

MAXIMUMS AND MINIMUMS:

In this section of the notes we will give a series of definitions which will allow us to precisely describe many interesting features of graphs. We will find that the critical #'s of f tell us much about its graph. (def's taken from §4.2 & §4.3 of Stewart's 2nd Ed.)

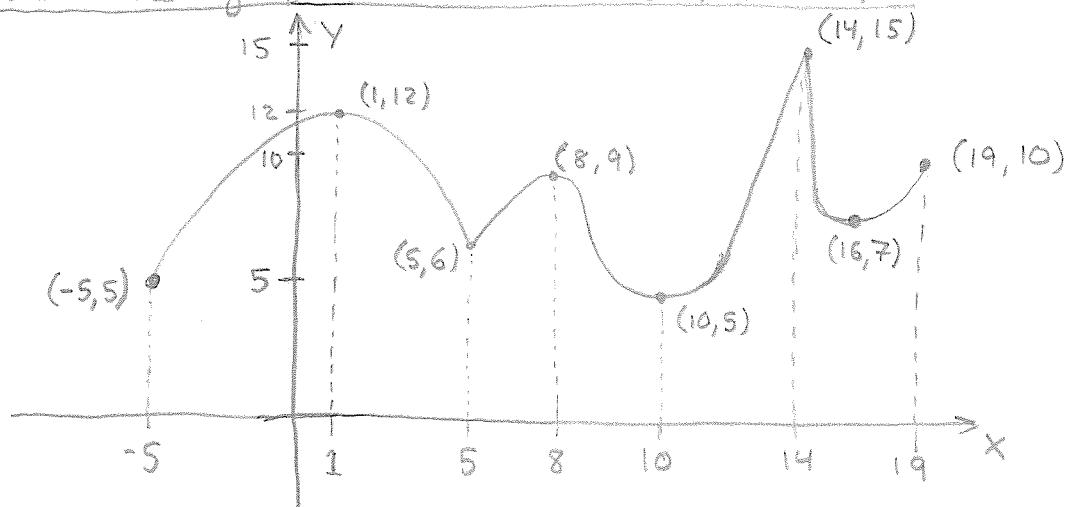
Defⁿ A function f has an absolute maximum at c if $f(c) \geq f(x) \quad \forall x \in \text{Dom}(f)$. The absolute maximum is $f(c)$. Likewise f has an abs. min. at d if $f(d) \leq f(x) \quad \forall x \in \text{Dom}(f)$. The absolute minimum is $f(d)$. We call the abs. max/min values the extreme values of f .

- Sometimes the word "absolute" is replaced with "global". We also can talk about local max/min.

Defⁿ A function f has a local max. at c if \exists some interval J containing c so that $f(c) \geq f(x) \quad \forall x \in J$. Likewise f has a local min. at d if \exists some interval I containing d so that $f(d) \leq f(x) \quad \forall x \in I$.

Th^m (EXTREME VALUE TH^m) If f is continuous on $[a, b]$ then f reaches its abs. max. $f(c)$ and an abs. min. $f(d)$ at some #'s c and d in $[a, b]$.

E1 Consider the graph below of some function f :



- f has local maximums at: $x = 1, 8, 14$ and 19
the values of the local max's are $= 12, 9, 15$ and 10 respectively.
- f obtains its absolute max at $x = 14$, its abs. max is $f(14) = 15$
- f obtains its abs. min at $x = -5$ and 10 , its abs. min is 5

Defⁿ A critical number of a function f is a number c in the domain of f such that one of the following holds,

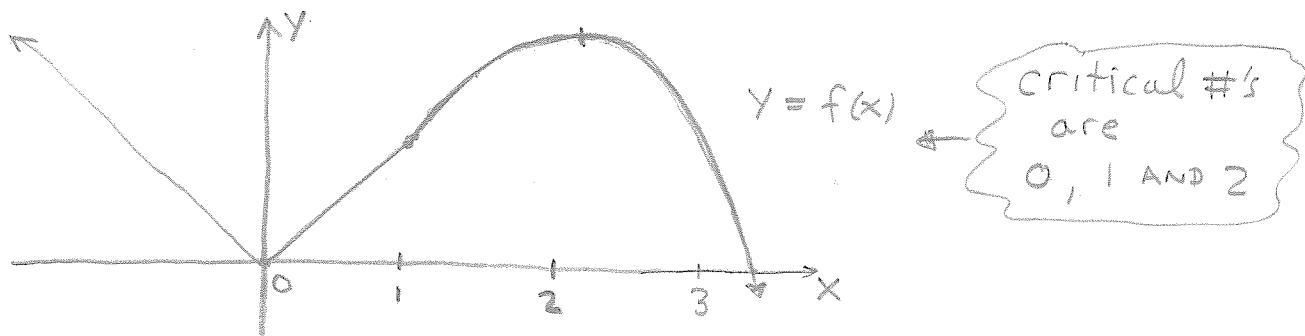
- 1.) $f'(c) = 0$.
- 2.) $f'(c)$ does not exist.

- Primarily 1.) is what we'll find in applications, but it's wise to remember 2.) also counts as a critical #.

E2 Lets Consider a piecewise defined function:

$$f(x) = \begin{cases} |x| & : x < 1 \\ -(x-2)^2 + 2 & : x \geq 1 \end{cases} \rightarrow f'(x) = \begin{cases} -1 & x < 0 \\ 2(x-2) & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

Notice that $f'(0)$ and $f'(1)$ d.n.e so $x=0, 1$ are critical #'. Also $2(x-2) = 0$ when $x=2$ which is also a critical #.



Notice that this function takes its local min at $x=0$ and its local max at $x=2$. This is no accident,

Thⁿ If f has a local max/min at c , then c is a critical number of f (Fermat's Thⁿ)

- The converse need not be true, for instance in the example above $x=1$ is a critical point (why? do you really know?) and $f(1) = 1$ is not a local max or min. Next we'll give a Thⁿ which gives a prescription to find max/min on a closed interval for continuous function.

Theorem / CLOSED INTERVAL METHOD: Given a continuous function f on an interval $[a, b]$ then find the abs. max/min values for f on $[a, b]$ as follows,

- ① locate all critical numbers c in (a, b) and calculate $f(c)$.
- ② calculate $f(a)$ and $f(b)$
- ③ compare the value(s) $f(c)$ with $f(a)$ and $f(b)$. The biggest is the abs. max of f on $[a, b]$ while the smallest is the abs. min. of f on $[a, b]$.

E3 Let $f(x) = \sin(x)$ for $0 \leq x \leq 2\pi$. Note f is continuous with,
 $f'(x) = \cos(x)$

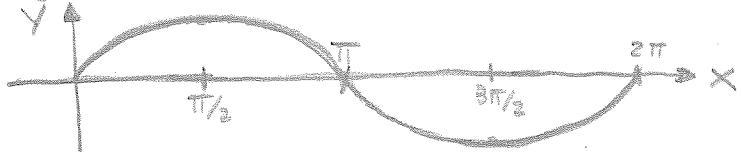
Critical numbers are sol's to $\cos(x) = 0$ for $0 \leq x \leq 2\pi$, namely,

$$x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2}$$

Now calculate:

$$\left. \begin{array}{l} f(0) = \sin(0) = 0 \\ f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1 \\ f(\frac{3\pi}{2}) = \sin(\frac{3\pi}{2}) = -1 \\ f(2\pi) = \sin(2\pi) = 0 \end{array} \right\} \begin{array}{l} \text{Comparing we find that} \\ \text{the max is 1 at } x = \frac{\pi}{2} \\ \text{and the min is -1 at } x = \frac{3\pi}{2} \end{array}$$

Hence the graph we know and love, $Y = \sin(x)$,

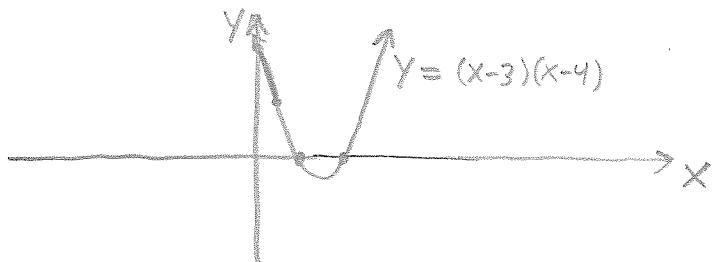


E4 Consider $f(x) = (x-3)(x-4)$ find max on $[0, 1]$. Notice

$$f'(x) = (x-4) + (x-3) = 2x - 7 \Rightarrow f'(x) = 0 \text{ for } x = \frac{7}{2}$$

Thus there are no critical #'s for f on $[0, 1]$ we need only check the endpts.

$$\left. \begin{array}{l} f(0) = (-3)(-4) = 12 \\ f(1) = (-2)(-3) = 6 \end{array} \right\} \Rightarrow f(0) = 12 \text{ is max of } f \text{ on } [0, 1]$$



Examples: find absolute maximum and minimum values of f on the given interval.

E5

$$f(x) = 3x^2 - 12x + 5 \text{ on } [0, 3]$$

① $f'(x) = 6x - 12$ well-defined on $[0, 3] \Rightarrow$ critical #'s only with $f'(c) = 0$.

$$f'(c) = 6c - 12 = 0 \Rightarrow c = 2 \text{ which is on } [0, 3] \quad (\text{Critical Number})$$

$$f(2) = 3(2)^2 - 12(2) + 5 = 12 - 24 + 5 = -7$$

② $f(0) = 5$
 $f(3) = 3(9) - 12(3) + 5 = 27 - 36 + 5 = -4$ } values of f at the endpoint.

③ Thus

$f(2) = -7$ is the absolute minimum of f on $[0, 3]$

$f(0) = 5$ is the absolute maximum of f on $[0, 3]$

E6

$$f(x) = x^4 - 2x^2 + 3 \text{ on } [-2, 3]$$

① $f'(x) = 4x^3 - 4x$ so $f'(c) = 0$ are the only type of critical #'s the derivative exists everywhere.

$$f'(c) = 4c^3 - 4c = 0$$

$$4c(c^2 - 1) = 0 \therefore c=0 \text{ & } c = \pm 1 \text{ are critical #'s}$$

$$f(0) = 3$$

$$f(1) = 1 - 2 + 3 = 2$$

$$f(-1) = 1 - 2 + 3 = 2$$

② Endpoints $f(-2) = (-2)^4 - 2(-2)^2 + 3 = 16 - 8 + 3 = 11$

$$f(3) = 3^4 - 2(3)^2 + 3 = 81 - 18 + 3 = 66$$

③ Comparing ① & ② we find

$$f(\pm 1) = 2 \text{ is the abs. min on } [-2, 3]$$

$$f(3) = 66 \text{ is the abs. max. on } [-2, 3]$$

Derivatives & the shape of curves

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Increasing and Decreasing test

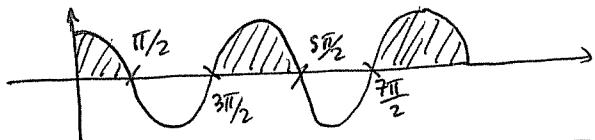
- If $f'(x) > 0$ for all x on (a, b) then f is increasing on (a, b)
- If $f'(x) < 0$ for all x on (a, b) then f is decreasing on (a, b)

Examples:

[E7] Let $f(x) = e^x$ then $f'(x) = e^x > 0$ for $x \in \mathbb{R}$ so $f(x) = e^x$ is increasing on all of \mathbb{R} .

[E8] Let $f(x) = x^2 - 2x + 1$ then $f'(x) = 2x - 2 = 2(x-1)$
So $f'(x) > 0$ when $x > 1$ and $f'(x) < 0$ when $x < 1$
Thus f is increasing on $(1, \infty)$ and f is decreasing on $(-\infty, 1)$.

[E9] Let $f(x) = \sin(x)$ then $f'(x) = \cos(x)$ so we draw a graph

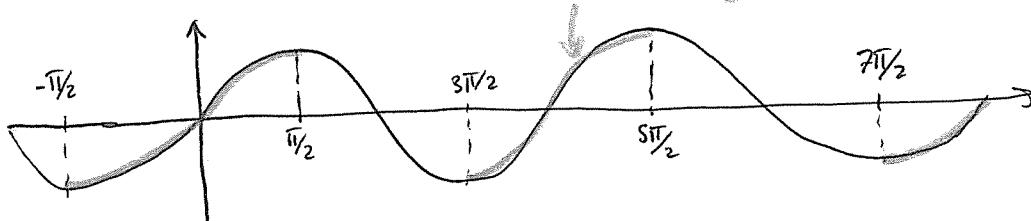


$f'(x) > 0$ when x is in $(-\pi/2, \pi/2)$, $(3\pi/2, 5\pi/2)$, $(7\pi/2, 9\pi/2)$, ...

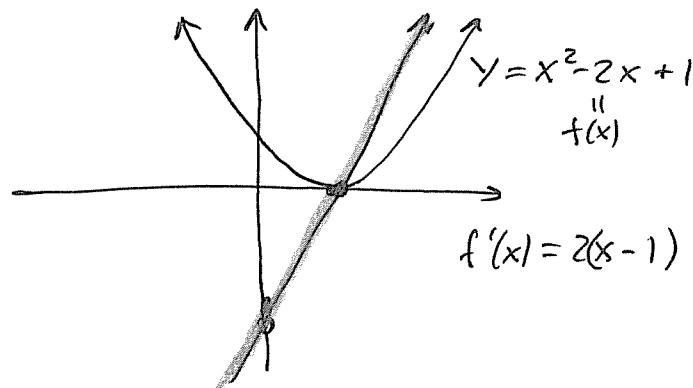
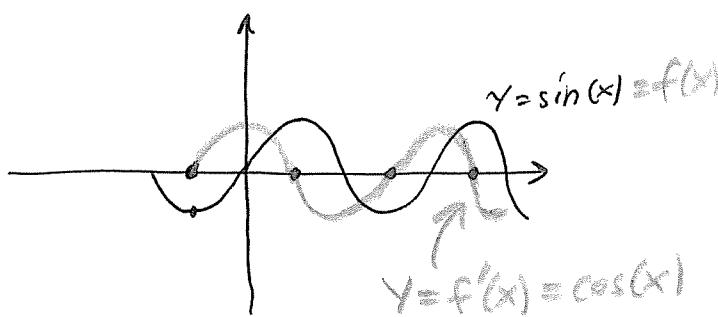
$f'(x) < 0$ when x is in $(\pi/2, 3\pi/2)$, $(5\pi/2, 7\pi/2)$, $(4\pi/2, 11\pi/2)$, ...

Does this make sense?

Increasing



Notice that for our examples we have local maximums or minimums where $f'(x)$ changes sign



Although for our examples the derivative exists everywhere we can generalize slightly to:

Th^m (First Derivative Test) Suppose that c is a critical # of a continuous fnct. f .

a.) If f' changes from pos. to neg. at c then $f(c)$ is a local maximum

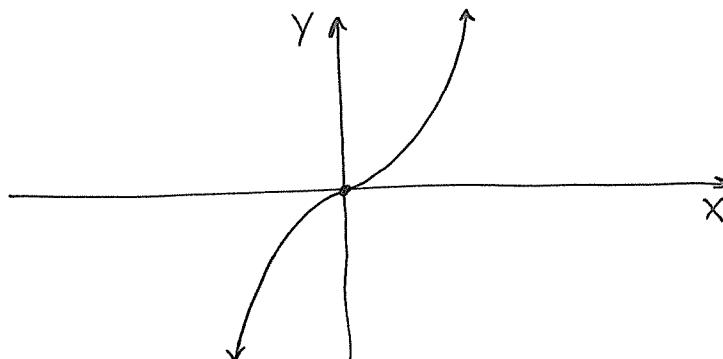
b.) If f' changes from neg. to pos. at c then $f(c)$ is a local minimum

c.) If f' doesn't change signs at c then f does not have a local max/min at c .

E10 Example of c.) Let $f(x) = x^3$ then $f'(x) = 3x^2$
thus $c=0$ is a critical # for f but

$$f'(x) = 3x^2 > 0$$

So $f'(x)$ does not change signs at $c=0$. Why?



Defⁿ/ Concavity: A function f is concave upward on I if f' is an increasing function on I . A function f is concave down on I if f' is a decreasing fnct. on I .

In other words:

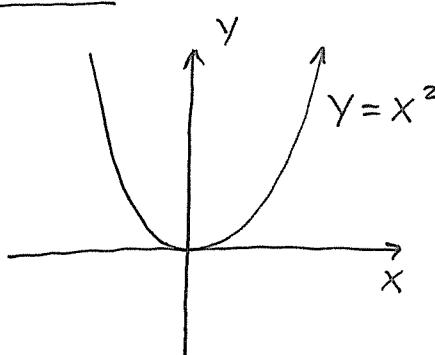
- 1.) If $f''(x) > 0 \quad \forall x \in I$ then f is concave up on I .
- 2.) If $f''(x) < 0 \quad \forall x \in I$ then f is concave down on I .

Defⁿ/ A point where f'' changes sign is called an inflection point. It is where the concavity of f changes.

- on pg. 70 in E10 $x=0$ was an inflection point, it was a critical number which was neither a max nor a min.

QUESTION: We have seen that critical #'s give min/max and inflection points possibly, is that all a critical # can give? Bonus Point: give me a function which has a critical point which is neither a max, min or inflection point.

- Remark: functions can be concave up, down or neither,

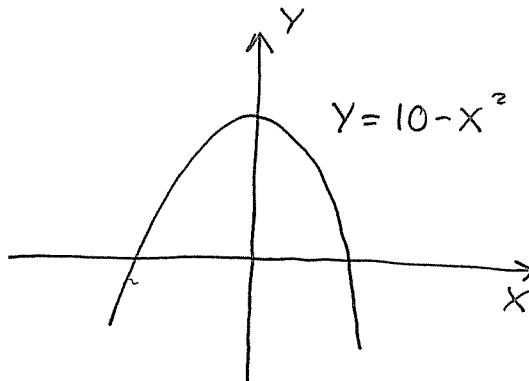


$$y' = 2x$$

$$y'' = 2$$

$$(f'' > 0)$$

Concave up

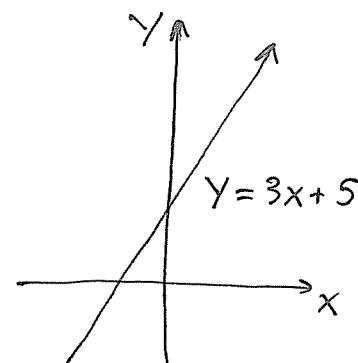


$$y' = -2x$$

$$y'' = -2$$

$$(f'' < 0)$$

Concave down



$$y' = 3$$

$$y'' = 0$$

$$(f'' = 0)$$

neither

Th^m / SECOND DERIVATIVE TEST:

- 1.) If $f'(c) = 0$ and $f''(c) < 0$ then $f(c)$ is a local max of f at $x=c$.
- 2.) If $f'(c) = 0$ and $f''(c) > 0$ then $f(c)$ is a local min of f at $x=c$.
- 3.) If $f'(c) = 0$ and $f''(c) = 0$ you should try the 1st DERIVATIVE TEST INSTEAD.

• Notice that this is consistent with the remark on (71).

E11 Consider $f(x) = x^3 - 12x - 5$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$

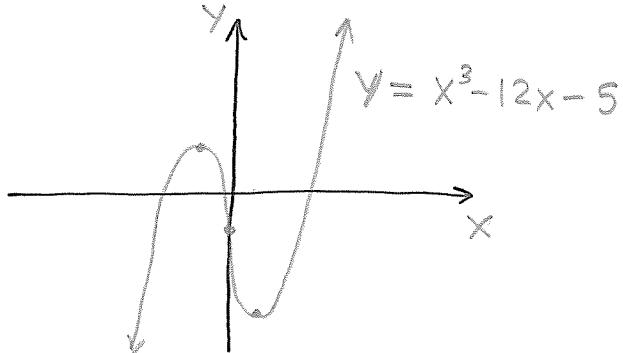
$$f''(x) = 6x$$

Critical #'s are $x = \pm 2$ where $f'(\pm 2) = 0$. Then

$$f''(2) = 6(2) = 12 \quad \therefore f(2) = 8 - 24 - 5 = -21 \quad \underline{\text{local min.}}$$

$$f''(-2) = 6(-2) = -12 \quad \therefore f(-2) = -8 + 24 - 5 = 11 \quad \underline{\text{local max}}$$

We can additionally note $x=0$ is an inflection point. Sketching f ,



PUTTING IT ALL TOGETHER

E12 We will study a # of questions about $f(x) = \frac{x}{(1+x)^2}$. I will introduce some notational helps to keep track of signs of f , f' and f'' . If you have your own system for finding zeroes etc... from precalculus then feel free to use it. Find the following,

- a.) CRITICAL POINTS
- b.) intervals on which f is decreasing/increasing
- c.) local maximums/minimums
- d.) intervals on which f is concave up/down
- e.) inflection points
- f.) zeroes of function (precalculus here)
- g.) graph f carefully using a $\rightarrow f$.

Let's begin by calculating f' and f'' ,

$$\begin{aligned}
 f'(x) &= \frac{\frac{dx}{dx}(1+x)^2 - x \frac{d}{dx}(1+x)^2}{(1+x)^4} \\
 &= \frac{1+2x+x^2 - 2x-2x^2}{(1+x)^4} \\
 &= \frac{1-x^2}{(1+x)^4} \quad : \quad 1-x^2 = (1+x)(1-x) \text{ thus,} \\
 &= \boxed{\frac{1-x}{(1+x)^3}} = f'(x)
 \end{aligned}$$

- a.) Notice $f'(1) = 0$ and $f'(-1)$ d.n.e thus the critical #'s of f are $C = \pm 1$. Now we know that f' can only change sign at critical #'s thus we can just check a pt. from each region to completely describe f' positive/neg.



- b.) from the sign-chart of f' we can read off that

- f is decreasing on $(-\infty, -1)$ and $(1, \infty)$
- f is increasing on $(-1, 1)$

E12 continued

c.) We use the 1st derivative test to conclude that since f' changes from (+) to (-) at $x = 1$ the value $f(1) = \frac{1}{2^2} = \frac{1}{4}$ is a local max of f at $x = 1$. ($x = -1$ is a vertical asymptote so although it's a critical point $f(-1)$ can't be a local max, it d.n.e.)

d.) We need to find f'' and how it behaves,

$$f''(x) = \frac{d}{dx} \left(\frac{1-x}{(1+x)^3} \right) = \frac{2(x-2)}{(x+1)^4}$$

Notice that f'' can only change sign at its zeroes and vertical asymptotes which occur at $x = 2$ and $x = -1$. We need to check 3 points to generate the following,

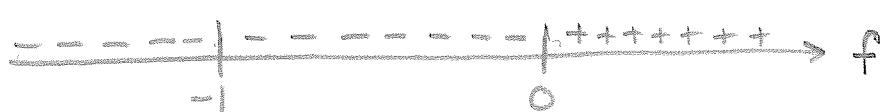


We can read off the sign chart for f'' that

f is concave down on $(-\infty, 2)$
 f is concave up on $(2, \infty)$

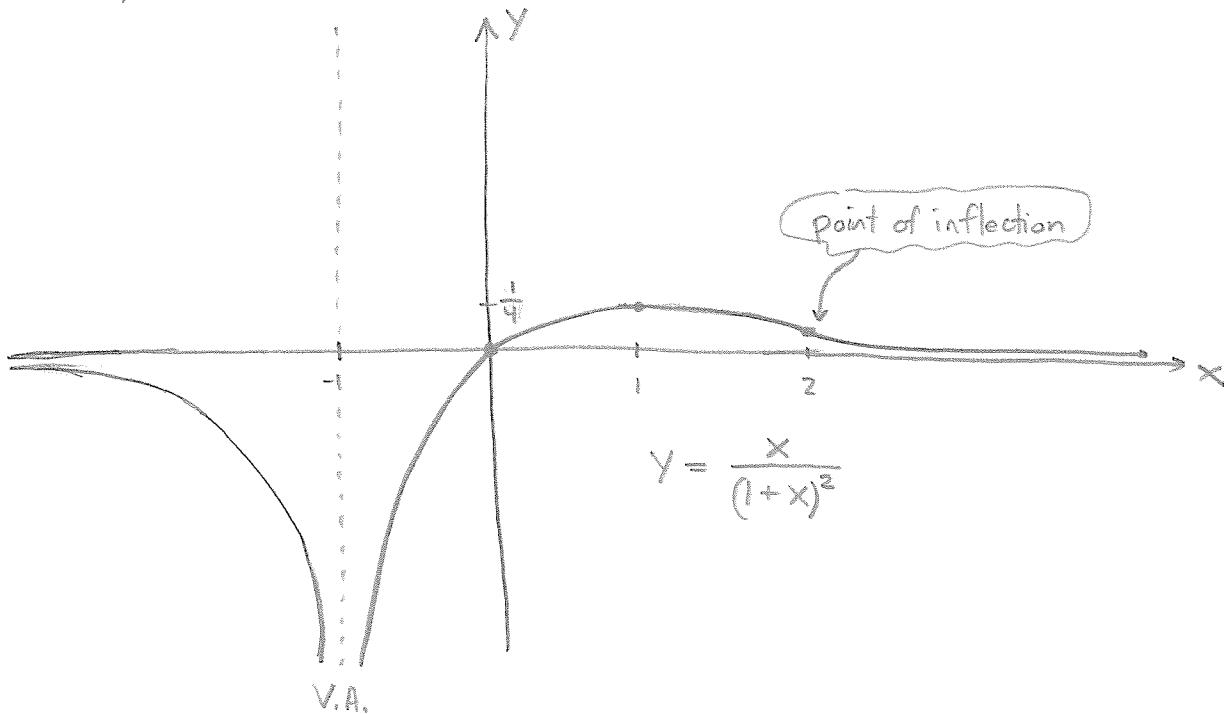
e.) f changes concavity at $x = 2$ which is the only inflection point.

f.) arguably I should've put this 1st, oh well anyway $f(0) = 0$ and $f(-1)$ d.n.e., f must change signs either 0 or -1 so we again just need to check 3 test pts.



ok, I only asked for the zeroes of f but this will help us graph f next

E12 conclusion Now let's assemble the graph of f using all the data.



E13 Do the same as E12 for $x e^{-x}$,

$$f(x) = x e^x \quad \Rightarrow \quad \text{graph of } f$$

$$f'(x) = e^x(1-x) \Rightarrow \overbrace{\quad\quad\quad}^{\text{Multiplikation}} f''$$

$$f''(x) = e^x(x-2) \Rightarrow \overbrace{\dots + \dots + \dots + \frac{1}{2}}_{2} \rightarrow f''$$

Therefore,

- a.) critical point is $x=1$

b.) f increases on $(-\infty, 1)$ and decreases on $(1, \infty)$

c.) f has a local max of $f(1) = \frac{1}{e}$ at $x=1$ by 1st Der. Test.

d.) f concave up on $(2, \infty)$ and concave down on $(-\infty, 2)$.

e.) $x=2$ is an inflection point of f .

f.) $x=0$ is the only zero of f

g.) graph

The graph shows a curve starting from the left, decreasing towards a local maximum at the point $(1, \frac{1}{e})$, which is marked with a dot and labeled. After the maximum, the curve decreases again. At the point $x=2$, there is a sharp change in the curvature: it goes from being concave down to being concave up, which is highlighted with a callout box labeled "inflection point". The y-axis is labeled with a vertical arrow pointing upwards, and the x-axis is labeled with a horizontal arrow pointing to the right.

