isomorphism (Φ, ϕ) from π_1 to π_2 such that for each $m \in M$ the restriction of Φ to $\pi_1^{-1}(m)$ is a vector space isomorphism from $\pi_1^{-1}(m)$ onto $\pi_2^{-1}(\phi(m))$.

Exercise 1.4 If $\pi : E \rightarrow M$ is a fiber bundle and $p \in E$ then there is an adapted coordinate system at p .

Note that if $u \in E$ and $w \in T_u E$ such that $d_u \pi(w) = 0$ then

$$w = \sum_{a=1}^N w^a \left(\frac{\partial}{\partial y^a} \Big|_w \right).$$

Indeed, in general, $w = \sum_{\mu=1}^n w^\mu \left(\frac{\partial}{\partial x^\mu} \Big|_w \right) + \sum_{a=1}^N w^a \left(\frac{\partial}{\partial y^a} \Big|_w \right)$. But $d_u x^\mu(w) = d_u \bar{x}^\mu(d_u \pi(w)) = 0$ and also

$$d_u x^\mu(w) = dx^\mu \left(\sum_\nu w^\nu \left(\frac{\partial}{\partial x^\nu} \Big|_w \right) + \sum_a w^a \left(\frac{\partial}{\partial y^a} \Big|_w \right) \right) = w^\mu.$$

Thus $w^\mu = 0$ for $1 \leq \mu \leq n$ and

$$w = \sum_{a=1}^N w^a \left(\frac{\partial}{\partial y^a} \Big|_w \right)$$

as asserted.

Definition 1.8 If $\pi : E \rightarrow M$ is a fiber bundle then a tangent vector $w \in T_u E$ at $u \in E$ is vertical iff $d_u \pi(w) = 0$. A curve $\gamma : I \rightarrow E$ in E is vertical iff $\gamma'(t) \in T_{\gamma(t)} E$ is vertical for all $t \in I$.

Exercise 1.5 Show that a curve $\gamma : I \rightarrow E$ is vertical iff the image of γ lies in a single fiber of E .

Definition 1.9 If $\pi : E \rightarrow M$ is a fiber bundle and $y_0 \in E$ then $J_{y_0} E$ denotes the set of all linear mappings $\gamma : T_{\pi(y_0)} M \rightarrow T_{y_0} E$ such that

$$d_{y_0} \pi \circ \gamma = \text{id}_{T_{\pi(y_0)} M}.$$

If $JE = \{(y, \gamma) \mid y \in E \text{ and } \gamma \in J_y E\}$ then we will show that the mapping $\pi_E : JE \rightarrow E$ defined by $\pi_E(y, \gamma) = y$ defines a fiber bundle structure. This fiber bundle is called the first order jet bundle of the fiber bundle $\pi : E \rightarrow M$.

Take Home Test:

10/13/04

①

$$\text{SU}(2) \longrightarrow \text{O}(3)$$

$$\varphi: \text{SU}(2) \rightarrow \text{O}(3)$$

$$\varphi(A): \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\varphi(A)(e_i) = (\varphi(A))_i^j e_j$$

$$\tilde{\varphi}(A)(\hat{e}_i) = \tilde{\varphi}(A)_i^j \hat{e}_j$$

$\tilde{\varphi}(A)_i^j$ and $\varphi(A)_i^j$ are in principle different but we can show they're the same
 Want to argue $A \xrightarrow{\varphi} \det(\varphi(A)_i^j)$
 is continuous.

$$\varphi(A)^T \varphi(A) = I$$

$$\det(\varphi(A)^2) = 1 \therefore \det \varphi(A) = \pm 1$$

$$\psi: \text{SU}(2) \rightarrow \{1, -1\}$$

$$A \xrightarrow{\psi} \det(\tilde{\varphi}(A)_i^j)$$

$$\tilde{\varphi}(A)(\hat{e}_i) = A \hat{e}_i A^{-1} \text{ by def' of } \tilde{\varphi}$$

Take a seq. of matrices in $\text{SU}(2)$, $A_n \rightarrow A_0$
 same top. as $\text{gl}(n)$, and we know mat. mult.
 & mat. inv. is continuous here

$$A_n \hat{e}_i A_n^{-1} \rightarrow A_0 \hat{e}_i A_0^{-1}$$

$$\tilde{\varphi}(A_n)(\hat{e}_i) \rightarrow \tilde{\varphi}(A_0)(\hat{e}_i)$$

$$\tilde{\varphi}(A_n)_i^j \rightarrow \tilde{\varphi}(A_0)_i^j$$

$$\tilde{\varphi}: \text{SU}(2) \rightarrow \text{End}(\mathbb{R}^3)$$

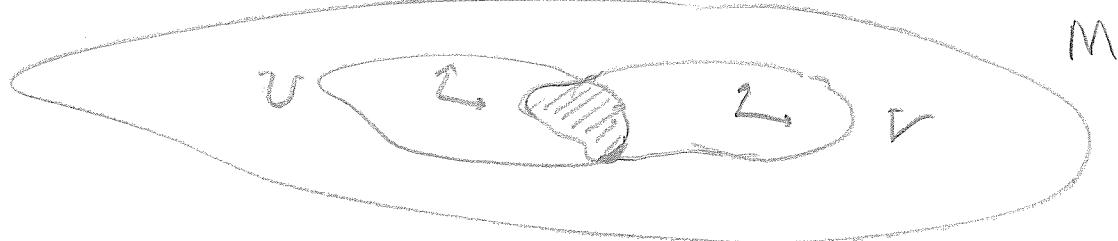
$$\psi_v : \pi_g^{-1}(U) \rightarrow U \times O(p, k)$$

$$\psi_v : \pi_g^{-1}(V) \rightarrow V \times O(p, k)$$

It's always possible to cover M with covering such that
 $\exists \dim(m)$ linearly independent vector fields on U

$$g_m(\chi_i(m), \chi_j(m)) = G_{ij} \quad \text{for } m \in U$$

$$g_p(\bar{\chi}_i(m), \bar{\chi}_j(m)) = G_{ij} \quad \text{for } p \in V$$



$$\psi_v(m, \lambda_i) = (m, \lambda) \quad \text{for } \lambda \in O(p, k)$$

$$e_i = \lambda_i^* \chi_j(m)$$

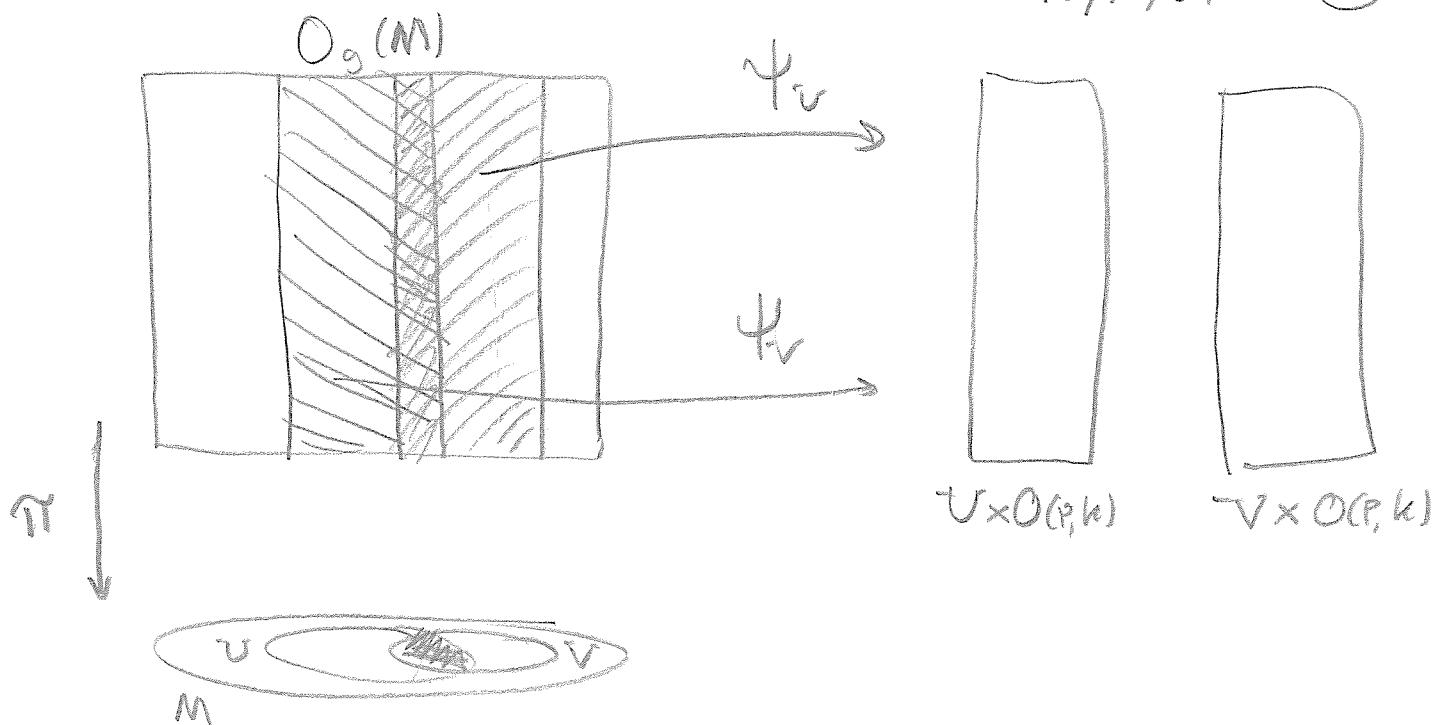
last time showed that
 λ existed exc...

$$(\psi_v \circ \psi_v^{-1})(m, \lambda) = \psi_v(m, \lambda_i^* \chi_j(m))$$

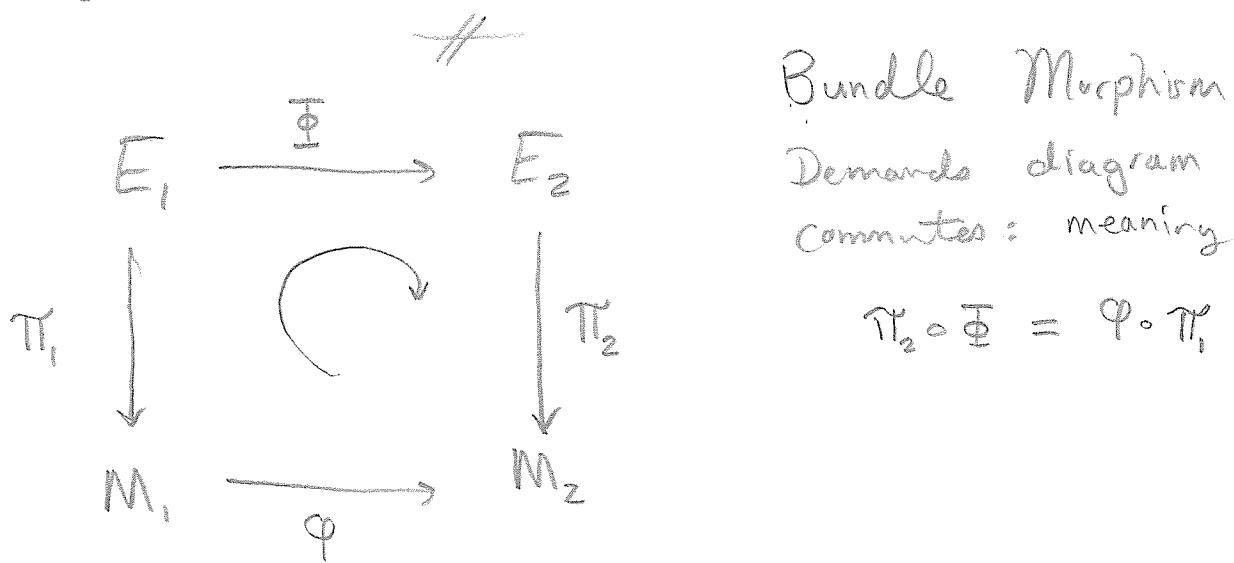
$$\chi_j(m) = \mu_j^k(m) \bar{\chi}_k(m) \quad \begin{matrix} \text{smoothness of } \chi \text{ and } \bar{\chi} \\ \text{demands that } \mu \text{ smooth} \end{matrix}$$

$$\begin{aligned} (\psi_v \circ \psi_v^{-1})(m, \lambda) &= \psi_v(m, \lambda_i^* \mu_j^k(m) \bar{\chi}_k(m)) \quad \text{true } \forall i \\ &= (m, \lambda_i^* \mu_j^k(m)) \\ &= (m, \lambda \mu(m)) \end{aligned} \quad (AB)_l^k = A_k^k B_l^k$$

So clearly $\psi_v \circ \psi_v^{-1}$ is smooth as $\lambda \mapsto \lambda \mu(m)$ is smooth.
 When checking compatibility of charts we needed this is smooth //



the trivializing maps form a bundle atlas in some sense.



$$\Phi(\pi_1^{-1}(m_1)) \subseteq \pi_2^{-1}(\varphi(m_1))$$

A bundle isomorphism has φ and Φ invertible so the diagram commutes the other direction.

- Gauge Transformations are Bundle Automorphisms.

Example : Newtonian Space time

N : featureless Newtonian Space

α_N an atlas on N

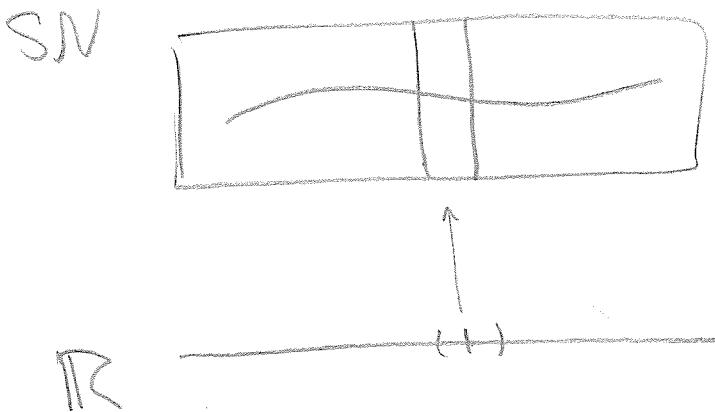
$x \in \alpha \quad x : N \rightarrow \mathbb{R}^3$

For Newtonian Space $x, y \in \alpha$

a.) $\varphi = y \circ x^{-1} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a rigid motion.

b.) $y = \varphi \circ x$ is also a rigid motion

Find them a trivial fibre bundle to model
newtonian mechanics is $SN = \mathbb{R} \times N$



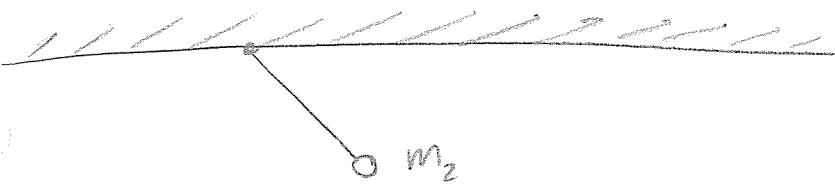
$$\gamma(t) = (t, \gamma_N(t))$$

Newtonian Spacetime is a fibre bundle over the real,
while Minkowski Space time has no such simple way
to write it as a fiber bundle. Question
how to write Mink. space as fibre bundle.

Configuration Manifold

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(5)



$$\mathbb{R} \times S^1 = Q$$

$$\mathbb{R} \times TQ \xrightarrow{L} \mathbb{R}$$

$$\begin{array}{ccc} \pi & \downarrow & s \\ \mathbb{R} & & \gamma(t) = (t, \gamma_Q(t)) \end{array}$$

Take $\Lambda^1 M$

$$\downarrow \\ M$$

~~A~~

$$\Lambda^1 M = \{ (m, \alpha) \mid \alpha \in T_m^* M \} \quad \text{a particular covector}$$

$$A(x) = A_\mu(x) dx^\mu \quad \text{diff form on Minkowski}$$

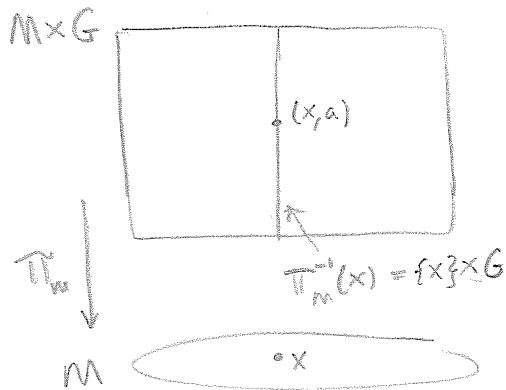
$$\Lambda^1 M \cong M \times (\mathbb{R}^m)^* \quad \text{not a very natural const.}$$

Def^b/ Let M be a manifold and G a Lie Group. The projection $\pi_m: M \times G \rightarrow M$ is a fiber bundle with fiber G . Define an right action of G on $M \times G$ by

$$(x, a) \cdot g = (x, ag)$$

Remark this is a free action and the orbit is

$$(x, a) \cdot G = \{(x, a) \cdot g \mid g \in G\} = \{(x, ag) \mid g \in G\} = \{x\} \times G = \pi_m^{-1}(x)$$



Well any fiber bundle obtained in this way is called a trivial principal G -bundle.

Def^b/ Assume $\pi: P \rightarrow M$ is a fiber bundle with fiber G . We say $\pi: P \rightarrow M$ is a principal G -bundle iff

- (1.) \exists a free right action of G on P such that $P \times G \rightarrow P$
- (2.) \exists a local trivialization $\{\psi_v\}_{v \in U}$ of π such that

$$\psi_v(x \cdot g) = \psi_v(x) \cdot g$$

and ψ_v is a G -bundle isomorphism.

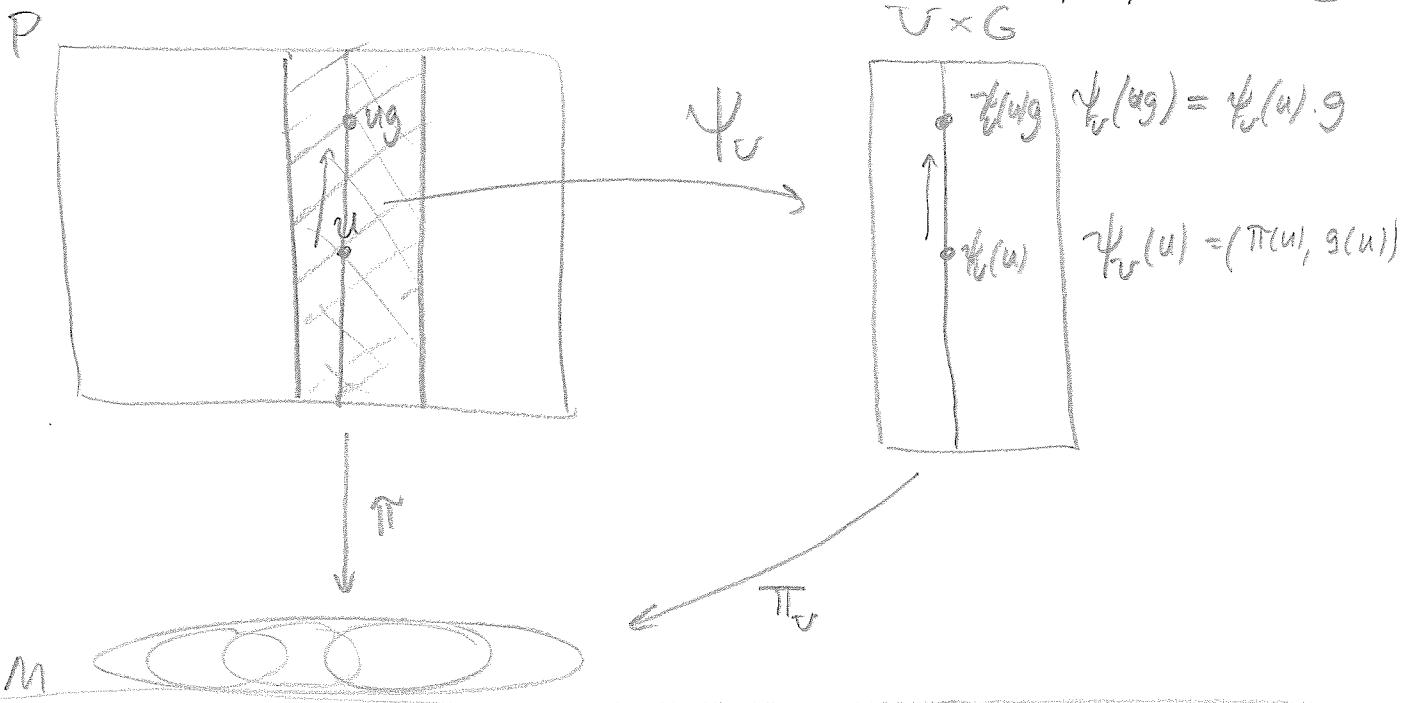
$U \cdot g \leftarrow$ right action on P

$$\psi_v: \pi^{-1}(U) \rightarrow U \times G$$

$$(u, h) \cdot g = (u, hg)$$

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(2)



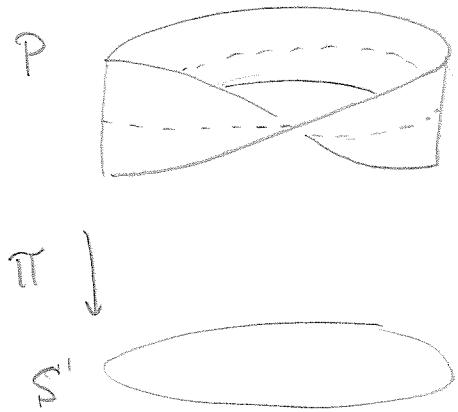
$$\begin{array}{ccc} \pi^{-1}(v) & \xrightarrow{\psi_v} & U \times G \\ \pi \downarrow & & \downarrow \pi_U \\ U & \xrightarrow{id} & U \end{array}$$

Bundle isomorphism makes this a commutative graph.
So $\pi^{-1}(v)$

Möbius Band

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(3)



$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi_U} & U \times G \\ \pi \downarrow & & \downarrow \pi_U \\ U & \longrightarrow & U \end{array}$$

The Spin Bundle is a Double cover of $SO(3)$ or it's the Double Cover of Lorentz Group. But does \exists a bundle with group $SL(2, \mathbb{C})$ then

$$SU(2) \times V \longrightarrow V$$

$$\downarrow$$

$$SO(3) \times V \longrightarrow V$$

$$SU(2) \xrightarrow{\rho} GL(V) \quad \text{or} \quad SO(3) \longrightarrow GL(V)$$

well if ρ can be factored into double cover map ... See Ryder.

Example 4

Let $\pi: E \rightarrow M$ be any vector bundle, with fiber V and let us denote the frame bundle.

$$\mathcal{F}E = \{(x, \{v_i\}) \mid x \in M, \{v_i\} \text{ is a basis of } E_x = \pi^{-1}(x)\}$$

If $n = \dim(E_x)$, independent of x , we can define a right action of $GL(n)$ on $\mathcal{F}E$ by $g \in GL(n)$;

$$(x, \{v_i\}) \cdot g = (x, \{v_j g_j^i\})$$

Notice that if $(x, \{v_i\}) \cdot g = (x, \{v_i\})$ yields

$$v_j g_j^i = v_j s_i^j$$

$$\therefore g_j^i = s_i^j \quad \therefore g = I$$

Hence the action is free.

Example: Vector Bundles

$\pi: E \rightarrow M$ is a vector bundle with fiber V

$$\mathcal{F}E = \{ (x, \{v_i\}) \mid x \in M, \{v_i\} \text{ an ordered basis of } \pi^{-1}(x) \}$$

$$n = \dim(V)$$

$\mathcal{F}E \times GL(n) \rightarrow \mathcal{F}E$ is the action given by

$$(x, \{v_i\}) \cdot g = (x, \{v_i g_i^j\})$$

This is a free action.

Let $\{\psi_v\}_{v \in \mathcal{U}}$ be a set of local trivializing of $\pi: E \rightarrow M$

$$\psi_v: \pi^{-1}(V) \rightarrow V \times V$$

$$\psi_v|_{\pi^{-1}(x)}: \pi^{-1}(x) \rightarrow \{x\} \times V$$

Let $\mathcal{F}V$ denote the set of all bases of V .

Define then

$$\mathcal{F}U = \{(x, \{v_i\}) \mid x \in U, \{v_i\} \text{ is a basis of } \pi^{-1}(x)\}$$

We want a map from $\mathcal{F}U \rightarrow U \times GL(n)$

$$\mathcal{F}U \rightarrow U \times \mathcal{F}V$$

Well two basis of V are related by an element of $GL(n)$.
 So if we choose a specific basis of V that is fix a specific
 basis $\{v_i^0\}$ of V and let $\{v_i^0\}$ be the dual basis for V^*
 (dual to $\{v_i^0\}$).

Continuing from ①

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②

Define then a mapping \hat{g} from $\mathcal{F}V$ to $GL(n)$ by

$$\hat{g}(\{v_i\}) = (v_i^i(v_j)) \quad (*)$$

Now \hat{g} is a bijection, for now the frames have no additional structure. Fine, we're trying to define the local trivialization of the frame bundle,

$$\tilde{\psi}_v : \mathcal{F}U \longrightarrow U \times GL(n)$$

$$\tilde{\psi}_v(x, \{v_i\}) = (x, \hat{g}\{\pi_v(\psi_v(v_i))\})$$

Notice that

$$\pi'(x) \subseteq \pi'(v)$$

$$v_i \in \pi'(x) \subseteq \pi'(v) \subseteq E$$

and recall

$$\pi_v : v \times v \rightarrow v$$

$$\pi_v(y, w) = w.$$

Knocks the x off.

Well is $\{\pi_v(\psi_v(v_i))\}$ a basis truly well yes

$$\{\pi_v(\psi_v(v_1)), \pi_v(\psi_v(v_2)), \dots, \pi_v(\psi_v(v_n))\}$$

then \hat{g} converts this to a matrix as $(*)$ describes

- We can give $\mathcal{F}V$ a diff structure if we wanted by requiring \hat{g} to be a diffeomorphism, but it doesn't matter...

We give $\mathcal{F}U$ a differentiable structure by requiring that $\tilde{\psi}_v$ be a diffeomorphism.

$$f : \mathcal{F}U \rightarrow \mathbb{R}$$

\downarrow ← fixed to be diffeomorphism

$$U \times GL(n)$$

Let $U_1, U_2 \in \mathcal{U}$ be such that $U_1 \cap U_2 \neq \emptyset$

We must show that $\tilde{\psi}_{U_2} \circ \tilde{\psi}_{U_1}^{-1} : (U_1 \cap U_2) \times GL(n) \rightarrow (U_1 \cap U_2) \times GL(n)$ is smooth.

Why is $\tilde{\psi}_{v_2} \circ \tilde{\psi}_v^{-1}: (U_i \cap U_j) \times GL(n) \rightarrow (U_i \cap U_j) \times GL(n)$ smooth? 10/18/04

Note that $\hat{g}^{-1}(a) = \{fa_i^j v_j^0\} \in \mathcal{F}V$ and $\tilde{\psi}_v: \pi^{-1}(U_i) \rightarrow U_i \times V$

$\{\tilde{\psi}_v^{-1}(x, fa_i^j v_j^0)\} \leftarrow$ basis of $\pi^{-1}(x)$.

$\tilde{\psi}_v^{-1}: U_i \times GL(n) \rightarrow \mathcal{F}U_i$

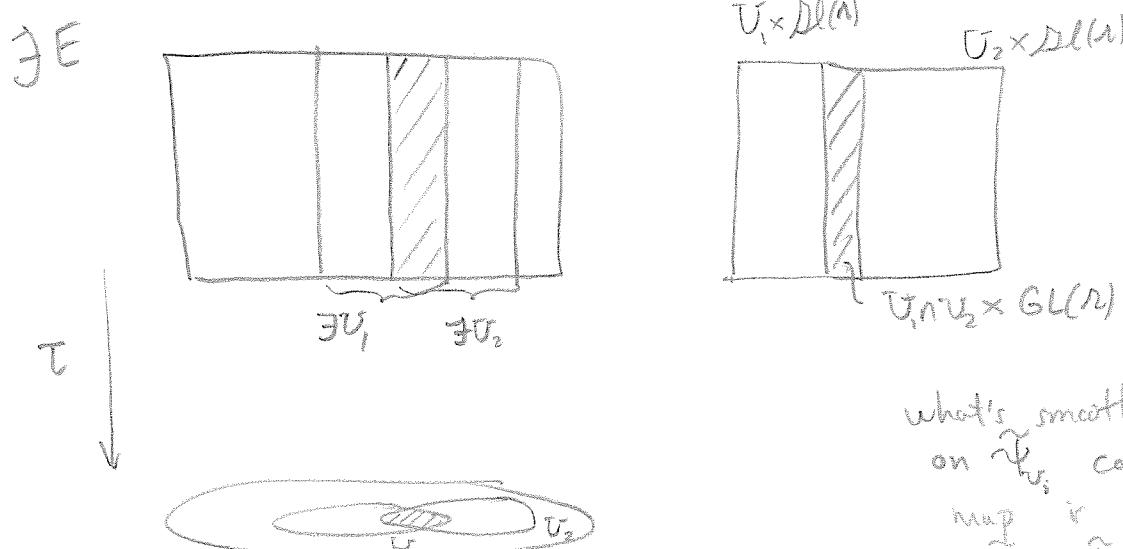
$\tilde{\psi}_v^{-1}(x, a) = (x, \tilde{\psi}_v^{-1}(x, a^i v_j^0))$

Check it by just taking $\tilde{\psi}_{v_2}$ of both sides. Fine then check,

$$\begin{aligned} (\tilde{\psi}_{v_2} \circ \tilde{\psi}_v^{-1})(x, a) &= \tilde{\psi}_{v_2}(x, \tilde{\psi}_v^{-1}(x, a^i v_j^0)) \\ &= (x, \hat{g}\{\underbrace{\pi_v(\tilde{\psi}_v(\tilde{\psi}_v^{-1}(x, a^i v_j^0)))}\}_{\substack{\text{set of L.I. vector} \\ \text{in } V}}) \in U_i \cap U_j \times GL(n) \end{aligned}$$

Composite of smooth maps is smooth!

We just constructed a "bundle atlas" which isn't quite an atlas but technically it wouldn't be hard to get.



what's smooth depends
on $\tilde{\psi}_v$, composed with
map is smooth, then
 $\tilde{\psi}_{v_1}, \tilde{\psi}_{v_2}$ better be
composable.

Consider in order to see the example is a Principal Bundles.

$$\begin{aligned}
 \hat{g}(\{v_i\} \cdot h) &= \hat{g}(\{v_i h_i\}) && \text{how } GL(1) \text{ acts} \\
 &= (v_0^k (v_j h_j))^k && \text{on } \mathcal{F}V \\
 &= (h_i^j v_0^k (v_j))^k && \text{this is a matrix } (A_{ij}^k) \\
 &= (v_0^k (v_j)) \circ (h_i^j) && \text{linearity of } v_0^k \\
 &= \hat{g}(\{v_i\}) \circ h &&
 \end{aligned}$$

this is an
equivariant map
can use this to
show action is free...

$$\hat{g} : \mathbb{F}V \rightarrow \mathrm{GL}(n)$$

$$\hat{g}(\{v_i\} \cdot h) = \hat{g}(\{v_i\})h \quad (*)$$

Metrics are equivariant
with respect to Adjoint.

$$\begin{array}{l} M \times G \rightarrow M \\ N \times G \rightarrow N \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{right actions}$$

$$f : M \rightarrow N$$

$$f(mg) = f(m)g$$

$$M \times G \rightarrow M$$

$$g * m = mg^{-1}$$

$$G \times N \rightarrow N$$

$$f(mg) = g^{-1} \cdot f(m) = f(m) * g$$

equivariant cohomology is modulo the group action
the orbits are points in the quotient space. A
fact. is equivariant & we can push it down to quotient.
• still needed to prove that

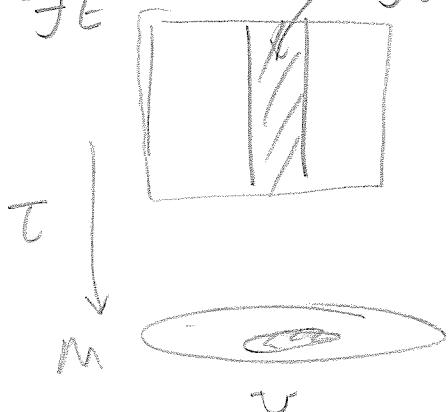
$$\tilde{\psi}_v : \mathbb{F}U \rightarrow U \times \mathrm{GL}(n)$$

must have transition facts that satisfy certain &
property in & to the fibres being groups.
it's not enough to just know fibre is group.

$$\tilde{\psi}_v((x, \{v_i\}) \cdot h) = \tilde{\psi}_v(x, \{v_i h_i^{-1}\})$$

defth of group
 $\mathrm{GL}(n)$ acts on
 $\mathbb{F}E$, it changes
basis.

$$\mathbb{F}E \quad \mathbb{F}U = T^*(U)$$



• need trans. facts
to be equivariant
w.r.t. to group
action.

$$\begin{aligned}
 \tilde{\psi}_v((x, \delta v_i) \cdot h) &= \tilde{\psi}(x, \delta v_i h_i^i) \\
 &= (x, \hat{g}(\{\pi_v \psi_v(v_i h_i^i)\}) \cdot h) \quad \text{def* of } \tilde{\psi}_v \text{ from last time.} \\
 &= (x, \hat{g}(\{\pi_v \psi_v(v_i)\} \cdot h)) \quad \text{if } \pi_v: \pi_v(v) \rightarrow U \times V \\
 &= (x, \hat{g}(\{\pi_v(\psi_v(v_i))\}) \cdot h) \quad \text{vect. space isomorphism} \\
 &= (x, \hat{g}(\{\pi_v(\psi_v(v_i))\}) \cdot h) \quad \pi'(x) \rightarrow \{x\} \times V \\
 &= \tilde{\psi}(x, \{v_i\}) \cdot h \quad \text{linear so scalars come out} \\
 &\quad \text{by * from ①} \\
 &= \tilde{\psi}(x, \{v_i\}) \cdot h \quad \text{group action on } U \times \mathrm{GL}(n)
 \end{aligned}$$

Hence $\tilde{\psi}_v$ is an equivariant with respect to the group action.

$$FU \xrightarrow{\tilde{\psi}_v} U \times \mathrm{GL}(n)$$

Hence it is indeed a principal fiber bundle.

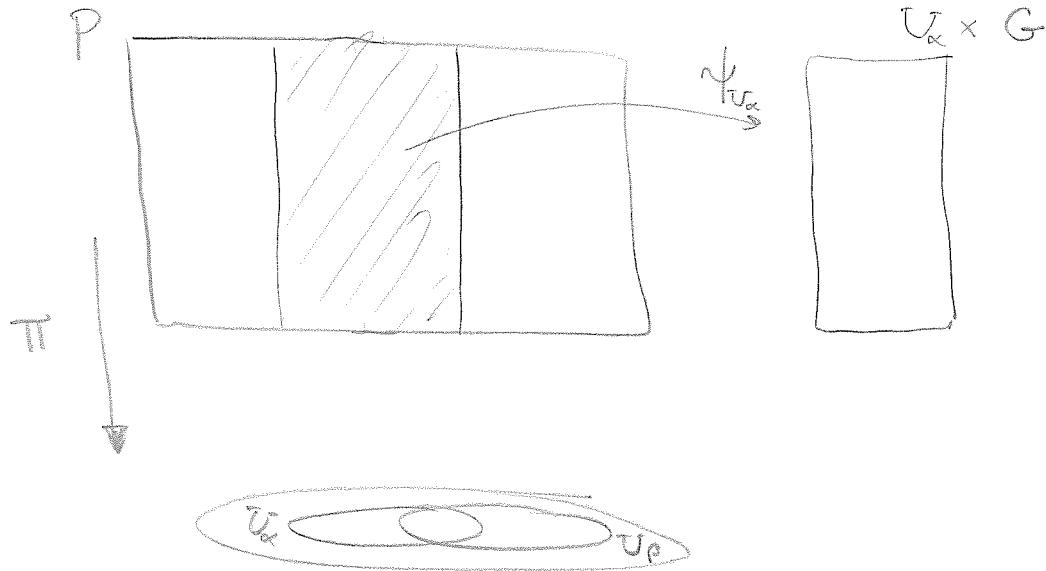
Now another question is how to construct local trivializing maps for

$$\begin{array}{c} G \\ \downarrow \\ G/H \end{array}$$

But this is better addressed once we know more about sections.

Let $P \xrightarrow{\pi} M$ be the Principal Fiber Bundle (PFB) with structure group G . Let $\{f_\alpha\}_{\alpha \in \Lambda}$ be a local (equivariant) trivialization. Write

$\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ an open cover of manifold.



If $U_\alpha \cap U_\beta \neq \emptyset$, we can get group-valued facts, called cocycles, conversely we can build fiber bundles from cocycles. Local \rightarrow Global

$$\psi_{U_\alpha}(ug) = \psi_{U_\alpha}(u) g \quad \text{for all } u \in U_\alpha \text{ and note } \psi_{U_\alpha}(u) \in \pi^{-1}(U_\alpha) \\ \text{so } g \text{ just moves it up/down the fiber.}$$

And for $x \in U_\alpha \cap U_\beta$ and $a \in G$. Note $\psi_{U_\alpha}(u) = (\pi(u), a)$

$$\begin{aligned} (\psi_{U_\beta} \circ \psi_{U_\alpha}^{-1})(x, a) &= \psi_{U_\beta}(\psi_{U_\alpha}^{-1}(x, e) \cdot a) \\ &= \psi_{U_\beta}(\psi_{U_\alpha}^{-1}(x, e) \cdot a) \\ &= \psi_{U_\beta}(\psi_{U_\alpha}^{-1}(x, e)) a \\ &= (x, g_{\alpha\beta}(x) a) \end{aligned}$$

Consequently $g_{\alpha\beta}(x) = \pi_{\text{second factor}}(\psi_{U_\beta} \circ \psi_{U_\alpha}^{-1}(x, e))$

Intuitively $\pi^{-1}(U_\alpha) \approx U_\alpha \times G$. Now on $\pi^{-1}(U_\alpha \cap U_\beta)$

there are 2 ways to identify it with $(U_\alpha \cap U_\beta) \times G$

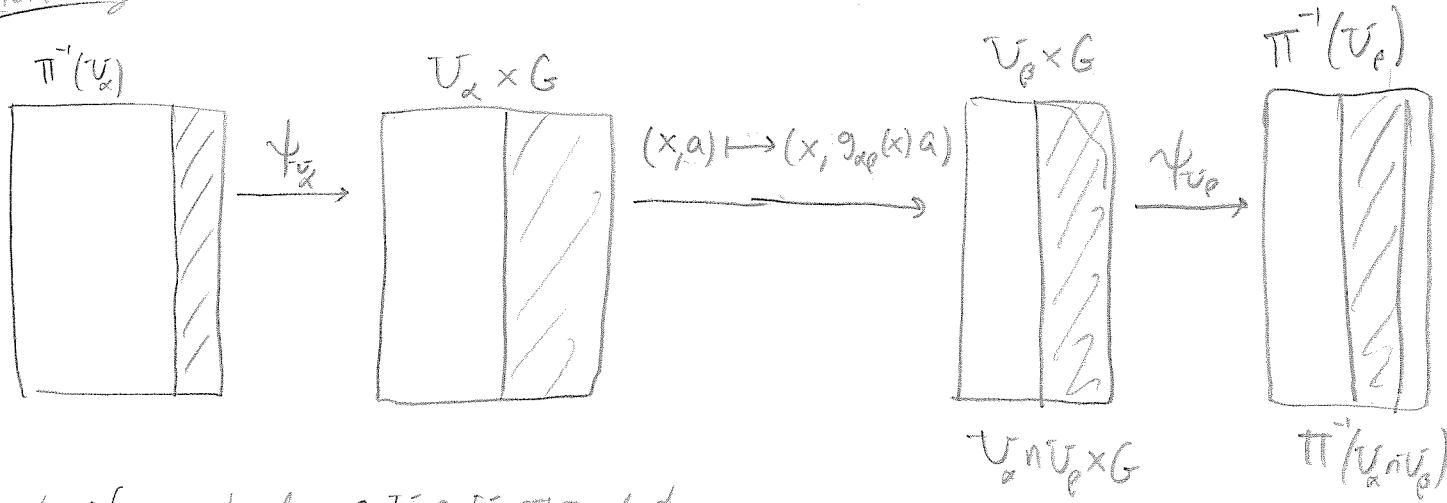
Due to the fact that $\varphi_{U_\alpha}, \varphi_{U_\beta}$ are equivariant we can "move" φ_{U_α} along its fibers with the group to get φ_{U_β} . Meaning that (since $\varphi_{U_\beta} \circ \varphi_{U_\alpha}^{-1} \approx g_{\alpha\beta}$ which moves up/down fiber by the group)

$$\varphi_{U_\beta} = (\varphi_{U_\beta} \circ \varphi_{U_\alpha})^{-1} \circ \varphi_{U_\alpha}$$

Where

$\varphi_{U_\beta} \circ \varphi_{U_\alpha}^{-1}$ translates $(x, a) \in U_\alpha \cap U_\beta \times G$ to $(x, g_{\alpha\beta}(x)a)$
this is a smooth transition along the fibers

Pictorially



Next if we had $x \in U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

$$\varphi_{U_\gamma} \circ \varphi_{U_\alpha}^{-1} = \varphi_{U_\gamma} \circ \varphi_{U_\beta}^{-1} \circ \varphi_{U_\beta} \circ \varphi_{U_\alpha}$$

$$\Rightarrow g_{\alpha\gamma}(x) = g_{\beta\gamma}(x) g_{\beta\alpha}(x) \quad \begin{cases} \text{(Changed convention from its defn)} \\ \forall x \text{ in triple intersection.} \end{cases}$$

$$U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} G$$

$$U_\beta \cap U_\gamma \xrightarrow{g_{\beta\gamma}} G$$

$$U_\gamma \cap U_\alpha \xrightarrow{g_{\gamma\alpha}} G$$

If we have $g_{\alpha\beta}$ for each pair of intersecting cover sets & the condition above holds then there are Czech cycles.

Consequences of $g_{\alpha\alpha}(x) = g_{\gamma\beta}(x)g_{\beta\alpha}(x)$

10/20/04 (3)

$$g_{\alpha\alpha}(x) = e$$

$$g_{\beta\alpha}(x) = g_{\alpha\beta}(x)^{-1}$$

these are the local transition functions. The proof probably came from Habayashi & Nomizu. (Need to buy this book.).

Manifold, Lie Group, open cover, and $g_{\beta\alpha}$ that are the check cocycles then you can build a principal fiber bundle from this data. This detail will help make sense of the bundle reduction for example $\mathcal{F}M$ reduces to $Og(M)$ thanks to the factoring of the metrics. In part. phys we use Higgs field to reduce bundle.

pick a vacuum



Thⁿ (On Handout from last time)

Let M be a manifold & G a Lie group. Assume

⑤ $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of M

such that $\forall \alpha, \beta$ such that $U_\alpha \cap U_\beta \neq \emptyset$

$\exists g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$. If

$\{g_{\alpha\beta}\}$ satisfy the cocycle condition then

\exists a PFB $P \xrightarrow{\pi} M$ with group G

which has a local triv. whose transition functions are the $\{g_{\alpha\beta}\}$:

$$g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha}$$

$c^0 : U_\alpha \rightarrow G$ cochains of type 0

$c^1 : U_\alpha \cap U_\beta \rightarrow G$ cochains of type 1

$c^2 : U_\alpha \cap U_\beta \cap U_\gamma \rightarrow G$ cochains of type 2

$$c^0 \xrightarrow{d} c^1 \xrightarrow{d} c^2$$

Check cohomology diff.
the differential is zero
if the condition is met.

Precisely those cocycles which are exact \Rightarrow Bundle Trivial.

Principal Bundles \longleftrightarrow Geck Cohomology. (See Bott & Tu)

Proof / $S = \bigcup_{\alpha \in \Lambda} (U_\alpha \times G)$ disjoint union. Meaning we can think of it

as $S = \{(x, g) \mid \alpha \in \Lambda, x \in U_\alpha, g \in G\}$ hence

$U_\alpha \times G \supseteq$ disjoint even if $U_\alpha \cap U_\beta = \emptyset$.

$U_\beta \times G \supseteq$

Let \sim be an atlas for $\{U_\alpha \times G\} \subseteq S$. Define an equivalence relation on S by

$$(\alpha, x, g) \sim (\beta, y, h) \iff x, y \in U_\alpha \cap U_\beta \subseteq M, y = x, g_{\alpha\beta}(x) = y$$

Should show \sim is equivalence relation. Would need to use cocycle cond. on $g_{\alpha\beta}$ to make it work.

Proof (Continued)

Let $[\alpha, x, g]$ = equivalence class of (α, x, g)

$$P = \frac{S}{\alpha}$$

this is how to glue together all the U_α 's together while respecting the structure of the transition functions. The details are in the notes (likely from Kobayashi & Nomizu) ~~✓~~

#

Remark: We know that the transition functions came from a local trivialization, yet the cocycle condition builds the principal bundle. Think...
 transition functions \rightarrow Unique Principal Bundle?

Def/ If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M

then ② $\pi: P \rightarrow M$ a PFB with group G

③ $\{g_{\beta\alpha}\}$ is a family of maps $g_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow G$
satisfying the cocycle condition

Then $\{g_{\beta\alpha}\}$ are called transition functions
of P if they are defined by a local
trivialization of $P \xrightarrow{\pi} M$.

Def/ Assume $\pi: P \rightarrow M$ is a PFB with group G
and that $H \subseteq G$ is a Lie subgroup of G .

A principal H -bundle $\tau: Q \rightarrow M$
is called a reduction of $\pi: P \rightarrow M$ iff

① Q is submanifold of P

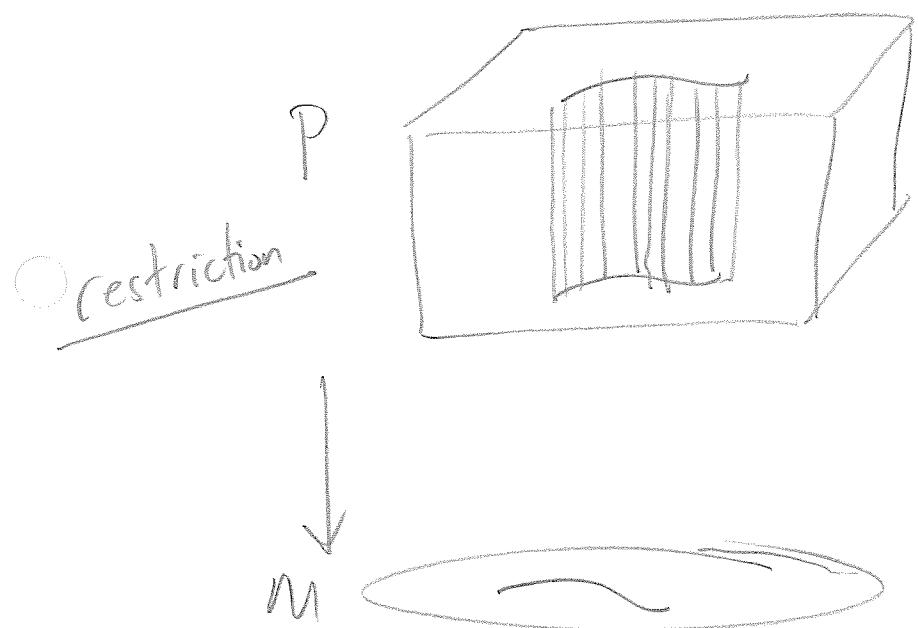
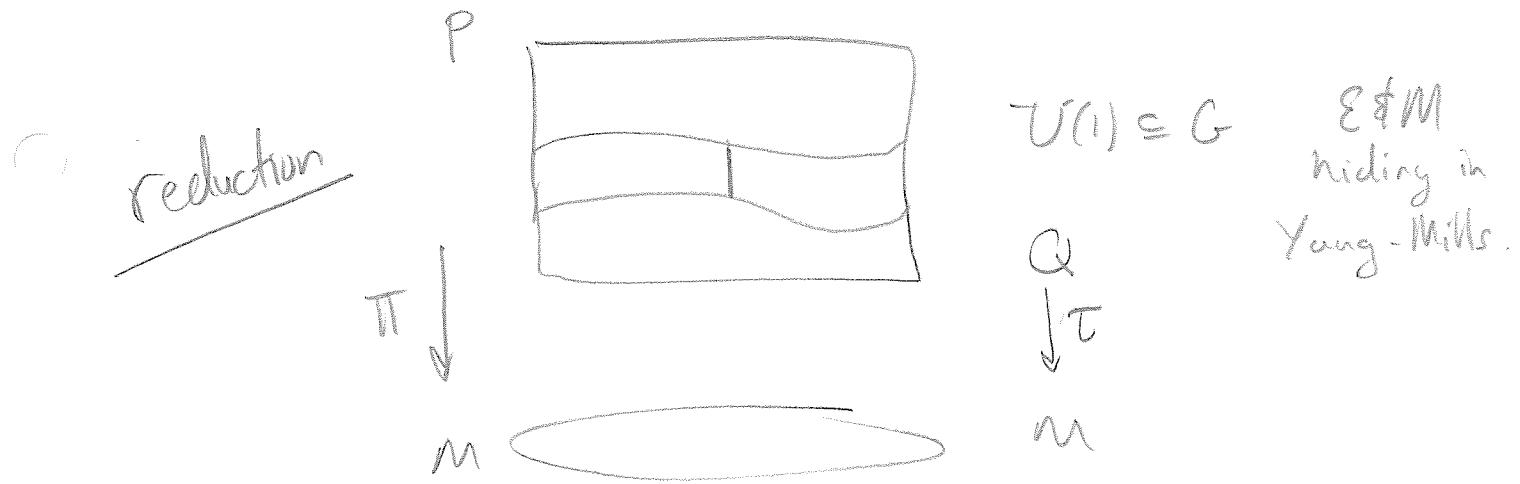
② $\tau = \pi|_Q$

We say Q is a reduced bundle of P . We say
such a P is reducible.

If $N \subseteq M$ is a submanifold of M
such that $Q = \pi^{-1}(N)$ is a submanifold of P
then $\pi|_Q: Q \rightarrow N$ is a principal G -bundle
which is called a restriction of P .

If $\tau: Q \rightarrow M$ is either a reduced
bundle or a restricted bundle of $\pi: P \rightarrow M$
then it is a sub-bundle of π .

10/22/04 ⑨



Thm Given a PFB $\pi: P \xrightarrow{G} M$ (has group G)

and a Lie subgroup $H \subseteq G$ then π is

reducible to H iff \exists open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of M and a family of transition maps $\{\varphi_{\beta\alpha}\}$ of π such that $\varphi_{\beta\alpha}(x) \in H \quad \forall x \in U_\alpha \cap U_\beta$

Pf First assume that $\pi: P \rightarrow M$ is reducible to $\tau: Q \rightarrow M$

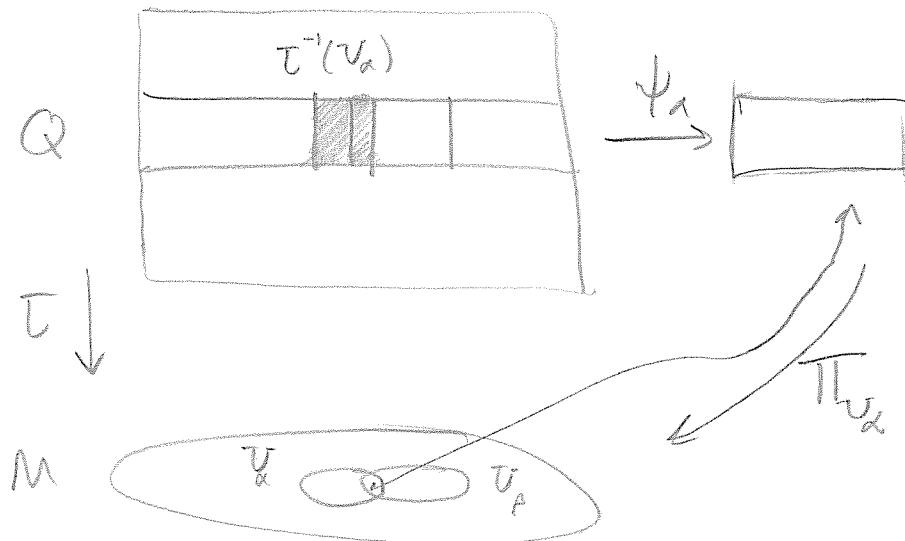
Let $\{\psi_\alpha\}_{\alpha \in A}$ be a local triv. of τ as a principal fiber bundle with group H .

Let $\psi_\alpha = \psi_{U_\alpha}$ to ease notation, $U_\alpha \in \mathcal{U}$,

Let $\{h_{\beta\alpha}\}$ be transition functions of $\{\psi_\alpha\}$ defined by the formula

$$h_{\beta\alpha}(x) = \underset{\substack{\uparrow \\ \text{drops 1st coordinate}}}{\pi_H^{-1}(\psi_\beta(\psi_\alpha^{-1}(x, e)))} \quad \forall x \in U_\alpha \cap U_\beta.$$

$U_\alpha \times H$



Define then:

$$d_\alpha(x) = \psi_\alpha^{-1}(x, e)$$

gives a section of

$$\text{Def}^{\frac{1}{2}} \quad s_{\alpha}(x) = \tilde{\psi}_{\alpha}^{-1}(x, e)$$

$$\circ \quad T(s_{\alpha}(x)) = T(\tilde{\psi}_{\alpha}^{-1}(x, e)) = x$$

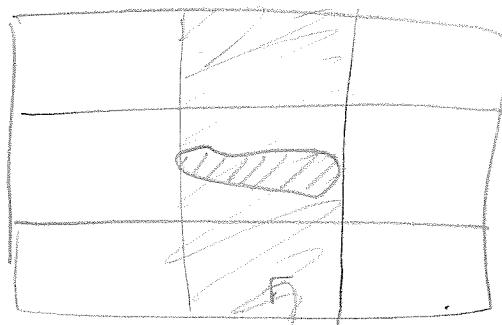
So s_{α} is a local section. Then

$$s_{\alpha}(x) \in T^{-1}(x) \subseteq \pi^{-1}(x)$$

So s_{α} is a local section of both π and T

$$\tilde{\psi}_{\alpha}(s_{\alpha}(x) \cdot g) = (x, g)$$

defines a trivializing map on π .



$$t \downarrow \quad t \downarrow \quad \text{trying to define } \tilde{\psi} \text{ on the whole fiber here. Well just translate the section vertically up & down the fibers.}$$

M

$$s_{\alpha}(x) \cdot g = \tilde{\psi}_{\alpha}^{-1}(x, g)$$

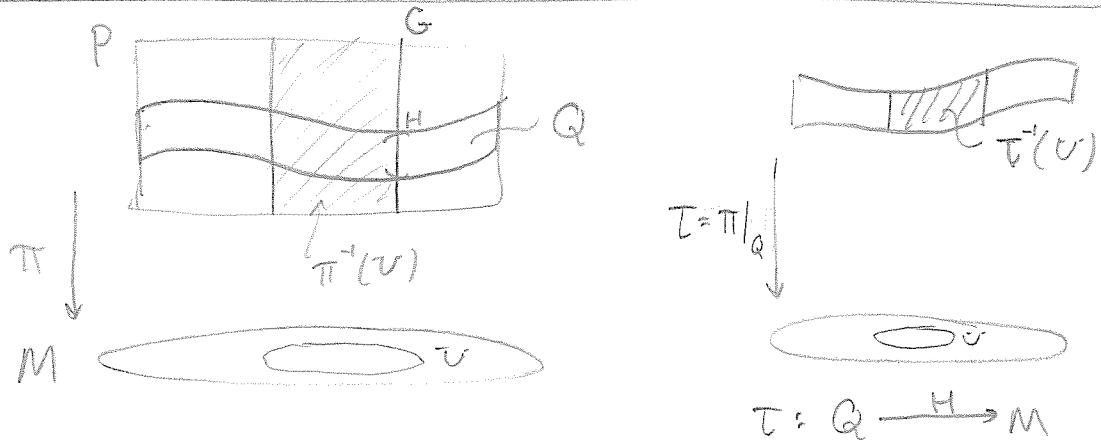
$\underbrace{\text{smooth}}$

$\overset{t}{\nearrow}$
smooth

thus by inverse fact.
the map is smooth....

10/25/04 ①

Th^b Given a PFB $\pi: P \xrightarrow{G} M$ and a Lie subgroup H of G , then π is reducible to H iff \exists an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of M and a family of transition functions $\{\varphi_{\alpha\beta}\}$ of π on \mathcal{U} with values of H .



• Motivated by the frame bundle reduction to $O_g(M)$ where $GL(n)$ reduces to $O(p, k)$

Proof: Assume $\pi: P \xrightarrow{G} M$ is reducible. Meaning we're given the existence of Q such that $T: Q \xrightarrow{H} M$ forms a bundle. So \exists then $\{\psi_\alpha\}$ local trivialization maps of $T = \pi|_Q: Q \rightarrow M$ where $V \in \mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ and $\psi_\alpha: T(V) \rightarrow U_\alpha \times H$

Define then $\psi_{\alpha\beta} = \psi_\beta \circ \psi_\alpha^{-1}$. Now there is a right action of

$P \times G \rightarrow G$ via $(P, g, g_2) = (P, g_1) \cdot g_2$ & $(P, e) = P \cdot e = P$ and

also $\psi_\alpha(P \cdot g) = \psi_\alpha(P) \cdot g$ that is the trivializing maps are equivariant

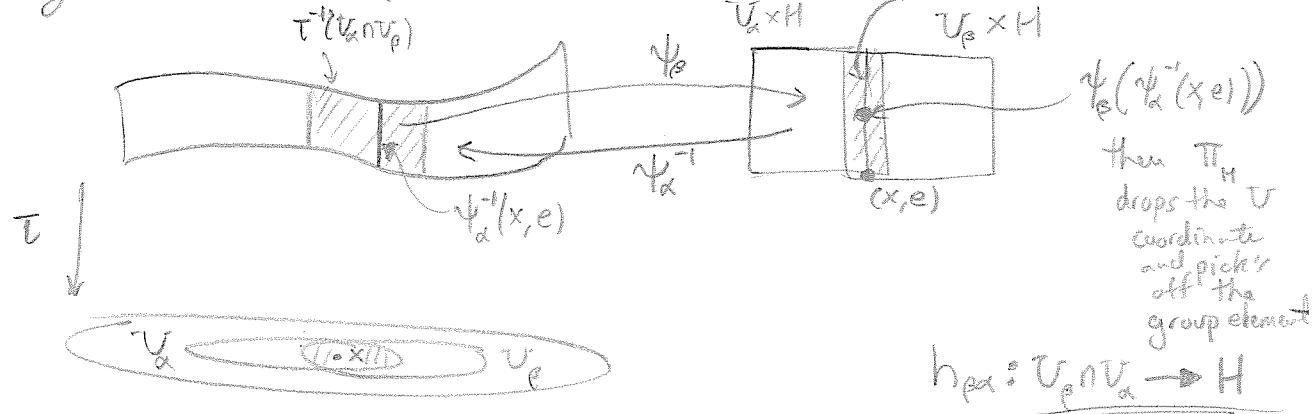
$$\psi_\alpha(P) = (x, h)$$

$$\psi_\alpha(P \cdot g) = (x, hg) = (x, h) \cdot g$$

- move in fiber or
- move in diffeomorphic image same results.

The transition functions of the ψ_α 's are maps $h_{\alpha\beta}(x) = \pi_H[(\psi_\beta \circ \psi_\alpha^{-1})(x, e)]$

Could define for Q or P here we def' them for Q and above we needed to say $x \in U_\alpha \cap U_\beta$.



then π_H drops the V coordinate and picks off the group element

$$h_{\alpha\beta}: U_\beta \cap U_\alpha \rightarrow H$$

Consequently, continuing proof: since $h_{\alpha}(x) = \Pi_H(N_\alpha \circ \psi_\alpha^{-1})(x, e)$ 10/25/04 (2)

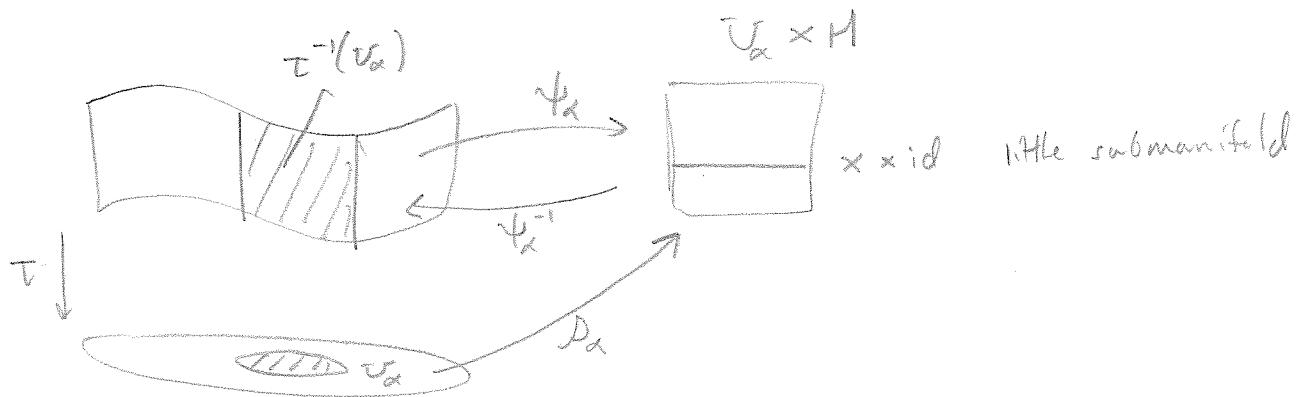
$$(\psi_\alpha \circ \psi_\alpha^{-1})(x, e) = (x, h_{\alpha}(x))$$

Now how to go from transition functions h_{α} on H to transition functions $g_{\alpha\beta}$ on G .

Observe that if $\Delta_\alpha : U_\alpha \rightarrow \tau^{-1}(U_\alpha) \subseteq \Pi^{-1}(U_\alpha)$ is def by

$$\Delta_\alpha(x) = \psi_\alpha^{-1}(x, e)$$

Then Δ_α is a section of $\tau^{-1}(U_\alpha) \rightarrow U_\alpha$.



Since clearly $\tau(\Delta_\alpha(x)) = x$.

In fact it is a section of $\Pi^{-1}(U_\alpha) \rightarrow U_\alpha$ since

$$\Pi(\Delta_\alpha(x)) = x \text{ as well.}$$

Now define a special local triv. induced on the bigien bundle.

$$\tilde{\psi}_\alpha : \Pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$$

$$\tilde{\psi}_\alpha(\Delta_\alpha(x)g) = (x, g)$$

If I let g vary I get the entire fiber. Note $\tilde{\psi}_\alpha$ is one-one and onto, and it's a diffeomorphism

$$\tilde{\psi}_\alpha^{-1}(x, g) = \Delta_\alpha(x) \cdot g$$

$$(x, g) \mapsto \Delta_\alpha(x) \cdot g = R_g(\Delta_\alpha(x)) \quad \begin{matrix} \text{smooth } G \text{ is a} \\ \text{Lie group.} \end{matrix}$$

$$R_g(p) = \sigma(g, p)$$

$$(R_g \circ \Delta_\alpha)(x) = \sigma(\Delta_\alpha(x), g) = \sigma \circ (\Delta_\alpha \times \text{id}_G)(x, g) \leftarrow \underline{\text{smooth}}$$

Inv. fact. Thⁿ will show $\tilde{\psi}_\alpha$ is smooth should define $\tilde{\psi}_\alpha^{-1}$ 1st
show it's smooth then to get $\tilde{\psi}_\alpha$ --

Continuing

Let $u \in \pi^{-1}(U_\alpha)$ and let $x = \pi(u)$ and $g \in G$ such that $u = s_\alpha(x) \cdot g$. Sensible since $s_\alpha(x) \cdot G = \pi^{-1}(x)$ so as $u \in \pi^{-1}(x) \Rightarrow \exists g$ so that $s_\alpha(x) \cdot g = u$.

$$\begin{aligned}\tilde{\psi}_\alpha(u \cdot \tilde{g}) &= \tilde{\psi}_\alpha((s_\alpha(x) \cdot g) \cdot \tilde{g}) \\ &= \tilde{\psi}_\alpha(s_\alpha(x) \cdot (g \tilde{g})) \\ &= (x, g \tilde{g}) \\ &= (x, g) \cdot \tilde{g} \\ &= \tilde{\psi}_\alpha(u) \cdot \tilde{g} \quad \text{so } \tilde{\psi}_\alpha \text{ is equivariant.}\end{aligned}$$

$\{\tilde{\psi}_\alpha\}$ are local trivialization maps of the PFB $\pi: P \xrightarrow{G} M$. We just showed $\tilde{\psi}_\alpha$ is equivariant.

We show now that the transition functions of $\{\tilde{\psi}_\alpha\}$ are just $f_{\beta\alpha}$. Let $\{g_{\beta\alpha}\}$ be the local transition functions of $\{\tilde{\psi}_\alpha\}$. Note that

$$\tilde{\psi}_\alpha(s_\alpha(x) \cdot g) = (x, g) \Rightarrow \tilde{\psi}_\alpha|_{\tilde{\psi}_\alpha^{-1}(U_\alpha)} = \psi_\alpha$$

Since s_α is section of q_α , $\psi_\alpha(s_\alpha(x) \cdot h) = \psi_\alpha(s_\alpha(x)) \cdot h = (x, h)$ but that's also $\tilde{\psi}_\alpha(s_\alpha(x) \cdot h)$ so $\tilde{\psi}_\alpha(s_\alpha(x) \cdot h) = (x, h)$ true for all x and all h in H . $\therefore \tilde{\psi}_\alpha|_{\tilde{\psi}_\alpha^{-1}(U_\alpha)} = \psi_\alpha$.

In particular then

$$s_\alpha(x) = \tilde{\psi}_\alpha^{-1}(x, e)$$

Hence we find;

$$\begin{aligned}g_{\beta\alpha}(x) &= (\pi_G \circ \tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e) \\ &= (\pi_G \circ \tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1} \circ \psi_\alpha \circ s_\alpha)(x) \\ &= (\pi_G \circ \tilde{\psi}_\beta)(s_\alpha(x)) \\ &= \pi_G(\psi_\beta(s_\alpha(x))) \quad \text{since } s_\alpha(x) \in \tilde{\psi}_\alpha^{-1}(U_\alpha) \\ &= \pi_G(x, h_{\beta\alpha}(x))\end{aligned}$$

$\boxed{g_{\beta\alpha}(x) = h_{\beta\alpha}(x)}$ transition functions match cause we picked the local triv. maps the way we did.

didn't prove th^m on 227, but it gives you the necessary conditions on the cocycles to construct PFB.
On page 233 we proved \Rightarrow now we'll do the converse.

Converse: Assume \exists a local trivialization, $\{\tilde{\Psi}_\alpha\}$ of $\pi: P \xrightarrow{G} M$ with transition functions $\{h_{\beta\alpha}\}$ with values in $H < G$. Want to build another bundle Q that imbeds in P such that G reduces to H , in the new bundle Q .

Parallel transport: Holonomy is minimal group which can reduce to is in fact the holonomy group for the fiber. We show π is reducible to H .

$$\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$$

$$\tilde{\Psi}_\alpha = \tilde{\Psi}_{U_\alpha}$$

$\{h_{\beta\alpha}\}$ satisfy the cocycle condition.

By th^m on 227. We can build a PFB $\tau: Q \xrightarrow{H} M$ with transition functions $\{h_{\beta\alpha}\}$

$$\bigsqcup_{\alpha \in \Lambda} U_\alpha \times H = \boxed{\quad} \quad \boxed{\quad} \quad \dots \quad \boxed{\quad}$$

$$U_\alpha \times H \quad U_\beta \times H \quad U_\gamma \times H$$

$$(\alpha, p, h) \sim (\beta, p_2, h_2) \Leftrightarrow p_1, p_2 \in U_\alpha \cap U_\beta \quad \& \quad p_1 = p_2 \\ h_2 = h_{\beta\alpha}(p)h_1,$$

\exists local trivializing functions

$$\{\Psi_\alpha\}$$
 for $\tau: Q \rightarrow M$

with $\{h_{\beta\alpha}\}$ as transition functions.

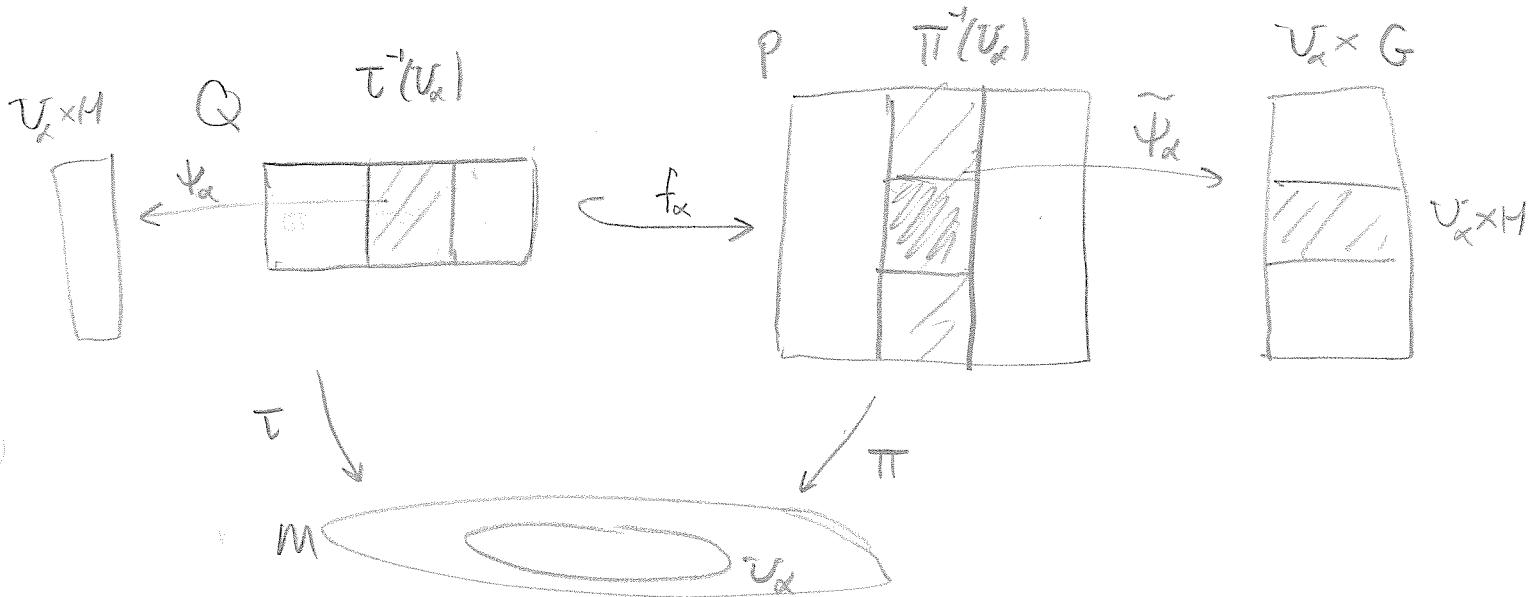
$$\psi_\alpha : \tau^{-1}(U_\alpha) \rightarrow U_\alpha \times H$$

$$f_\alpha = \tilde{\psi}_\alpha^{-1} \circ i_\alpha \circ \psi_\alpha$$

where $i_\alpha : U_\alpha \times H \longrightarrow U_\alpha \times G$ (an inclusion of sets.)

So then notice

$$f_\alpha : \tau^{-1}(U_\alpha) \rightarrow \pi^{-1}(U_\alpha)$$



Now f_α is smooth, injective and

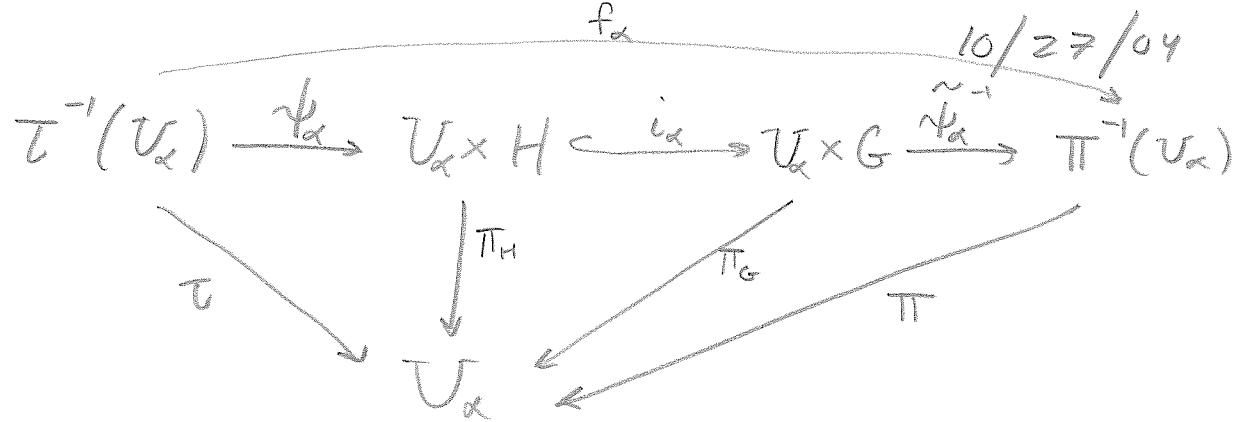
$$\begin{aligned} f_\alpha(uh) &= \tilde{\psi}_\alpha^{-1}(i_\alpha(\psi_\alpha(uh))) \\ &= \tilde{\psi}_\alpha^{-1}(i_\alpha(\psi_\alpha(u)h)) \\ &= \tilde{\psi}_\alpha^{-1}(i_\alpha(\psi_\alpha(u)) \cdot h) \\ &= \tilde{\psi}_\alpha^{-1}(i_\alpha(\psi_\alpha(u))) \cdot h \\ &= f_\alpha(u)h \end{aligned}$$

inverse is equivariant
as well.

Hence f_α is a fiber bundle morphism, in fact its a diffeomorphism. --

10/27/04

(3)



Simply:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{f_\alpha} & \pi^{-1}(U_\alpha) \\ \pi \searrow & & \swarrow \pi \\ & & U_\alpha \end{array}$$

this bundle mapping commutes with the projections.
we've shown they're locally imbeddible now

we show that $f_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)} = f_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)}$

so on the overlaps the f_α 's paste together
correctly to make $f_\alpha: Q \rightarrow P$. Note that

$$\begin{aligned} (\psi_\beta \circ \psi_\alpha^{-1})(x, e) &= (x, h_{\beta \alpha}(x)) \\ &= (\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e) \end{aligned}$$

action on
bundle is
equivariant

$$\text{Thus } (\psi_\beta \circ \psi_\alpha^{-1})(x, h) = (\psi_\beta \circ \psi_\alpha^{-1})(x, e) \cdot h$$

$$= (\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, e) \cdot h \quad \text{by above}$$

$$= (\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, h) \quad \text{for } x \in U_\alpha \cap U_\beta \text{ of course.}$$

$$\begin{aligned}
 (f_\beta \circ \tilde{\psi}_\alpha^{-1})(x, h) &= (\tilde{\psi}_\beta^{-1} \circ i_\beta \circ \tilde{\psi}_\alpha)(\tilde{\psi}_\alpha^{-1}(x, h)) \\
 &= (\tilde{\psi}_\beta^{-1} \circ i_\beta)(\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, h) \\
 &= (\tilde{\psi}_\beta^{-1} \circ \tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(x, h) \quad \text{since } i_\beta \text{ is identity} \\
 &= \tilde{\psi}_\alpha^{-1}(x, h) \quad \text{on H type pairs.}
 \end{aligned}$$

True $\forall (x, h)$ thus,

$$f_\beta \circ \tilde{\psi}_\alpha^{-1} = \tilde{\psi}_\alpha^{-1}$$

$$\begin{aligned}
 f_\beta &= \tilde{\psi}_\alpha^{-1} \circ \tilde{\psi}_\alpha \\
 &= \tilde{\psi}_\alpha^{-1} \circ i_\alpha \circ \tilde{\psi}_\alpha \quad \text{since } \tilde{\psi}_\alpha \text{ goes into} \\
 &= f_\alpha \quad (T_\alpha, H) \text{ and}
 \end{aligned}$$

Hence $f_\beta = f_\alpha$.

Next time: pg. 237 then onto sections